Notes

- Reading: Ch. 2 of text and Appendix A of text (complex numbers).

Skill Problems

1. **[15]** Concept(s): correlation and the effect of signal operations
   In the readings, it was stated that \( C_N(ax, by) = C_N(x, y) \). This is correct if \( ab > 0 \), but not quite if \( ab < 0 \).
   Let \( x(t) \), \( y(t) \), and \( z(t) \) denote signals, and let \( a \) and \( b \) denote nonzero real numbers.
   Show the following relationships.
   (a) \([5]\) \( C(ax, by) = abC(x, y) \)
   (b) \([5]\) \( C(x, y + z) = C(x, y) + C(x, z) \). (These first two properties are called bilinearity.)
   (c) \([0]\) \( C(x, y) = C(y, x) \) (for real signals)
   (d) \([5]\) \( C_N(ax, by) = \begin{cases} C_N(x, y), & ab > 0 \\ -C_N(x, y), & ab < 0. \end{cases} \)
   \([0]\) What happens if \( ab = 0 \)?
   (e) \([0]\) \( C_N(x, \alpha x) = \begin{cases} 1, & \alpha > 0 \\ -1, & \alpha < 0. \end{cases} \) (In fact, \( C_N(x, y) = \pm 1 \) if and only if \( y \) is an amplitude-scaled version of \( x \), so \( y \) and \( x \) have identical “shapes.”)
   (Think about how correlation is affected by other signal operations, e.g., amplitude shift.)

2. **[25]** Concept(s): representations of sinusoidal signals
   When relevant below, use the principal value for the phase: \(-\pi < \phi \leq \pi\).
   (a) \([5]\) A sinusoidal signal \( x(t) \) has amplitude=5, frequency 40Hz, and phase = \( \pi/3 \) radians.
      Sketch \( x(t) \) carefully by hand, labeling your axes.
   (b) \([5]\) Express the following signal in the standard form, i.e., in the form \( A \cos(2\pi f_0 t + \phi) \):
      \[ y(t) = -7 \sin(8\pi(t - 3) + 13\pi/4) \]
   (c) \([5]\) Find an expression in standard form for the following sinusoidal signal.
      ![Image of z(t) signal]
   (d) \([5]\) Simplify the following sum of sinusoidal signals into standard form:
      \[ s(t) = 5 \sin(8t) + 5 \cos(8t - \pi/3) \]
   (e) \([5]\) Find a complex-valued signal \( \bar{x}(t) \) such that \( x(t) = \text{Re}(\bar{x}(t)) \), for \( x(t) \) as defined in part (a).
3. [30] Concept(s): **complex arithmetic**
   (a) [15] Convert the following complex numbers from cartesian form to complex exponential form and plot in the complex plane: 
   \[ z_1 = \sqrt{3} + j, \quad z_2 = -\sqrt{2} + j\sqrt{2}, \quad z_3 = -2 - j. \]
   (b) [10] Determine the product of \( z_1 \) and \( z_2 \) by:
   - performing multiplication entirely in cartesian coordinates,
   - performing multiplication entirely with the exponential forms of these complex variables.
   (c) [10] Determine the ratio \( z_1/z_2 \) by:
   - performing division by first converting \( z_1 \) and \( z_2 \) to exponential form,
   - performing division by multiplying the numerator and denominator of \( z_1/z_2 \) by \( z_2^* \).
   (d) [0] Which form is easier for multiplication and division? What about for addition and subtraction?

4. [30] Concept(s): **complex arithmetic**
   Simplify the following complex-valued expressions.
   For (a)-(d), give answers in both Cartesian form and exponential (or polar) form.
   (a) [4] \( 2e^{j\pi/3} + 4e^{-j\pi/6} \)
   (b) [4] \((\sqrt{3} - j 3)^9 \)
   (c) [4] \((\sqrt{3} - j 3)^{-1} \)
   (d) [4] \((\sqrt{3} - j 3)^{1/3}\). Hint: there are three answers.
   (e) [4] \( \text{Re}(je^{-j\pi/3}) \)
   (f) [10] Determine all solutions \( \theta \) (in radians) to the following equation: \( \text{Re}((1 - j)e^{j\theta}) = -1. \)

5. [15] Concept(s): **sums of sinusoidal signals with same frequency and phasors, effect of time shift/scale**
   Simplify the following sums of sinusoidal signals into standard form.
   (a) [0] \( x_1(t) = 5\cos(2t + \pi/4) + 5\cos(2t + 3\pi/4) - \cos(2t + \pi/2). \)
      Hint. Using phasors, \( x_1(t) = (5\sqrt{2} - 1)\cos(2t + \pi/2) \approx 6.071\cos(2t + \pi/2). \)
   (b) [10] \( x_2(t) = 5\cos(\pi t + \pi/2) + 5\sin(\pi t - \pi/6) - \cos(\pi t - 2\pi/3) \)
   (c) [5] \( x_3(t) = x_1(-3(t - 2)), \) where \( x_1(t) \) was defined in (a).

   **Mastery Problems**

6. [20] Concept(s): **sinusoids and linear systems**
   Sinusoidal signals are particularly important because when a sinusoid is the input to a linear time-invariance (LTI) system, the output is also a sinusoid, and this property is unique to sinusoids!
   Consider a system with an “echo”: the output signal \( y(t) \) is the sum of the input signal \( x(t) \) and a delayed version of \( x(t) \). (You may have experienced something like this in some cell phone calls.) Assume that the following input/output relationship describes the system: \( y(t) = x(t) + x(t - 1) \).
   (a) [10] If \( x(t) = A\cos(2\pi f_1 t) \), show that the output \( y(t) \) can be written as \( B\cos(2\pi f_2 t + \phi) \).
      Relation \( B, \phi \) and \( f_2 \) to \( A \) and \( f_1 \). This is called the “sine in, sine out” property.
      Hint. Use this “phase splitting” trick: \( 1 + e^{j\gamma} = e^{-j\gamma/2} \left[e^{j\gamma/2} + e^{-j\gamma/2}\right] = e^{-j\gamma/2}\cos(\gamma/2). \)
   (b) [10] Now instead of a sinusoidal input, suppose that the input signal \( x(t) \) is periodic with period \( 4 \) and \( x(t) = 1 \) for \( 0 < t < 2 \) and \( x(t) = 0 \) for \( 2 < t < 4 \). Sketch the input \( x(t) \) and the output signal \( y(t) \).
      [0] Is there a “square wave in, square wave out” property?

7. [15] Concept(s): **Euler’s formula**
   (a) [10] Prove the following equality, called DeMoivre’s formula, using Euler’s formula.
      \( \cos(\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta). \)
   (b) [5] Use this result to evaluate \( (3 + j 4)^{99} \).