3.4.5 Fourier Series Derivation

The analysis formula[1] for the Fourier Series coefficients (3.4.2) is based on a simple property of the complex exponential signal: the integral of a complex exponential over one period is zero. In equation form:

$$\int_0^{T_0} e^{j(2\pi/T_0)t}\,dt = 0 \tag{3.4.7}$$

where T_0 is the period of the complex exponential whose frequency is $\omega_0 = 2\pi/T_0$. This fact is obvious if we use Euler's formula to separate the integral into its real and imaginary parts which integrate cosine and sine over one period:

$$\int_0^{T_0} e^{j(2\pi/T_0)t}\,dt = \int_0^{T_0} \cos((2\pi/T_0)t)\,dt + j\int_0^{T_0} \sin((2\pi/T_0)t)\,dt = 0 + j0$$

The vowel signal and the square-wave are both examples that suggest the idea of *approximating* a periodic signal with a sum of complex exponentials.

$$x(t) \approx \sum_{k=-N}^{N} C_k e^{j(2\pi k/T_0)t}$$

Where $2N+1$ is the number of frequency components used. In fact, we might hope that with enough complex exponentials we could make the approximation perfect. This leads to the notion of an infinite series expansion for a periodic signal:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j(2\pi k/T_0)t} \tag{3.4.8}$$

A key ingredient in the series expansion is the form of the complex exponentials, which all have the same period as the signal, T_0. If we define $v_k(t)$ to be the complex exponential of frequency $\omega_k = 2\pi k/T_0$, then

$$v_k(t) = e^{j(2\pi k/T_0)t} \tag{3.4.9}$$

Even though the minimum length period of $v_k(t)$ is smaller than T_0, we can prove that $v_k(t)$ does repeat with a period of T_0:

$$\begin{aligned}\Longrightarrow \qquad v_k(t+T_0) &= e^{j(2\pi k/T_0)(t+T_0)}\\ &= e^{j(2\pi k/T_0)t}\,e^{j(2\pi k/T_0)T_0}\\ &= e^{j(2\pi k/T_0)t}\,e^{j2\pi k}\\ &= e^{j(2\pi k/T_0)t} = v_k(t)\end{aligned}$$

because $e^{j2\pi k} = 1$ for any integer k (positive or negative).

The only step in the derivation of the Fourier Series is that of going from the series expansion (3.4.8) to the analysis integral (3.4.2). To do this, we generalize the zero-integral property (3.4.7) of the complex exponential. Here is the form that we need:

$$\int_0^{T_0} v_k(t)v_\ell^*(t)\,dt = \begin{cases} 0 & \text{if } k \neq \ell \\ T_0 & \text{if } k = \ell \end{cases} \tag{3.4.10}$$

[1]The page numbering and section numbering of this insert corresponds to Chapter 3 of *DSP First.*

where the * in $v_\ell^*(t)$ denotes the conjugate. Proving this fact is straightforward:

$$\begin{aligned}
\int_0^{T_0} v_k(t)v_\ell^*(t)\,dt &= \int_0^{T_0} e^{j(2\pi k/T_0)t} e^{-j(2\pi\ell/T_0)t}\,dt \\
&= \int_0^{T_0} e^{j(2\pi(k-\ell)/T_0)t}\,dt
\end{aligned}$$

There are two cases for the last integral: when $k = \ell$ the exponent becomes zero, so the integral is

$$\int_0^{T_0} e^{j(2\pi(k-\ell)/T_0)t}\,dt = \int_0^{T_0} e^{j0t}\,dt = \int_0^{T_0} 1\,dt = T_0$$

Otherwise, when $k \neq \ell$ the exponent is non-zero and we can invoke Euler's formula to see that we are integrating cosine and sine over an integral number of cycles:

$$\begin{aligned}
\int_0^{T_0} e^{j(2\pi(k-\ell)/T_0)t}\,dt &= \int_0^{T_0} e^{j(2\pi m/T_0)t}\,dt \\
&= \int_0^{T_0} \cos((2\pi m/T_0)t)\,dt + j\int_0^{T_0} \sin((2\pi m/T_0)t)\,dt = 0 + j0
\end{aligned}$$

where $m = k - \ell$. Equation (3.4.10) is called the *orthogonality property* of complex exponentials. It is often quite helpful in solving problems that involve integrals with complex exponentials.

Now we are ready for the last step in the "proof." If we assume that (3.4.8) is valid, then we can multiply both sides by $v_\ell^*(t)$ and integrate over the period T_0.

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} C_k e^{j(2\pi k/T_0)t} \\
\Longrightarrow \qquad \int_0^{T_0} x(t)e^{-j(2\pi\ell/T_0)t}\,dt &= \int_0^{T_0} \sum_{k=-\infty}^{\infty} C_k e^{j(2\pi k/T_0)t} e^{-j(2\pi\ell/T_0)t}\,dt \\
&= \sum_{k=-\infty}^{\infty} C_k \int_0^{T_0} e^{j(2\pi(k-\ell)/T_0)t}\,dt \qquad \text{CRUCIAL STEP !!} \\
&= C_\ell T_0
\end{aligned}$$

The last step relies on the "orthogonality property" stated in (3.4.10), so that the only non-zero case for the integral occurs when $k = \ell$.

In the crucial step, the order of the infinite summation and the integration have been swapped. This is a delicate manipulation that depends on convergence properties of the infinite series expansion. It was also a topic of research that occupied mathematicians for a good part of the early 19th century. For our purposes, we assume that $x(t)$ is either smooth or has a finite number of discontinuities so that the swap is permissible.

The final analysis formula is obtained by writing C_ℓ on one side of the equation:

$$C_\ell = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi\ell/T_0)t}\,dt$$

Since ℓ is just a "dummy" index, we can replace ℓ with k. In addition, we would like to compare the Fourier Series coefficients C_k to the complex amplitudes $X_k = A_k e^{j\phi_k}$ in the spectrum, but we notice that there is a factor of two that must be used:

$$\begin{aligned}
X_0 &= C_0 \\
X_k &= 2C_k \qquad \text{for } k \neq 0
\end{aligned}$$

This factor of 2 comes from our definition of X_k as $X_k = A_k e^{j\phi_k}$ in (3.4.1) using the amplitude of the cosine signal directly. On the other hand, when the positive and negative frequency terms in the Fourier Series are combined we add a complex number and its conjugate, so we get twice the real part.

Convergence

The infinite sum in (3.4.8) actually means that there is a limiting process that gives equality between the right-hand and lefthand sides of (3.4.8). There are two possibilities:

1. The usual way to interpret the limit is *pointwise*

$$x(t) = \lim_{N\to\infty} \sum_{k=-N}^{N} C_k e^{j(2\pi k/T_0)t} \tag{3.4.11}$$

   but this presents a difficulty when $x(t)$ has a point of discontinuity, as in the square wave example.

2. A better way to define the limiting process is to use what is called the squared error.

$$\lim_{N\to\infty} \int_0^{T_0} \left| x(t) - \sum_{k=-N}^{N} C_k e^{j(2\pi k/T_0)t} \right|^2 \to 0 \tag{3.4.12}$$

   In this definition the error is the difference between the lefthand and righthand sides of (3.4.8), and it becomes small because the total *error energy* (over one period) goes to zero. This interpretation depends on the following definition of the average energy of $s(t)$ over one period:

$$E_s^2 = \frac{1}{T_0} \int_0^{T_0} |s(t)|^2 \, dt$$

   The integrand in (3.4.12) should be called the approximation error when a finite number of Fourier Series terms are used to represent $x(t)$. If we define $x_N(t)$ to be the signal formed by a finite sum of complex exponentials:

$$x_N(t) = \sum_{k=-N}^{N} C_k e^{j(2\pi k/T_0)t}$$

   then the error signal is $e_N(t) = x(t) - x_N(t)$.

   The best example of convergence under these two interpretations is given by the square wave which has a point of discontinuity. Figure 3.13 allows us to compare $x(t)$ and $x_N(t)$ for several values of N. As N increases, the signal $x_N(t)$ has higher frequency oscillations, but it also gets closer to $x(t)$ over most of the time interval. On the other hand, when we focus our attention on the regions near the discontinuous edges ($t = 0, 0.02, 0.04, 0.06, \ldots$), we notice that the size of the last oscillation is *not* decreasing. This "overshoot" is called the Gibbs' Phenomenon, after J. Gibbs who first proved that the size of the overshoot does not decrease with N, and in fact is always equal to about 9% of the size of the discontinuity in the square wave. Also we notice that $x_N(t)$ is always equal to 0 at the edges, but the definition of $x(t)$ is ambiguous at those points (it should be either $+1$ or -1). These two observations are the specific reasons why pointwise convergence is not obtained for the Fourier Series. However, we can notice in Fig. 3.13 that the overshoot is getting narrower even as its maximum amplitude stays the same. Thus the energy in the overshoot is decreasing—in other words, there is convergence according to the squared error measure.