1 Review of LTI Filters in the Time-Domain

Table 1: Summary of LTI filter description in the time-domain

<table>
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<tr>
<th>Input signal description</th>
<th>Filter description</th>
<th>Input/output relationship</th>
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</thead>
<tbody>
<tr>
<td>$x[n]$ as a discrete-time sequence</td>
<td>coefficients ${b_k}$ of a difference equation</td>
<td>$y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$</td>
</tr>
<tr>
<td>$x[n] = x[n] \ast \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$</td>
<td>impulse response $h[n]$</td>
<td>$y[n] = x[n] \ast h[n]$</td>
</tr>
<tr>
<td></td>
<td>block diagram of delays, multipliers, adders</td>
<td></td>
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</table>

• The filter described by $y[n] = \sum_{k=-M_1}^{M_2} b_k x[n-k]$ is LTI and has the following impulse response $h[n]$

$$h[n] = \begin{cases} b_n, & -M_1 \leq n \leq M_2, \\ 0, & \text{else.} \end{cases}$$

• Any signal $x[n]$ can be represented as

$$x[n] = x[n] \ast \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

• An LTI filter with the impulse response $h[n]$ has the response $y[n]$ to input $x[n]$,

$$y[n] = x[n] \ast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

• Cascaded LTI filters act as one LTI filter with its overall impulse response

$$h[n] = h_1[n] \ast \cdots \ast h_K[n],$$

where $h_i[n]$ are the impulse responses of cascaded sub-filters.
2 Frequency Response of an LTI Filter

Read 6.1, 6.4, and 6.5

• A different approach to filters is possible.
  (a) Find the filter response to sinusoids/exponentials.
  (b) Express \( x[n] \) as a sum of sinusoids/exponentials.
    Tools available are DFT or other spectral decompositions.
  (c) Express \( y[n] \) as a sum of sinusoid/exponential responses.

• This is called the frequency domain approach, because we work with the frequency domain description of signals.

2.1 The Response of an LTI Filter to an Exponential Signal

• Suppose \( x[n] = Ae^{j(\hat{\omega}n + \phi)} = Ae^{j\phi}e^{j\hat{\omega}n} \) is applied to a filter with impulse response \( h[n] \)

Then the output \( y[n] \)

\[
y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]
\]

\[
= \sum_{k=-\infty}^{\infty} h[k]e^{j\phi}e^{j\hat{\omega}(n-k)}
\]

\[
= \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}e^{j\hat{\omega}n}
\]

\[
= \left( \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} \right) Ae^{j\phi}e^{j\hat{\omega}n}
\]

a constant determined by \( h \) and \( \hat{\omega} \)

\[
= H(\hat{\omega})x[n].
\]

• Conclusion:
  (a) (Form) The output is complex exponential
    When input \( x[n] \) is a complex exponential, the output \( y[n] \) is complex exponential with the same frequency.
  (b) Later, we will see that sinusoidal input \( \implies \) sinusoid output.
  (c) (Amplitude and phase) What about the amplitude and phase of \( y[n] \)?

\[
H(\hat{\omega}) \triangleq \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}.
\]

This function (of \( \hat{\omega} \)) is called the frequency response (function) of the (LTI) filter/system. Then we see that

\[
y[n] = H(\hat{\omega}) Ae^{j\phi}e^{j\hat{\omega}n}
\]

\[
= |H(\hat{\omega})|e^{j\angle H(\hat{\omega})} Ae^{j\phi}e^{j\hat{\omega}n}
\]

\[
= A|H(\hat{\omega})|e^{j(\phi+\angle H(\hat{\omega}))}e^{j\hat{\omega}n}
\]

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(i) The magnitude is scaled by $|H(\hat{\omega})|$.  
(ii) The phase is added by $\angle H(\hat{\omega})$.

- $|H(\hat{\omega})|$ is called the magnitude (or gain) of the frequency response.
- $\angle H(\hat{\omega})$ is called the phase (or angle) of the frequency response.
- Note that both are dependent on frequency $\hat{\omega}$.

### 2.2 Example

An LTI filter has the following impulse response:

$$h[n]$$

$$\begin{array}{c}
\text{2} \\
\text{1} \\
\text{n}
\end{array}$$

Find the magnitude and phase of the frequency response of the filter.

$$H(\hat{\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k}$$

$$= 2 + e^{-j\hat{\omega}}$$

$$= 2 + \cos(\hat{\omega}) - j \sin(\hat{\omega}).$$

$$|H(\hat{\omega})| = \sqrt{(2 + \cos(\hat{\omega}))^2 + \sin^2(\hat{\omega})}$$

$$= \sqrt{4 + 4 \cos(\hat{\omega}) + \cos^2(\hat{\omega}) + \sin^2(\hat{\omega})}$$

$$= \sqrt{5 + 4 \cos(\hat{\omega})}.$$ 

$$\angle H(\hat{\omega}) = \tan^{-1}\left(\frac{-\sin(\hat{\omega})}{2 + \cos(\hat{\omega})}\right).$$

(a) It is a lowpass filter.

(b) Why?

Note that $\hat{\omega} = \pi$ is the highest frequency.

(c) The filter is not sharp. Note that the order of the filter is 1.