• Response to a suddenly-applied sinusoid
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1 Response to a Suddenly-Applied Sinusoid

In general, the response of an LTI filter to a suddenly-applied signal consists of two signal parts:

(a) the **transient** part
   - (i) the signal pattern determined by the poles of the filter
   - (ii) the scale factor by the poles and the input

(b) the **input-caused** part
   - (i) the signal pattern determined by the input
   - (ii) the scale factor by the poles and the input

(1) **(Stable Filters)** The transient part dies out eventually.
(2) **(Unstable Filters)** The transient part does not die out.

**Example 1.1** A first-order IIR filter is given in the following block diagram.

![Block Diagram]

Find the response of the filter to \( x[n] = K \cos(\hat{\omega}n)u[n] \).

(a) **System function:** The system function \( H(z) \) is found

\[
H(z) = \frac{1}{1 - az^{-1}}.
\]

Note that the filter may or may not be stable depending on the value of \( a \). It will be stable if \(|a| < 1\) and unstable otherwise.
(b) **Response:** First note that

\[
\cos(\hat{\omega} n) = \frac{1}{2} (e^{j \hat{\omega} n} + e^{-j \hat{\omega} n}) \iff \frac{1}{2} \left( \frac{1}{1 - e^{j \hat{\omega} z^{-1}}} + \frac{1}{1 - e^{-j \hat{\omega} z^{-1}}} \right) = \frac{1 - \cos(\hat{\omega}) z^{-1}}{1 - 2 \cos(\hat{\omega}) z^{-1} + z^{-2}} \]

\[= \frac{z^2 - \cos(\hat{\omega}) z}{z^2 - 2 \cos(\hat{\omega}) z + 1} = \frac{1 - \cos(\hat{\omega}) z^{-1}}{(1 - e^{j \hat{\omega} z^{-1}})(1 - e^{-j \hat{\omega} z^{-1}})}.\]

The the z-transform \(Y(z)\) of the output is given by

\[Y(z) = H(z)X(z) = \frac{1}{1 - az^{-1}} \cdot K \frac{1 - \cos(\hat{\omega}) z^{-1}}{(1 - e^{j \hat{\omega} z^{-1}})(1 - e^{-j \hat{\omega} z^{-1}})} = \frac{A_1}{1 - az^{-1}} + \frac{A_2}{1 - e^{j \hat{\omega} z^{-1}}} + \frac{A_3}{1 - e^{-j \hat{\omega} z^{-1}}},\]

where

\[A_1 = K \frac{1 - \cos(\hat{\omega}) z^{-1}}{(1 - e^{j \hat{\omega} z^{-1}})(1 - e^{-j \hat{\omega} z^{-1}})} \bigg|_{z=a} = K \frac{1 - \cos(\hat{\omega})/a}{1 - e^{j \hat{\omega}}/a} \frac{1 - \cos(\hat{\omega})/a}{1 - 2 \cos(\hat{\omega})/a + 1/a^2},\]

\[A_2 = K \frac{1 - \cos(\hat{\omega}) z^{-1}}{(1 - az^{-1})(1 - e^{-j \hat{\omega} z^{-1}})} \bigg|_{z=e^{j \hat{\omega}}} = \frac{K}{1 - ae^{-j \hat{\omega}}(1 - e^{-2j \hat{\omega}})},\]

\[A_3 = K \frac{1 - \cos(\hat{\omega}) z^{-1}}{(1 - az^{-1})(1 - e^{j \hat{\omega} z^{-1}})} \bigg|_{z=e^{-j \hat{\omega}}} = \frac{K}{1 - ae^{j \hat{\omega}}(1 - e^{2j \hat{\omega}})}.\]

We note that \(A_2 = A_3^*\) and hence

\[|A_3| = |A_2| \quad \text{and} \quad \angle A_3 = -\angle A_2.\]

Upon inverse z-transforming \(Y(z)\), we get

\[y[n] = A_1 \cdot a^n u[n] + A_2 \cdot e^{j \hat{\omega} n} u[n] + A_3 \cdot e^{-j \hat{\omega} n} u[n] \]

\[= A_1 \cdot a^n u[n] + |A_2| \cdot e^{j(\hat{\omega} n + \angle A_2)} u[n] + |A_2| \cdot e^{-j(\hat{\omega} n + \angle A_2)} u[n] \]

\[= A_1 \cdot a^n u[n] + 2|A_2| \cdot \cos(\hat{\omega} n + \angle A_2) u[n].\]

We note that, when a sinusoid is suddenly applied, the response has

(i) the transient part, which dies out for a stable filter,
(ii) the steady state sinusoid of the same frequency.

(c) **Conclusion:** In general, for stable filters, a suddenly-applied sinusoid causes a sinusoidal output eventually.
## 2 Stability of an LTI Filter and Poles

We have used the following that

An IIR filter is stable ⇐⇒ all poles lie inside the unit circle.

### BIBO Stability

(a) (Definition) A system is BIBO-stable if the output is bounded for a bounded input, i.e., there exists a number $M_Y$ such that

$$|y[n]| \leq M_Y$$

for $|x[n]| \leq M_x$.

(b) An LTI system is BIBO-stable iff

$$\sum_n |h[n]| < \infty.$$ 

(c) An LTI system is BIBO-stable iff all poles of $H(z)$ lie inside the unit circle.

### Sketch

(i) If all poles of $H(z)$ are inside the unit circle, then $|p_k| < 1$ and from

$$H(z) = G \frac{(z - z_1)(z - z_2) \cdots (z - z_K)}{(z - p_1)(z - p_2) \cdots (z - p_L)}$$

$$= \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \cdots + \frac{A_L}{1 - p_L z^{-1}},$$

we get

$$h[n] = A_1 p_1^n u[n] + \cdots + A_L p_L^n u[n].$$

Then

$$\sum_n |h[n]| = \sum_n |A_1 p_1^n u[n] + \cdots + A_L p_L^n u[n]|$$

$$\leq \sum_n \left( |A_1| \cdot |p_1|^n + \cdots + |A_L| \cdot |p_L|^n \right) < \infty.$$

(ii) Suppose that any of the poles of $H(z)$ is outside of the unit circle. Let $p_k$ denote the pole that has the largest magnitude, i.e., $|p_k|$ is the largest among all the poles outside the unit circle. Then, eventually as $n \to \infty$, $A_k p_k^n u[n]$ will be a dominant term of $h[n]$. That is, for large $n$,

$$h[n] \approx A_k p_k^n u[n].$$

Therefore,

$$|h[n]| \to \infty \text{ as } n \to \infty.$$ 

(iii) Suppose that any of the poles of $H(z)$ is on the unit circle. Let $p_k$ denote such a pole. (Assume that it is the farthest pole from the origin.) (Note $|p_k| = 1$.) Then a bounded signal $x[n] = p_k^n u[n]$ will produce an unbounded signal:

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=0}^n A_k |p_k|^2 + \text{ other terms}$$

$$= A_k (n + 1) + \text{ other terms}.$$
Therefore, \(|y[n]| \to \infty\) as \(n \to \infty\).

3 Implementation Structures

(a) **Direct Form I**: feedforward loop followed by the feedback loop

\[
x[n] \rightarrow z^{-1} b_0 \rightarrow y[n]
\]

(b) Intermediate Form

\[
x[n] \rightarrow z^{-1} a_1 \rightarrow w[n] \rightarrow H_B(z) \rightarrow y[n]
\]

(c) **Direct Form II**

\[
x[n] \rightarrow z^{-1} b_0 \rightarrow w[n] \rightarrow a_1 \rightarrow y[n]
\]