1 The Inverse z-Transform

The inverse z-transform is the process of finding a discrete-time sequence that corresponds to a z-domain function.

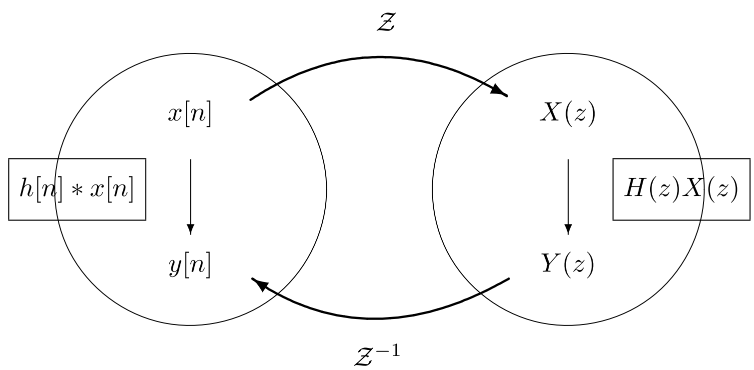
$$w[n] \quad \rightleftharpoons \quad W(z).$$

There are several methods available for the inverse z-transform.

- The inspection method
- The division method
- The partial fraction expansion method
- The contour integration method

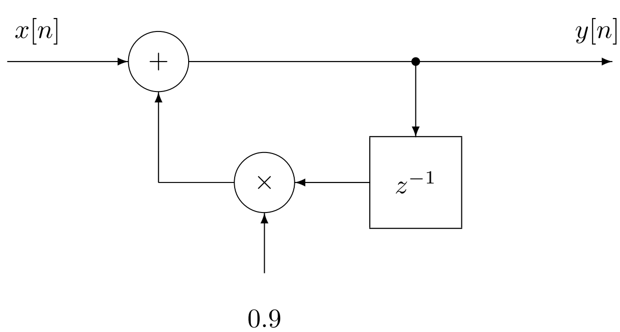
Usefulness of the inverse z-transform The output of an LTI filters can be found in the z-domain.

$$y[n] = h[n] * x[n] \quad \Longleftrightarrow \quad Y(z) = H(z)X(z).$$



2 The Partial Fraction Expansion

Example 2.1 A first-order IIR filter is given in the following block diagram.



Find the response of the filter to $x[n] = a^n u[n]$, where $a \neq 0.9$.

(a) **Difference equation:** We note that the difference equation of the filter is

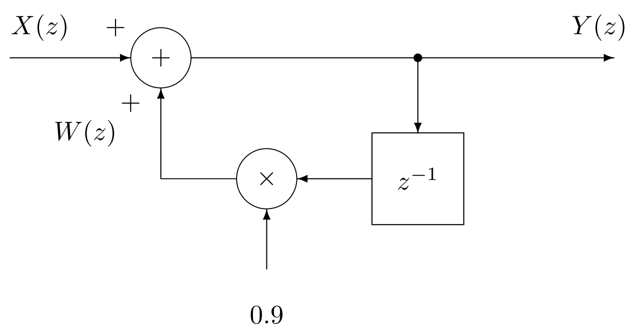
$$y[n] = 0.9y[n-1] + x[n].$$

(b) **System function:** The system function $H(z)$ is found

(i) either from taking the z-transform of the difference equation

$$Y(z)\big(1 - 0.9z^{-1}\big) = X(z) \qquad \Longrightarrow \qquad H(z) = Y(z)/X(z) = \frac{1}{1 - 0.9z^{-1}}.$$

(ii) or from abgeraic manipulations of the system in the z-domain.



$$Y(z) = X(z) + W(z)$$
$$W(z) = 0.9Y(z)z^{-1}$$
$$\Longrightarrow \qquad H(z) = Y(z)/X(z) = \frac{1}{1 - 0.9z^{-1}}.$$

Note that there are a pole $z = 0.9$ and a zero $z = 0$. And the filter is stable because all the poles (there is only one pole) reside inside the unit circle.

(c) **Impulse response:** The impulse response can be found

(i) either from the difference equation with $\delta[n]$ as the input.

(ii) or by the inverse z-transforming $H(z)$.

$$H(z) = \frac{1}{1 - 0.9z^{-1}} \qquad \Longrightarrow \qquad h[n] = 0.9^n u[n].$$

(d) **General response:** The filter response to $x[n] = a^n u[n]$ can be found

(i) either from the difference equation with $x[n]$

(ii) or by convolution in the time-domain

$$y[n] = h[n] * x[n]$$

(iii) (the preferred method) or by tranform/inverse transforming in the z-domain

$$Y(z) = H(z)X(z) = \frac{1}{1 - 0.9z^{-1}} \cdot \frac{1}{1 - az^{-1}}.$$

Watch how it proceeds.

$$
\begin{aligned}
Y(z) &= \frac{1}{1-0.9z^{-1}} \cdot \frac{1}{1-az^{-1}} \\
&= \frac{A}{1-0.9z^{-1}} + \frac{B}{1-az^{-1}}. \\
y[n] &= A \cdot 0.9^n u[n] + B \cdot a^n u[n].
\end{aligned}
$$

The constants A and B are found in the following way.

$$
\begin{aligned}
A &= Y(z)(1-0.9z^{-1}) \big|_{z \text{ at } (1-0.9z^{-1})=0} \\
&= \frac{1}{1-0.9z^{-1}} \cdot \frac{1}{1-az^{-1}} \cdot (1-0.9z^{-1}) \big|_{z \text{ at } (1-0.9z^{-1})=0} \\
&= \frac{1}{1-az^{-1}} \big|_{z \text{ at } (1-0.9z^{-1})=0} \\
&= \frac{1}{1-a/0.9}, \\
B &= Y(z)(1-az^{-1}) \big|_{z \text{ at } (1-az^{-1})=0} \\
&= \frac{1}{1-0.9z^{-1}} \cdot \frac{1}{1-az^{-1}} \cdot (1-az^{-1}) \big|_{z \text{ at } (1-az^{-1})=0} \\
&= \frac{1}{1-0.9z^{-1}} \big|_{z \text{ at } (1-az^{-1})=0} \\
&= \frac{1}{1-0.9/a}.
\end{aligned}
$$

The partial fraction expansion is a method of expressing a given fraction as a sum of fractions whose denominators are factors of the given fraction.

Procedure in the case of simple poles:

(a) Given $W(z)$, identify the degrees M and N of the numerator and the denominator polynomials in negative powers of z, respectively. Check if $M < N$, then proceed to the next step. (Otherwise, express $W(z)$, by long division, as a sum of a quotient polynomial and a proper rational function.)

(b) Factor the denominator:

$$
\begin{aligned}
W(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_N z^{-M}}{1 - a_1 z^{-1} - \cdots - a_M z^{-N}} \\
&= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{(1-p_1 z^{-1}) \cdots (1-p_N z^{-1})}.
\end{aligned}
$$

(c) See that the poles are simple (i.e., the roots are of multiplicity 1.)

(d) Express $W(z)$ as a sum of partial fractions.

$$
\begin{aligned}
W(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{(1-p_1 z^{-1}) \cdots (1-p_N z^{-1})} \\
&= \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \cdots + \frac{A_N}{1-p_N z^{-1}},
\end{aligned}
$$

where A_k are found as

$$
A_k = W(z)(1-p_k z^{-1}) \big|_{z=p_k}, \quad k = 1, \cdots, N.
$$

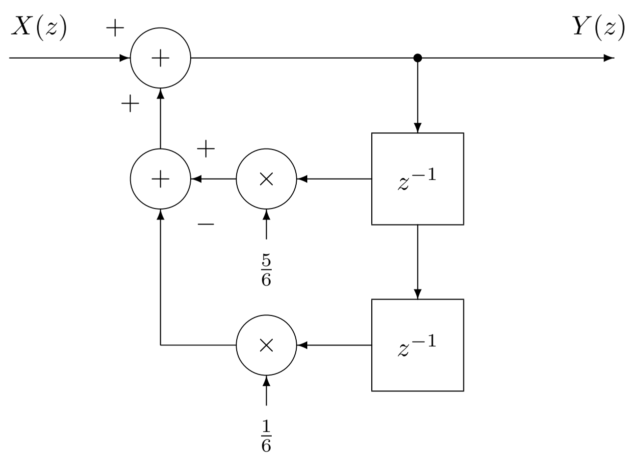
(e) Then

$$w[n] = A_1 p_1^n u[n] + \cdots A_N p_N^n u[n]$$
$$= \sum_{k=1}^{N} A_k p_k^n u[n].$$

(f) The signal $w[n]$ may reduce further, e.g., a complex conjugate pair may reduce to a sinusoid. □

Example 2.2 Answer for the following 2nd-order IIR filter.

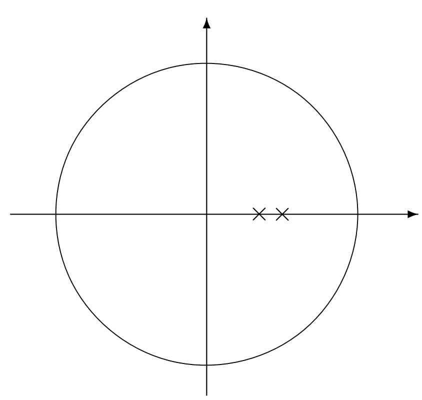
$$y[n] = \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2] + x[n].$$



(a) Find the impulse response.

First find the system function $H(z)$ either from the difference equation or from the block diagram.

$$H(z) = \frac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$
$$= \frac{1}{(1 - \frac{1}{3}z^{-1}) \cdot (1 - \frac{1}{2}z^{-1})}.$$



We observe that there are two poles inside the unit circle, which implies the stability of the filter. Also we gather that the filter is a lowpass filter.

To find the impulse response, we note that $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.

$$
\begin{aligned}
H(z) &= \frac{1}{(1-\frac{1}{3}z^{-1})\cdot(1-\frac{1}{2}z^{-1})} \\
&= \frac{A_1}{1-\frac{1}{3}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}},
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= H(z)(1-\frac{1}{3}z^{-1})\,\big|_{z=1/3} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}}\,\big|_{z=1/3} = -2, \\
A_2 &= H(z)(1-\frac{1}{2}z^{-1})\,\big|_{z=1/2} \\
&= \frac{1}{1-\frac{1}{3}z^{-1}}\,\big|_{z=1/2} = 3.
\end{aligned}
$$

Therefore,

$$
h[n] = 3\big(\frac{1}{2}\big)^n u[n] - 2\cdot\big(\frac{1}{3}\big)^n u[n].
$$

(b) Find the response of the filter to $x[n] = Ka^n u[n]$, where $a \neq \frac{1}{3}$ or $\frac{1}{2}$.

Let's try the z-domain approach:

$$
\begin{aligned}
Y(z) = H(z)X(z) &= \frac{1}{(1-\frac{1}{3}z^{-1})\cdot(1-\frac{1}{2}z^{-1})}\cdot\frac{1}{1-az^{-1}} \\
&= \frac{A_1}{1-\frac{1}{3}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}} + \frac{A_3}{1-az^{-1}},
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= H(z)(1-\frac{1}{3}z^{-1})\,\big|_{z=1/3} \\
&= \frac{1}{(1-\frac{1}{2}z^{-1})(1-az^{-1})}\,\big|_{z=1/3} = -\frac{2}{1-3a}, \\
A_2 &= H(z)(1-\frac{1}{2}z^{-1})\,\big|_{z=1/2} \\
&= \frac{1}{(1-\frac{1}{3}z^{-1})(1-az^{-1})}\,\big|_{z=1/2} = \frac{3}{1-2a}, \\
A_3 &= H(z)(1-az^{-1})\,\big|_{z=1/a} \\
&= \frac{1}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}\,\big|_{z=1/a} = \frac{1}{(1-\frac{a}{3})(1-\frac{a}{2})}.
\end{aligned}
$$

Then

$$
y[n] = \underbrace{\frac{3}{1-2a}\cdot\big(\frac{1}{2}\big)^n u[n] - \frac{2}{1-3a}\cdot\big(\frac{1}{3}\big)^n u[n]}_{\text{transient part}} + \frac{1}{(1-\frac{a}{3})(1-\frac{a}{2})}\cdot a^n u[n].
$$

We note that the total response consists of two parts

(i) the transient part: the signal pattern determined by the poles of the filter; the scale factor by the poles and the input

(ii) the input-caused part: the pattern determined by the input; the scale factor by the poles and the input.

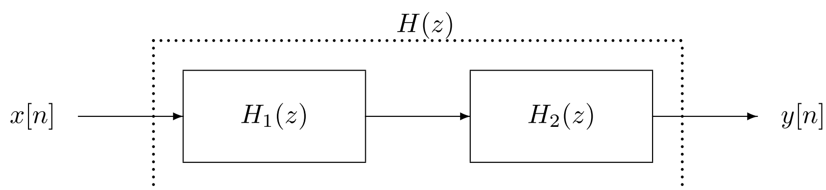
Example 2.3 (Case of $M \geq N$) Find the impulse response of the following first-order IIR filter

$$y[n] = -\frac{1}{2}y[n-1] + x[n] + \frac{1}{3}x[n-1] + \frac{1}{4}x[n-2].$$

First find the system function $H(z)$:

$$\begin{aligned} H(z) &= \frac{1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2}}{1 + \frac{1}{2}z^{-1}} \\ &= -\frac{1}{3} + \frac{1}{2}z^{-1} + \frac{\frac{4}{3}}{1 + \frac{1}{2}z^{-1}} \quad \text{by long division.} \\ h[n] &= -\frac{1}{3}\delta[n] + \frac{1}{2}\delta[n-1] + \frac{4}{3}\left(-\frac{1}{2}\right)^n u[n]. \end{aligned}$$

Example 2.4 (Case of $M \geq N$) Find the impulse response of the following cascaded filters.



Assume that

$$H_1(z) = \frac{1}{1 + 0.5z^{-1}}, \quad H_2(z) = \frac{1 + z^{-2}}{1 - 0.8z^{-1}}.$$

The overall system function $H(z) = H_1(z)H_2(z)$:

$$\begin{aligned} H(z) &= \frac{1}{1 + 0.5z^{-1}} \cdot \frac{1 + z^{-2}}{1 - 0.8z^{-1}} \\ &= \frac{1 + z^{-2}}{1 - 0.3z^{-1} - 0.4z^{-2}} \\ &= -\frac{5}{2} + \underbrace{\frac{\frac{7}{2} + \frac{3}{4}z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}}}_{W(z)} \quad \text{by long division.} \\ W(z) &= \frac{\frac{7}{2} + \frac{3}{4}z^{-1}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})} \\ &= \frac{A_1}{1 + 0.5z^{-1}} + \frac{A_2}{1 - 0.8z^{-1}}. \\ h[n] &= -\frac{5}{2}\delta[n] + A_1 \cdot 0.5^n u[n] + A_2 \cdot 0.8^n u[n]. \end{aligned}$$