Lectures on Spectra of Continuous-Time Signals

Principal questions to be addressed:

1. What, in a general sense, is the "spectrum" of a signal?

2. Why are we interested in spectra? ("spectra" = plural of "spectrum")

3. How does one assess the spectrum of a given signal?

All of these are for continuous-time signals. The next part of the course discusses spectra of discrete-time signals.

Outline of Coverage of the Spectra of Continuous-Time Signal

A. Rough definition of spectrum and motivation for studying spectra.

B. The spectrum of a signal that is a finite sum of sinusoids

C. The spectrum of a periodic signal via Fourier series

D. Spectra of segments of signals.

Note:

- The spectra plays important two roles:
  - a. Analysis and design: The spectra is a theoretical tool that enables one to understand, analyze and design signals and systems.
  - b. System component: The computation and manipulation of spectra is a component of many important systems.

II. Lectures on Spectra of Continuous-Time Signals

Notes:

- Our coverage of spectra goes significantly beyond the coverage in Chapter 3.
- See the list of errata for Chapter 3.

A. Rough definition of spectrum and motivation for studying spectrum

1. Introduction to the concept of "spectrum"?

Definition:

Roughly speaking, the "spectrum of a signal" indicates how the signal may be thought of as being composed of sinusoids.

It describes the frequencies, amplitudes and phases of the sinusoids that "sum" to yield the signal.

The individual sinusoids that sum to give the signal are called "sinusoidal components".

Alternatively, the spectum describes the distributions of amplitude and phase vs. frequency of the sinusoidal components.

Since each sinusoid can be decomposed into the sum of two complex exponentials, the spectrum equivalently indicates how the signal may be thought of as being composed of complex exponentials.

It describes the frequencies, amplitudes and phases of the complex exponentials that "sum" to yield the signal.

The individual complex exponentials that sum to give the signal are called "complex exponential components".

Alternatively, the spectrum describes distribution of amplitude and phase vs. frequency of the complex exponential components.

Sinusoidal and complex exponential components are also called "spectral components".

Plotting the spectra

We like to plot and visualize spectra. We plot lines at the frequencies of the exponential components (at both positive and negative frequencies). The height of the line is the magnitude of the component. We label the line with the complex amplitude of the component, e.g. with $2e^{j.5}$.

Alternatively, sometimes we make two line plots, one showing the magnitudes of the components and the other showing the phases. These are called the "magnitude spectra" and "phase spectra", respectively.

Important note:

"Spectrum" is a broad collective noun, like "economy" or "health" for which there is no universal mathematically precise definition. Rather as with economy and health, there are a variety of specific ways to assess the spectrum of a signal.

For example, to assess the economy, one can measure gnp, average income, unemployment rate, poverty rate, djia, nasdaq, money supply, ...).

For example, to assess health one can measure body temperature, heart rate, blood pressure, blood chemistry, weight, etc.

Similarly, there are a variety of ways to assess the spectrum of a signal. A limited set will be discussed in this course: principally, Fourier series (FS) for periodic continuous-time signals, discrete Fourier transform (DFT) for periodic discrete-time signals. But there will also be some discussion and use (mainly in the labs) of FS and DFT to assess the spectra of finite segments of signals. The Fourier transform, which is another important method of assessing the spectrum of continuous-time signals, will be discussed in EECS 306.

Reasons for decomposing into sinusoids.

It's mainly that sinusoids into linear systems lead to sinusoids. (No other class of signals has this property.)

This causes the input-output relationship for linear sytems to be particularly simple for sinusoidal signals.

So representing signals with sinusoids simplifies analysis greatly.

Because analysis is simplified, efficient design methods can be developed.

2. Why are we interested in spectrum?

Here are some reasons:

- Signals with nonoverlapping spectrum do not interfere with one another. Thus many information carrying signals can be transmitted over a single communication medium (wire, fiber, cable, atmosphere, water, etc.).

  To design such systems, we need to be able to quantitatively determine the spectrum of signals to be able to assess whether or not they overlap, and if they do, by how much. Also, we need to able to develop systems (e.g. filters) that select one signal over another, based on its spectrum.

- Some signals can be recognized based on their spectra, e.g. vowels (Labs 8,9), touchtone telephone key presses, musical notes and chords, bird songs, whale sounds, mechanical vibration analysis, atomic/molecular makeup of sun and other stars, etc. To build systems that automatically recognize such signals, we need to able to quantatively determine the spectrum of a signal.

- Communication media, e.g. the atmosphere, the ocean, a wire, an optical fiber, often limit propagation to signals with components only in a certain frequency range (atmosphere is high frequency, ocean is low frequency, wire is low frequency, optical fiber is high frequency, but what is considered "high" or "low" depends on the media. We need to be able to assess the spectrum of a signal to see if it will propagate. We need to be able to design signals to have appropriate spectra for appropriate media.

- In many situations, the behavior of many natural and man-made linear systems is best analyzed in the "frequency domain", i.e. one determines the behavior in response to sinusoids (or complex exponentials) at various frequencies, and from this one can deduce the response to other signals. The previous bullet is a special case of this.

- In many situations, an undesired signal interferes with a desired signal, e.g. the desired signal might correspond to someone speaking and the undesired signal might be background noise. We wish to reduce or eliminate the background signal. In order to be able to reduce or eliminate the background signal it must have some characteristic that is distinctly different than the desired signal. Often it happens that the desired and undesired signals have distinctly different spectra (e.g. the noise has mostly high frequency components). In such cases, one can design systems, called "filters", that selectively reduce certain frequency components. These can be used to reduce the noise while having little effect on the desired signal.

- Many other signals and systems methods are based on spectra: e.g. control engineering, data compression, voice recognition, music processing.

- And ....

3. How does one assess the spectrum of a given signal?

The remainder of these notes are intended to make progress on this question, with occasional references to questions 1 and 2.

There is no single answer.

The answer/answers do not fit into one course.

We address this question in spiral fashion in EECS 206. The answer continues in EECS 306 and beyond. (Just like you don't learn all there is to know about the economics in Econ. 101)

We will develop several methods for continuous-time signals, several methods for discrete-time signals.

There is no universal spectral concept in wide use.

We use different measures of the spectrum for different types of signals.

We will discuss mainly

1. spectra of a sum of sinusoids (with support $(-\infty,\infty)$)
2. spectra of periodic signals (with support $(-\infty,\infty)$) via Fourier series

and briefly discuss

3. spectra of a segment of a signal via Fourier series, which leads to:
    - the spectra of signal with finite support
    - the spectra of signal with infinite support via Fourier series applied to successive segments

We won't discuss

4. spectra of a signal with infinite support and finite energy via Fourier transform. This will be disucssed in EECS 306.

We will have a similar discussion of spectra for discrete-time signals in the next part of the course.

We won't get rigorous in our treatment of Fourier series. We'll leave that to future courses such as EECS 306.

B. The spectrum of a finite sum of sinusoids

As in the text Section 3.1, we begin the discussion of how to assess a spectrum by considering signals that are finite sums of sinusoids, as in

$$x(t) = 4 + 3\cos(3t+.1) + 5\cos(7t+.3) - 4\cos(9t+.2)$$

More generally, consider a signal of the form

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t+\phi_k)$$

$$= A_0 + A_1\cos(2\pi f_1 t+\phi_1) + A_2\cos(2\pi f_2 t+\phi_2) + ...$$

$$+ A_N\cos(2\pi f_N t+\phi_N)$$

where $N, A_0, A_1, f_1, \phi_1, ..., A_N, f_N, \phi_N$ are parameters that specify $x(t)$.
Using Euler's formula, we can rewrite $x(t)$ as

$$x(t) = X_0 + \sum_{k=1}^{N} \mathrm{Re}\left\{ X_k e^{j2\pi f_k t}\right\}$$

where

$$X_k = A_k e^{j\phi_k}$$

is the phasor corresponding to $A_k \cos(2\pi f_k t+\phi)$. (It's a complex number.)
Using the inverse Euler formula, we can also rewrite this

$$x(t) = X_0 + \sum_{k=1}^{N} \left(\frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t}\right)$$

Finally, we can also rewrite this as

$$x(t) = \sum_{k=-N}^{N} \alpha_k e^{j2\pi f_k t}$$

where $\alpha_0 = X_0 = A_0$

$$\alpha_k = \begin{cases} \frac{X_k}{2}, & k\geq 1 \\ \frac{X_k^*}{2}, & k\leq -1 \end{cases}$$

- The (two-sided) spectrum of this signal is the list of pairs

$$\{(X_0,0), (\tfrac{1}{2}X_1,f_1), (\tfrac{1}{2}X_1^*,-f_1), (\tfrac{1}{2}X_2,f_2), (\tfrac{1}{2}X_2^*,-f_2),..., (\tfrac{1}{2}X_N,f_N), (\tfrac{1}{2}X_N^*,-f_N)\}$$

- We like to plot these, i.e we like to plot the spectrum.
- The spectrum, i.e. this list, is considered to be a "compact" representation of the signal $x(t)$, i.e. just a few numbers.
- The "spectrum" is also called the "frequency-domain representation" of the signal. In contrast $x(t)$ is the "time domain representation of the signal".
- The terms $A_k \cos(2\pi f_k t+\phi)$ are called the "sinusoidal components" of $x(t)$.

- The terms $\alpha_k e^{j2\pi f_k t}$ are called the "complex exponential component" or "spectral components" of x(t).
- It is equally valid to express the frequencies in Hz as in rad/sec.
- Often we're mainly interested in the magnitude of the spectrum.

Example: Sum of several sinusoids. Let

$$x(t) = 4 + 3\cos(3t+.1) + 5\cos(7t+.3) - 4\cos(9t+.2)$$

Show a plot of x(t)

Problem: Find the spectrum of x(t).

Solution: To do this, we decompose the x(t) into a sum of complex exponentials using inverse Euler

$$x(t) = 4 + 1.5\, e^{j(.1)} e^{j3t} + 2.5\, e^{j(.3)} e^{j7t} + 2\, e^{j(.1+\pi)} e^{j9t}$$

$$+ 1.5\, e^{-j(.1)} e^{-j3t} + 2.5\, e^{-j(.3)} e^{-j7t} + 2\, e^{-j(.1+\pi)} e^{-j9t}$$

The spectrum is

$$\{(4,0),\ (1.5\, e^{j(.1)},3),\ (1.5\, e^{-j(.1)},-3),\ (2.5\, e^{-j(.3)},7),$$

$$(2.5\, e^{-j(.3)},-7),(2\, e^{-j(.1+\pi)},9),(2\, e^{-j(.1+\pi)},-9)\}$$

Show a plot of the spectrum.

Compare the plot of the spectrum to the plot of the signal. Notice that the plot of the spectrum is "simpler, more compact and more intuitively informative" than the plot of x(t). This illustrates what we mean when we say that "the spectrum is a compact representation of the signal".

Example: Given that the spectrum of the signal x(t) is shown below, find x(t).

Show plot of a spectrum.

Pick off and sum the exponential components. Simplify so as to write x(t) as a sum of sinusoids in standard form. Plot x(t).

Example: Amplitude Modulation (AM)

Consider the form of a signal transmitted by an AM radio station

$$x(t) = (v(t)+1) \cos(2\pi f_c t)$$

where

$v(t)$ is the audio signal, which is scaled so that $-1 \le v(t) \le 1$ for all t

$\cos(2\pi f_c t)$ is the "carrier signal",

$2\pi f_c$ is usually a high freqeuncy, e.g. xxx khz., f_c is the frequency in hz to tune the radio to

Draw block diagram:

$$\begin{array}{l} v(t) \rightarrow \oplus \rightarrow \otimes \rightarrow x(t) \rightarrow \text{amplifier} \rightarrow \text{antenna} \\ \qquad\quad\;\; | \qquad | \\ \qquad\quad\;\; 1 \quad \cos(2\pi f_c t) \end{array}$$

Motivation:

Our audio signal is low frequency typically 0 to 5 khz.

Low frequencies don't propagate through the atmosphere.

Need to generate a high frequency signal that "carries" the audio signal.

$\cos(2\pi f_c t)$ is called the "carrier signal". It has high frequency.

$x(t)$ is obtained by "modulating" the carrier signal by the audio signal

Specifically, $x(t) = (v(t)+1) \cos(2\pi f_c t)$

$v(t)$ becomes the envelope of $x(t)$. (adding the +1 insures this

Plot: Show plot of typical $v(t)$, $\cos(2\pi f_c t)$, and $x(t)$

Problem:

Assuming $v(t) = \cos(2\pi f_v t)$, find and plot the spectrum of $x(t)$?
(A real radio station is not usually interested in transmitting a sinusoidal audio signal. The sinusoidal $v(t)$ is just a stand-in for a genuine audio signal. We're assuming this choice of $v(t)$, because so far it's all that we can analyze.)

Solution:

The spectrum has components at frequencies f_c-f_v, f_c, f_c+f_v .

Find the actual components, plot the spectrum, discuss how it depends on f_v and f_c, mention the "bandwidth".

Note: This example is intended as a simple example of using the concept of "spectrum" to do an "analysis".

Example: frequency multiplexing of AM signals (this example uses spectra to design a frequency multiplexing parameter)

Suppose:

Radio station 1 wants to transmit audio signal $v_1(t)$ at carrier frequency $f_{c,1}$

Radio station 2 wants to transmit audio signal $v_2(t)$ at carrier frequency $f_{c,2} > f_{c,1}$.

Question:

How far apart must $f_{c,1}$ and $f_{c,2}$ be in order that the two transmitted signals do not interfere with each other?

For concreteness assume:

$$v_1(t) = \cos(2\pi f_{a,1} t), \quad v_2(t) = \cos(2\pi f_{a,2} t)$$

Then

Radio station 1 transmits: $x_1(t) = (1+v_1(t)) \cos(2\pi f_{c,1} t)$

Radio station 2 transmits: $x_2(t) = (1+v_2(t)) \cos(2\pi f_{c,2} t)$

Solution:

The spectrum of $x_1(t)$ has components at frequencies

$$f_{c,1}-f_{a,1}, \; f_{c,1}, \; f_{c,1}+f_{a,1} \; .$$

The spectrum of $x_c(t)$ has components at frequency has components at frequencies

$$f_{c,2}-f_{a,2}, \; f_{c,2}, \; f_{c,2}+f_{a,2}$$

We need to choose $f_{c,1}$ and $f_{c,2}$ so that

$$f_{c,1}+f_{a,1} < f_{c,2}-f_{a,2}$$

i.e. so that

$$f_{c,2}-f_{c,1} > f_{a,1}+f_{a,2}$$

In a practical AM system, the audio signal has spectrum ranging from 0 khz to +5 khz. In fact they limit the audio signals to this range. So the AM radio signal has "bandwidth" about 10khz --- from f_c-5000 to f_c+5000. Because of this, AM radio stations are assigned frequencies increments of 10 khz. And the FCC avoids having two stations in the same area being separated only by 10 khz. This is because the limited audio signals don't actually have spectra fitting exactly between 0 and +5 khz. And because even if they did, a radio receiver cannot pick out the signal components in the range f_c-5000 to f_c+5000 without also accepting at a least some signal components outside this band. Sometimes you can hear two AM radio stations at once, especially if you've tuned to a weak one and a powerful one is at transmitting at a frequency only 10 khz away, especially if you have an old/cheap radio receiver.

Note: This example is intended to be a concrete example of the practical use of the concept of spectrum to do a simple design task.

C. The spectrum of a periodic signal

The main point of this section is the following theorem, which we won't prove, but which we will illustrate and use.

Fourier Series Theorem: (Fourier, 1768-1830, French mathematician and Egyptologist, see Oppenheim & Willsky for biosketch)

A periodic signal x(t) with period T can be written as an infinite sum of sinusoids, all of which have frequencies that are multiples of 1/T.

That is, there are a set of amplitudes and phases (A_0,ϕ_0), (A_1,ϕ_1), (A_2,ϕ_2), ... such that

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(\frac{2\pi}{T} kt+\phi_\kappa\right)$$

Equivalently,

$$x(t) = X_0 + \sum_{k=1}^{\infty} \left(\frac{X_k}{2} e^{j\frac{2\pi}{T}kt} + \frac{X_k^*}{2} e^{-j\frac{2\pi}{T}kt} \right)$$

where

$$X_0 = A_0 \text{ and } X_k = A_k e^{j\phi_k}$$

Eqivalently,

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{j\frac{2\pi}{T}kt}$$

where $\alpha_0 = X_0 = A_0$

$$\alpha_k = \begin{cases} \frac{X_k}{2}, & k \geq 1 \\ \frac{X_k^*}{2}, & k \leq -1 \end{cases}$$

Notes:

- This is an amazing, surprising, and deep theorem. (When it was first discovered, sime important mathematicians and scientists did not believe it.)
- The proof of the theorem is beyond the scope of this class, and EECS 306, too.
- The theorem says that ANY periodic signal can be represented as the sum of sinusoids. But it <u>may</u> take an infinite number of them.
- $A_k \cos(\frac{2\pi}{T} kt+\phi_k)$ is the sinusoidal component of x(t) at frequency $\frac{2\pi}{T}$ k.
- Note that all sinusoids in the above have frequencies that are multiples of $\frac{2\pi}{T}$.
- It also says that ANY periodic signal can be represented as a sum of complex exponentials. (It may take an infinite number.)
- $\alpha_k e^{j\frac{2\pi}{T}kt}$ is the complex exponential component (equivalently, the spectra component) of x(t) at frequency $\frac{2\pi}{T}$ k.
- It follows from the theorem that the spectrum of a periodic signal with period T is concentrated at frequencies

  $0, \pm 1/T, \pm 2/T, \pm 3/T, \ldots$

or some subset thereof, i.e. $x(t)$ has spectral components only at these frequencies.

- The frequency $1/T$ is called the "fundamental" or "first harmonic" frequency, the frequency k/T is called the "kth-harmonic" frequency. Likewise, the component at frequency $1/T$ is called the "fundamental" or "first harmonic" component, the frequency k/T is called the "kth-harmonic" component.

- The three sums given above are considered to be three forms of the "Fourier series". (A "series" is an infinite sum.)

- The book introduces the first two forms in Section 3.4 (equation (3.4.1)

- It is most common to use the third form, because it is easier to work with. We'll primarily use the third form.

- The A_k's, ϕ_k's, X_k's and α_k's are called "Fourier series coefficients" or just "Fourier coefficients".

- To "find the spectrum of a periodic signal", we need to find T and we need to find the Fourier coefficients.

- Here's the formula for the coefficients:

  $$\alpha_k = \frac{1}{T}\int_0^T x(t)\; e^{-j\frac{2\pi}{T}kt}\, dt$$

  This is often called the "analysis formula".

- The derivation of the analysis formula is well presented in the new section 3.4.5. Reading it is strongly recommended.

- Since the integrand is periodic with period T, the integral in the analysis formula could have limits consisting of any interval of length T.

- Notice that α_k is the correlation of $x(t)$ with $e^{j\frac{2\pi}{T}kt}$ normalized by $1/T$, which is the energy of one period of the exponential.

- Suggested reading. The discussion of "signal components" at the end of Section IIIB of "Introduction to Signals" by DLN.

  In the terminology of that discussion

  $\alpha_k\, e^{j2\pi kt/T}$ is the component of $x(t)$ that is like $e^{j\frac{2\pi}{T}kt}$

  α_k measures the similarity of $x(t)$ to the exponential.

  There's a similar interpretation that $A_k \cos(\frac{2\pi}{T} kt+\phi_k)$ is the component of $x(t)$ that is like a cosine at frequency $\frac{2\pi}{T}$.

- Illustrate the summing of sinusoids to obtain an arbitrary signal using the "sinsum" demo program from Lab 3 .

  Can also use the Matlab demo program called "xfourier.m".

Summary of Fourier series

Synthesis Formula: shows how $x(t)$ is a sum of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{j\frac{2\pi}{T}kt}$$

Analysis Formula: shows how to compute the α_k's, i.e. the Fourier coefficients

$$\alpha_k = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi kt/T}\, dt$$

Can alternatively integrate over any T-second interval.

Notes:

- Finding the spectrum of a periodic signal involves finding the period and the α_k's.
- Finding the α_k's is often called "taking the Fourier series".
- There is a one-to-one relationship between periodic signals with period T and sequences of Fourier coefficients.

  In the Fourier series theorem, it can be shown that there is one and only one set of coefficients that works in the synthesis formula, i.e.

  there is one and only one set of coefficients $\{\alpha_k\}$ such that

  $$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{j\frac{2\pi}{T}kt}$$

  Thus, if you find a set of coefficients α_k such that $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{j\frac{2\pi}{T}kt}$.

  Then these are necessarily the Fourier coefficients.

  Equivalently, if $\sum_{k=-\infty}^{\infty} \alpha_k \, e^{j\frac{2\pi}{T}kt} = \sum_{k=-\infty}^{\infty} \beta_k \, e^{j\frac{2\pi}{T}kt}$, then $\alpha_k = \beta_k$.

  This means that in some cases, the Fourier coefficients can be found by inspection.

  Another statement of the one-to-oneness is that if $x_1(t)$ and $x_2(t)$ are distinct signals[1], each with period T, then for at least one k, α_k for $x_1(t)$ does not equal α_k for $x_2(t)$.

- If a signal has period T, then it also has period 2T. So when applying Fourier analysis, we have a choice as to T. Often, but certainly not always, we choose T to equal the fundamental period. When we want to explicitly specify the value of T used, we will say "the T-second Fourier series".
- If you wish to find the other forms of the Fourier series, use the formulas:

  $A_0 = \alpha_0$, $A_k = |\alpha_k|$, $\phi_k = \text{angle}(\alpha_k)$, $k = 1, 2, ...$

  $X_0 = \alpha_0$, $X_k = 2\alpha_k$, $k = 1, 2, ...$

[1]Here, "distinctt" means that their difference has nonzero power.

Example: Find the spectrum of the following signal.

x(t) = some periodic signal like a square or sawtooth wave

find a closed form expression for the coefficients

Example: Find the spectrum of the following example

x(t) = finite sum of sinusoids.

in this example one computes the spectrum (i.e. the Fourier series coefficients) by inspection just as we did in the section on finite sums of sinusoids. The one-to-one-ness of the relation between Fourier coefficients and periodic signals means that the coefficients we obtain by inspection are the Fourier series coefficients.

Example: Find the signal corresponding to the following spectrum.

Show a spectrum with finite number of spectral lines. This is the same sort of problem as in the section on finite sums of sinusoids section.

Example: Show a real-world nearly periodic signal, like a vowel.

Show its spectrum, as computed by a computer.

More properties

- α_0 = DC value (important)
- One can compute the Fourier coefficients by integrating over any time interval of length T:

$$\alpha_k = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi kt/T}\, dt$$

$$= \frac{1}{T}\int_a^{a+T} x(t)\, e^{-j2\pi kt/T}\, dt \quad \text{for any value of } a \qquad \text{(important)}$$

$$= \frac{1}{T}\int_{<T>} x(t)\, e^{-j2\pi kt/T}\, dt \qquad \text{(new notation)}$$

- Conjugate symmetry (important)

$$\alpha_{-k} = \alpha_k^*$$

Derivation:

$$\alpha_k^* = \left(\frac{1}{T}\int_0^T x(t)\, e^{-j2\pi kt/T} dt\right)^* = \frac{1}{T}\int_0^T x^*(t)\, e^{j2\pi kt/T}\, dt$$

$$= \frac{1}{T}\int_0^T x(t)\, e^{j2\pi(-k)t/T}\, dt \qquad \text{because } x^*(t) = x(t)$$

$$= \alpha_{-k}$$

This property does not apply to complex signals.

- Conjugate pairs of coefficients synthesize a sinusoid (this is also important)

$$\alpha_k\, e^{j\frac{2\pi}{T}kt} + \alpha_{-k}\, e^{-j\frac{2\pi}{T}kt} = 2\,|\alpha_k| \cos\!\left(\frac{2\pi}{T} k\, t + \text{angle}(\alpha_k)\right)$$

Thus when looking at a spectrum one "sees" cosines --- one for every conjugate pair of coefficients.

Derivation:

$$\alpha_k e^{j\frac{2\pi}{T}kt} + \alpha_{-k} e^{-j\frac{2\pi}{T}kt} = \alpha_k e^{j\frac{2\pi}{T}kt} + \alpha_k^* e^{-j\frac{2\pi}{T}kt} \quad \text{by the previous property}$$

$$= \alpha_k e^{j\frac{2\pi}{T}kt} + \left(\alpha_k e^{j\frac{2\pi}{T}kt}\right)^*$$

$$= 2\,\mathrm{Re}\left(\alpha_k e^{j\frac{2\pi}{T}kt}\right)$$

$$= 2\,|\alpha_k| \cos\left(\frac{2\pi}{T} k\,t + \mathrm{angle}(\alpha_k)\right)$$

- Linearity: Suppose $x(t)$ and $y(t)$ are periodic with period T and with α_k and β_k as their T-second Fourier coefficients, respectively. Then the T-second Fourier coefficients of $x(t) + y(t)$ are $\alpha_k+\beta_k$. (useful)

- Parseval's theorem: (useful but not quite as critical)

$$\text{signal power} = \frac{1}{T}\int_0^T x^2(t)|\ dt$$

$$= \sum_{k=-\infty}^{\infty} |\alpha_k|^2$$

Recall: The power of a periodic signal $x(t)$ is

$$P(x) = \lim_{S\to\infty} \frac{1}{2S}\int_{-S}^{S} x^2(t)\,dt = \frac{1}{T}\int_0^T x^2(t)\,dt$$

Though useful, the remaining properties will not be emphasized in this class.

- Approximating the Fourier synthesis formula with a finite number of terms, i.e.

$$x(t) = \sum_{k=-N}^{N} \alpha_k e^{j\frac{2\pi}{T}kt}$$

For practical reasons, this is necessary in many cases. It can be shown that the difference signal

$$e(t) = x(t) - \sum_{k=-N}^{N} \alpha_k e^{j\frac{2\pi}{T}kt}$$

has power $2\sum_{k=N+1}^{\infty} |\alpha_k|^2$ which goes to zero as N increases. If possible, we choose N so large that this is small.

- Time shifting: If $x(t)$ has Fourier coefficients α_k, then $x'(t) = x(t-t_o)$ has Fourier coefficients

$$\alpha_k' = \alpha_k e^{-j\frac{2\pi}{T}kt_o}$$

This shows, not surprisingly, that a time shift causes a phase shift of each spectral component, where the phase shift is proportional to the frequency of the component.

- Frequency shifting: If $x(t)$ has Fourier coefficients α_k, then $x'(t) = x(t)\, e^{j\frac{2\pi}{T}k_o t}$ has Fourier coefficients

$$\alpha_k' = \alpha_{k-k_o} .$$

This shows that multiplying a signal by a complex exponential has the effect of shifting the spectrum of the signal.

- Time scaling: Let $a>0$. If $x(t)$ is periodic with period T with T-second Fourier coefficients α_k, then $x'(t) = x(at)$ is periodic with period T/a and T/a-second Fourier coefficients

$$\alpha_k' = \alpha_k .$$

This shows that the Fourier coefficients are not affected by a time scaling. However, a time scaling does affect the spectrum. Specifically, the Fourier coefficients of $x(t)$ are spaced at intervals of $1/T$ hz, whereas the Fourier coefficients of $x'(t)$ are spaced at intervals of a/T hz. For example, if $a>1$, then the Fourier coefficients are more widely spaced, and consequently, the spectrum of $x'(t)$ is expanded towards higher frequencies. This is consistent with the fact that making $a > 1$, means that $x'(t)$ fluctuates more rapidly than $x(t)$.

- Technicalities (mostly a warning that there are such)

In order that the integral in the anslysis formula be well defined and thta the synthesi formula holds, one needs to assume

$$\int_0^T |x(t)|\, dt < \infty \quad \text{and/or} \quad \int_0^T |x(t)|^2\, dt < \infty$$

When mathematicians prove

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt},$$

what they really show is that the power in the difference signal

$$e(t) = x(t) - \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt} \text{ is zero, assuming } \int_0^T |x(t)|^2\, dt < \infty.$$

So $x(t)$ and $\sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt}$ can differ at a isolated points.

Moreover, assuming $\int_0^T |x(t)|\, dt < \infty$ and the so-called "Dirichlet conditions[2]" the only points at which they can differ are points of discontinuity in $x(t)$.

Specifically, $\sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt} = x(t)$ if $x(t)$ is continuous at t.

and $\sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt} = \frac{1}{2}(x(t-)+x(t+))$ if $x(t)$ is discontinuous at t.

There's more discussion of "technicalities" in EECS 306.

[2]Dirichlet conditions: In addition to $\int_0^T |x(t)|\, dt < \infty$, in any one period $x(t)$ has only a finite number of maxima and minimum and only a finite number of discontinuities.

C. The spectra of segments of a signal

Motivating question: What is the spectra of a signal that is not periodic?

For example, what if the signal has finite support? Or what if the signal has infinite support, but is not periodic?

Observation: The Fourier series analysis formula works with a finite segment of a signal.

An approach to assessing the spectra of a signal with finite support:

If the signal has finite support $[t_1,t_2]$ apply the analysis formula

$$\alpha_k = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} x(t)\, e^{-j2\pi kt/(t_2-t_1)}\, dt$$

Now let

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt}, \quad \text{where } T = t_2-t_1.$$

Then $\tilde{x}(t)$ is periodic with period T, and

$$\tilde{x}(t) = x(t) \text{ when } t_1 \le t \le t_2$$

Therefore,

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j\frac{2\pi}{T}kt}, \quad t_1 \le t \le t_2$$

which is a synthesis formula for $x(t)$ that works only for the support interval $t_1 \le t \le t_2$.

$\tilde{x}(t)$ is called the "periodic extension" of $x(t)$.

Note: We have considered Fourier series to fundamentally apply to periodic signals and secondarily apply to signals with finite support. The reverse point of view is also valid.

An approach to assessing the spectrum of a periodic signal with infinite support:

If the signal has infinite support, choose a time interval length T, divide the support interval into segments of length T, as in $[0,T]$, $[T,2T]$, $[2T,3T]$, ... Apply the previous approach to each segment. We obtain a sequence of spectra, one for each segment. Notice that with this approach the spectra varies with time. There are lots of issues here --- for example, what choice of T?

Homework Problems:

Problems from Chapter 3

Problems from the Chapter 3 section of the CD ROM

Additional questions about spectra and Fourier series

1. Find the spectrum of the signal

$$x(t) = A\cos(2t)\sin(3t)$$

2. Find the T_o-second Fourier coefficients of the following signals, where T_o is the fundamental period of the signal.

   (a) $x(t) = t, \ 0 \le t \le T_o$

   (b) $x(t) = |\sin(t)|$

   (c) $x(t) = \sin^2(t)$

3. Consider the signal $x(t)$ whose spectrum is shown below. (All terms are real.)

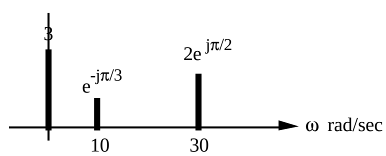
   (a) Is the signal periodic? If so, find its fundamental period.

   (b) Find its DC value.

   (c) Find its power.

   (d) Find $x(t)$. (Express the answer as a sum of sinusoids in standard form.)

4. The nonnegative frequency portion of the spectrum of a signal $x(t)$ is shown below



   (a) Is the signal periodic? If so, find its fundamental period.

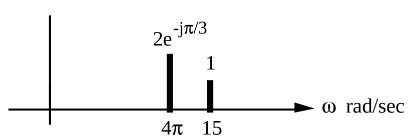
   (b) Find its DC value.

   (c) Find its power.

   (d) Find the negative frequency portion of the spectrum of $x(t)$.

   (e) Find $x(t)$. (Express the answer as a sum of sinusoids in standard form.)

5. The nonnegative frequency portion of the spectrum of a signal $x(t)$ is shown below



   (a) Is the signal periodic? If so, find its fundamental period.

   (b) Find its DC value.

(c) Find its power.

(d) Find the negative frequency portion of the spectrum of $x(t)$.

(e) Find $x(t)$. (Express the answer as a sum of sinusoids in standard form.)

6. Find the power of the signal

$$x(t) = 4 + 3\cos(3t+.1) + 5\cos(7t+.3) - 4\cos(9t+.2)$$

7. Do the signals $x(t)$ and $y(t)$ given below have overlapping spectra? That is, is there a spectral component of one whose frequency lies on top of one spectral component or between the frequencies of two spectral components of the other?

$$x(t) = \cos(20t) + \cos(22t) + \cos(23t)$$

$$y(t) = 2 + \cos(10t)\cos(11t)$$

8. Let $x(t)$ be a periodic signal with fundamental period T_o, and let α_k be the T_o-second Fourier coefficients of $x(t)$. Suppose we also calculate the $2T_o$-second Fourier coefficients, denoted α_k^2. Derive an expression for α_k^2 in terms of α_k.

9. (a) Show that if $x(t)$ is an even periodic function, i.e. if $x(-t) = x(t)$, then all of its Fourier coefficients are real.

(a) Show that if $x(t)$ is an odd periodic function, i.e. if $x(-t) = -x(t)$, then all of its Fourier coefficients are imaginary.

10. Let $x(t)$ and $y(t)$ be sinusoids with different frequencies and support $(-\infty,\infty)$. Show that the power of $x(t)+y(t)$ equals the sum of the powers of $x(t)$ and $y(t)$. (Do not assume that $x(t)$ and $y(t)$ are such that $x(t)+y(t)$ is periodic. This fact implies that Parseval's theorem can be used to compute the power of a sum of sinusoids, even if the sum is not periodic.)

11. Derive the linearity property of Fourier series: Suppose $x(t)$ and $y(t)$ are periodic with period T and with α_k and β_k as their T-second Fourier coefficients, respectively. Then the T-second Fourier coefficients of $x(t) + y(t)$ are $\alpha_k+\beta_k$.

12. Derive the time-shifting property of Fourier series: If $x(t)$ has Fourier coefficients α_k, then $x'(t) = x(t-t_o)$ has Fourier coefficients

$$\alpha_k' = \alpha_k\, e^{-j\frac{2\pi}{T}kt_o}$$

13. Derive the frequency-shifting property of Fourier series: If $x(t)$ has Fourier coefficients α_k, then $x'(t) = x(t)\, e^{j\frac{2\pi}{T}k_o t}$ has Fourier coefficients

$$\alpha_k' = \alpha_{k-k_o} .$$

14. Derive the time-scaling property of Fourier series: Let $a>0$. If $x(t)$ is periodic with period T with T-second Fourier coefficients α_k, then $x'(t) = x(at)$ is periodic with period T/a and T/a-second Fourier coefficients

$$\alpha_k' = \alpha_k .$$