Homework #7, EECS 206, W03. Due **Fri. Feb. 21**, by 11:30AM

Notes

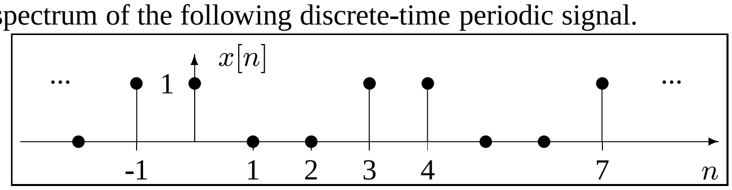
- Review the HW policies on HW1!
- Reading: "Part 3d" lecture notes. (Do not read Chapter 9!)
  As needed: Section 3 in Prof. Wakefield's "Primer on DFT."

Skill Problems

1. [10] Concept(s): ***period of discrete-time sinusoidal signals***
   For each of the following discrete-time sinusoidal signals, determine the frequency (in radians/sample). Also determine whether the signal is periodic. If so, determine its fundamental period.
   (a) [0] $x[n] = \cos\left(\frac{\pi}{2}n + \pi/3\right)$
   (b) [0] $x[n] = \cos\left(2\pi\frac{7}{6}n - \pi/4\right)$
   (c) [0] $x[n] = \cos(5n + \pi/7)$
   (d) [5] $x[n] = 3(-1)^n$
   (e) [5] $x[n] = 3\cos(7\pi n - \pi/3)$
   (f) [0] Is there a simple relationship between the frequency and the period? How do you explain the answers to (a) and (b) considering that the frequency in (b) is higher than the frequency in (a)?
   Hint. You may use MATLAB's `stem` command to plot some of these signals. However, show *analytically* how you determined the frequencies and periods (not just by looking at the graph). If you have a plotting calculator, you might compare what your calculator does to what MATLAB's `stem` command does.

2. [15] Concept(s): ***spectra of sums of discrete-time sinusoids***
   (a) [10] Sketch the two-sided spectrum of the following signal:
   $$y[n] = 29 + 8\cos(2\pi n/32) + 3\sin(2\pi 5n/32) + 8\sin(2\pi 33n/32).$$
   Hint: think carefully about that "33."
   (b) [5] Sketch the one-sided spectrum of $y[n]$.
   (c) [0] If the 32 (nonzero) values of $y[n]$ were stored in computer memory using 4-byte floating point numbers, how many bytes would be required?
   Alternatively, if the list of (frequency, complex amplitude) pairs (those with nonzero amplitude) were instead stored, how many bytes would be required? (Remember that complex numbers require two values to be stored. However, there is complex conjugate symmetry in the spectrum so you could be plan to be efficient in your storage scheme.
   You should find that for this type of signal the frequency domain representation is much more efficient for storage; these memory savings are fundamental to how MP3 digital audio compression works.

3. [15] Concept(s): ***spectra of discrete-time periodic signals***
   Sketch the two-sided spectrum of the following discrete-time periodic signal.



Mastery Problems

4. [30] Concept(s): ***finite Fourier series synthesis***

An engineer is working with the 6-periodic signal $x(t)$ whose Fourier series coefficients are given by $\alpha_k = 3^{-|k|}, \; k \in \mathbb{Z}$. She needs to plot this signal, but of course cannot do computation with an infinite number of coefficients. Instead, she synthesizes an *approximation* to $x(t)$ by using $2K+1$ coefficients as follows:

$$\hat{x}_K(t) = \sum_{k=-K}^{K} \alpha_k e^{\jmath 2\pi \frac{k}{T} t}.$$

To save computation, she would like to use the smallest value of K possible while maintaining adequate accuracy for her problem. Specifically, she wants the normalized RMS error to be less than 0.05%. In other words, she wants the smallest K for which $\mathrm{NRMS}(\hat{x}_K, x) = \dfrac{\mathrm{RMS}(\hat{x}_K - x)}{\mathrm{RMS}(x)} < 0.0005$.

The purpose of this problem is to provide and example of how spectra concepts arise in a *design* problem. Unlike many "number crunching" problems, design problems usually involve multiple steps. Here are the steps for this situation.

(a) [5] Sketch the spectrum of $x(t)$.

(b) [5] Sketch the spectrum of $\hat{x}_K(t)$ for $K = 2$.

(c) [5] Sketch the spectrum of $x(t) - \hat{x}_K(t)$ for $K = 4$ or for a "generic" value of $K \geq 4$.

(d) [5] Show that $\mathrm{RMS}(x) = \sqrt{5}/2$. Hint: use Parseval and a geometric series formula.

(e) [5] Show that

$$\mathrm{RMS}(\hat{x}_K - x) = \frac{1}{2 \cdot 3^K}.$$

(f) [5] Find the smallest K that satisfies the error criterion.

5. [30] One of the important topics that we will discuss later in the course is **filtering**, which means processing a signal in a way that amplifies or attenuates various frequency components. The tone controls (bass and treble) in a stereo system are examples of filters: when you “turn up the bass” you are amplifying the low frequency components, and when you “turn down the treble” you are attenuating the high frequency components. This problem is a preview of filters.

You will analyze what happens when a signal $x(t)$ passes through a special kind of filter called a **moving average** filter. This filter is used frequently in signal analysis to “smooth out” signals. A block diagram for a filter looks like

$$\text{(Input) } x(t) \to \boxed{\text{filter}} \to y(t) \text{ (Output).}$$

For this problem, assume that the input signal $x(t)$ has the spectrum described by the following (amplitude, frequency) pairs $\left\{(2+\jmath 2, -8), (4e^{-\jmath \pi/3}, -3), (1, 0), (4e^{\jmath \pi/3}, 3), (2-\jmath 2, 8)\right\}$, where the frequencies are in Hz.

(a) [5] Plot the magnitude spectrum of $x(t)$.

(b) [10] Express $x(t)$ as a sum of sinusoidal signals. Hint: there are three terms, and one of the amplitudes is $4\sqrt{2}$.

(c) [10] A T-second moving average filter is described by the following formula:

$$y(t) = \frac{1}{T}\int_0^T x(t+\tau)\, d\tau.$$

Using this formula with $T = 1/4$ s and the formula for $x(t)$ derived in the previous part, determine $y(t)$. Your integration should give you a sum of a few sinusoidal signals and you should simplify the result using phasor methods.

(d) [5] Plot the magnitude spectrum of $y(t)$.

(e) [0] Describe qualitatively how the spectrum of the output signal compares to the spectrum of the input signal.

(f) [0] Repeat the above (again for $T = 1/4$) for a generic sinusoidal input signal $\cos(2\pi f t)$ and make a plot of the output amplitude as a function of the input frequency f. Does this filter amplify or attenuate the high or low frequency components?