# Complex Signals: Properties, Statistics, and Operations

These notes summarize the properties, statistics and operations of complex signals, as discussed in lecture on 1/25/02.

## Signal Value Properties and Statistics

The following table shows the definitions of the signal characteristics mentioned previously, with the exception of signal value distribution, which is not easily summarized in table form. It also lists the analogous characteristics for discrete-time signals.

| | **Continuous-time signal z(t)**<br>**z(t) = x(t) + j y(t)** | **Discrete-time signal**<br>**z[n] = x[n] + j y]n]** |
|---|---|---|
| support interval | $[t_1,t_2]$ | $\{n_1,n_1+1,...,n_2\}$ |
| duration | $t_2-t_1$ | $n_2-n_1+1$ |
| average value: | $M(z) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} z(t)\,dt$<br>$= M(x) + j\,M(y)$ | $M(z) = \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} z[n]$<br>$= M(x) + j\,M(y)$ |
| magnitude: | $\lvert z(t)\rvert = \sqrt{x^2(t)+y^2(t)}$ | $\lvert z[n]\rvert = \sqrt{x^2[n]+y^2[n]}$ |
| squared value,aka instantaneous power: | $\lvert z(t)\rvert^2 = x^2(t)+y^2(t)$ | $\lvert z[n]\rvert^2 = x^2[n]+y^2[n])$ |
| mean-squared value, aka average power: | $MS(z) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} \lvert z(t)\rvert^2\,dt$<br>$= MS(x) + MS(y)$ | $MS(z) = \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} \lvert z[n]\rvert^2$<br>$= MS(x) + MS(y)$ |
| RMS value: | $RMS(z) = \sqrt{MS(z)}$ | $RMS(z) = \sqrt{MS(z)}$ |
| energy: | $E(z) = \int_{t_1}^{t_2} \lvert z(t)\rvert^2\,dt$<br>$= E(x) + E(y)$ | $E(x) = \sum_{n=n_1}^{n_2} \lvert z[n]\rvert^2$<br>$= E(x) + E(y)$ |

## Periodicity of continuous-time signals:

A complex continuous-time signal $z(t)$ is said to be *periodic with period* $T$ if $z(t+T) = z(t)$ for all values of $t$. This is equivalent to saying that both $x(t)$ and $y(t)$ are periodic with period $T$.

1. A continuous-time signal $z(t)$ with period $T$ is also periodic with period $nT$ for any positive integer $n$.
2. The *fundamental period* $T_o$ is the smallest period. The reciprocal of $T_o$ is called the *fundamental frequency* $f_o$ of the signal. That is, $f_o = 1/T_o$.
3. $z(t)$ is periodic with period $T$ if and only if $T$ is an integer multiple of $T_o$.
4. If signals $z(t)$ and $z'(t)$ are both periodic with period $T$, then the sum of these two signals, $w(t) = z(t) + z'(t)$ is also periodic with period $T$. This same property holds when three or more signals are summed.
5. The sum of two signals with fundamental period $T_o$ is periodic with period $T_o$, but its fundamental period might be less than $T_o$.
6. The sum of two signals with differing fundamental periods, $T_1$ and $T_2$, will be periodic when and only when the ratio of their fundamental periods equals the ratio of two integers. The fundamental period of the sum is the least common multiple of $T_1$ and $T_2$. The fundamental frequency of the sum is the greatest common divisor of the fundamemental frequencies of the two sinusoids.

**Periodicity of discrete-time signals:**

A discrete-time signal $z[n]$ is said to be *periodic with period* $N$ if $z[n+N] = z[n]$ for all integers $n$. This is equivalent to saying that both $x[n]$ and $y[n]$ are periodic with period $N$.

1. A discrete-time signal with period $N$ is also periodic with period $mN$ for any positive integer $m$.
2. The *fundamental period*, denoted $N_o$, is the smallest period. The reciprocal of $N_o$ is called the *fundamental frequency* $f_o$ of the signal. That is, $f_o = 1/N_o$.
3. $z[n]$ is periodic with period $N$ if and only if $N$ is an integer multiple of $N_o$.
4. If signals $z[n]$ and $z'[n]$ are both periodic with period $N$, then the sum of these two signals, $w[n] = z[n] + z'[n]$ is also periodic with period $N$. This same property holds when three or more signals are summed.
5. The sum of two signals with fundamental period $N_o$ is periodic with period $N_o$, but its fundamental period might be less than $N_o$.
6. The sum of two signals with differing fundamental periods, $N_1$ and $N_2$, is periodic with fundamental period equal to the least common multiple of $N_1$ and $N_2$ and fundamental frequency equal to the greatest common divisor of their fundamental frequencies $f_1$ and $f_2$. Note that unlike continuous-time case, the ratio of the fundamental periods of discrete-time periodic signals is always the ratio of two integers. Therefore, the sum is always periodic.

## A. Elementary Operations on One Signal.

These are illustrated for continuous-time signals, but apply equally to discrete-time signals.

**Adding a constant:** $z'(t) = z(t) + c$, where $c$ is a real or complex number.

**Amplitude scaling:** $z'(t) = c\, z(t)$, where $c$ is a real or complex number.

This has the effect of scaling both the average and the mean-squared values. Specifically, $M(z') = c\, M(z)$ and $MS(z') = |c|^2 MS(z)$.

**Time shifting:** If $z(t)$ is a signal and $T$ is some number, then the signal

$$z'(t) = z(t-T) = x(t-T) + j\, y(t-T)$$

is a *time-shifted* version of $x(t)$.

**Time reflection/reversal:** The time reflected or time reversed version of a signal $z(t)$ is

$$z'(t) = z(-t).$$

**Time scaling**: The operation of *time-scaling* a signal $x(t)$ produces a signal

$$z'(t) = z(ct)$$

where $c$ is some positive real-valued constant.

**Combinations of the above operations:** In the future we will frequently encounter signals obtained by combining several of the operations introduced above, for example,

$$z'(t) = 3\, z(-2(t-1)) .$$

## B. Elementary Operations on Two or More Signals

These are illustrated for continuous-time signals, but apply equally to discrete-time signals.

**Summing:** $w(t) = z(t) + z'(t)$ .

**Linear combining:** $w(t) = c_1\, z_1(t) + c_2\, z_2(t) + c_3\, z_3(t)$ , where $c_1,c_2,c_3$ are real or complex numbers.

**Multiplying:** $w(t) = z(t)\, z(t)$ .

**Concatenating:** *Concatenation* is the process of appending one signal to the end of another.

## C. Correlation

The correlation between continuous-time complex signals $z(t)$ and $z'(t)$ is

$$C(z,z') = \int_{t_1}^{t_2} z(t)\, z'^*(t)\, dt \ ,$$

where $(t_1,t_2)$ is the time interval of interest.

Similarly, the correlation between discrete-time complex signals $z[n]$ and $z'[n]$ is defined to be

$$C(z,z') = \sum_{n_1}^{n_2} z[n]\, z'^*[n] \ .$$

The correlation between a signal and itself is the signals energy:

$$C(z,z) = E(x) \ .$$

Correlation is not symmetric, i.e. $C(z,z') \neq C(z',z)$. However,

$$C(z',z) = C^*(z,z') \ .$$

The normalized correlation between signals $z$ and $z'$ is

$$C_N(z,z') = \frac{C(z,z')}{\sqrt{E(z)}\sqrt{E(z')}} \ .$$

*Schwarz Inequality*

$$|C_N(z,z')| \leq 1,$$

with equality if and only if one signal is an amplitude scaling of the other; i.e. $y(t) = c\, x(t)$ for some real or complex constant $c$.