EECS 206, Final Exam Solutions and Statistics 4/26/02

Average Score = 72.7 , Standard Deviation = 20.6, High Score = 101.

Average Scores for each problem:

1	2	3	4	5	6	7	8	9	10	11	EC
2.5/5	2.7/5	4.6/5	4.0/5	2.8/5	4.3/5	4.8/5	13.8/17	8.8/15	12.3/16	11.1/17	.8/1

Grade Ranges: A = 100-85 (70 A's), B = 84-67 (58 B's), C = 66-46 (46 C's), D = 45- (25 D's)

Note: Problem solutions are not consecutive.

1. Answer: (b)

$$E(x+y) = \int (x(t)+y(t))^2 dt \quad \int (x(t)^2+2x(t)y(t) + y(t)^2)\, dt = E(x) + 2\,C(x,y) + E(y).$$

Since $E(x+y) = 5$, $E(x) = 1$, and $E(y) = 2$, we have $2C(x,y) = 2$ and $\mathbf{C(x,y) = 1}$.

3. Answer: (a)

$$x_2[n] = 4 \cos(\frac{\pi}{3} n - \frac{2\pi}{3}) = 4 \cos(\frac{\pi}{3} (n-1) - \frac{\pi}{3}) = \frac{4}{7} x_1[n-1]$$

Therefore, by linearity and time-invariance,

$$\mathbf{y_2[n]} = \frac{4}{7} y_1[n-1] = \frac{4}{7}\sqrt{3} \cos(\frac{\pi}{3} (n-1)) = \boldsymbol{\frac{4}{7}\sqrt{3}\, \cos(\frac{\pi}{3} n - \frac{\pi}{3})}$$

7. Answer: (e)

By direct convolution.

8. (17 points) The continuous-time signal

$$x(t) = 4 \cos(2\pi(2)t) + 6 \cos(2\pi(3)t) + 2 \cos(2\pi(11)t)$$

is sampled at rate $f_s = 10$ samples per second creating the discrete-time signal $x[n]$.

(a) (5) Find and plot the magnitude spectrum of $x(t)$.

With frequency measured in hz, the plot shows spectral lines with the heights given below at the frequencies given below:

2 @ -11 3 @ -3 2 @ -2 2 @ 2 3 @ 3 2 @ 11

Alternatively, with frequency measured in radians/sec, the plot shows spectral lines with the heights given below at the frequencies given below:

2 @ -22π 3 @ -6 2 @ -4π 2 @ 4π 3 @ 6π 2 @ 22π

(b) (4) Find an expression for $x[n]$

$$\mathbf{x[n]} = x(nT_s) = x(\frac{n}{10}) = 4 \cos 4\pi \frac{n}{10} + 6 \cos 6\pi \frac{n}{10} + 2 \cos 22\pi \frac{n}{10}$$

$$= 4 \cos \frac{2\pi}{5} n + 6 \cos \frac{3\pi}{5} n + 2 \cos \frac{11\pi}{5} n$$

$$\boldsymbol{= 4 \cos \frac{2\pi}{5} n + 6 \cos \frac{3\pi}{5} n + 2 \cos \frac{\pi}{5} n} \quad \text{since } \frac{11\pi}{5} - 2\pi = \frac{\pi}{5}$$

(c) (8) Find and plot the magnitude spectrum of x[n] for $\hat{\omega}$ between π and $-\pi$.
The plot shows lines with the heights given below at the frequencies given below:

$$\mathbf{3\ @\ -\frac{3\pi}{5} \quad 2\ @\ -\frac{2\pi}{5} \quad 1\ @\ -\frac{\pi}{5} \quad 1\ @\ \frac{\pi}{5} \quad 2\ @\ \frac{2\pi}{5} \quad 3\ @\ \frac{3\pi}{5}}$$

Note that the cosine $\cos(2\pi(11)t)$ has the same samples as $\cos(2\pi t)$. Thus the spectral ine at frequency $\pi/5$ is due to aliasing.

9. (15 points) Consider the filter described by the difference equation

$$y[n] = 2\,x[n] + 2\,x[n-1]$$

A periodic input signal x[n] with period 4 is applied to this system, resulting in output y[n]. The 4-point DFT of y[n] is:

$$Y[0] = 0 \qquad Y[1] = \sqrt{2}\, e^{-j\pi/4} \qquad Y[2] = 0 \qquad Y[3] = \sqrt{2}\, e^{j\pi/4}$$

Find an expression for the input x[n].

Answer: $\mathbf{x[n] = \cos(\frac{\pi}{2} n)}$

Approach 1: The filter has frequency response $H(\hat{\omega}) = 2 + 2\,e^{-j\hat{\omega}}$.

We know $Y[k] = X[k]\,H(\frac{2\pi}{4}k)$. Therefore, $X[k] = \dfrac{Y[k]}{H(\frac{2\pi}{4}k)}$. We find

$$X[0] = 0,\ X[1] = \tfrac{1}{2},\ X[2] = 0,\ X[3] = \tfrac{1}{2}\ .$$

Taking the inverse DFT of X[k], or using the DFT synthesis formula gives

$$x[n] = \frac{1}{2} e^{j\frac{2\pi}{4}n} + \frac{1}{2} e^{j\frac{2\pi}{4}3n} = \frac{1}{2} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{2}n} = \cos \frac{\pi}{2} n$$

Approach 2: Taking the inverse DFT of Y[k], or using the DFT synthesis formula gives

$$y[n] = \sqrt{2}\, e^{-j\pi/4}\, e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}\sqrt{2}\, e^{j\pi/4} = \sqrt{2}\,(e^{j(\frac{\pi}{2}n-\pi/4)} + e^{-j(\frac{\pi}{2}n-\pi/4)})$$

$$= 2\sqrt{2}\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)$$

$$= 2\,|Y[1]|\cos(\tfrac{2\pi}{4}1n + \text{angle}(Y[1])) = 2\sqrt{2}\cos\left(\frac{\pi}{2}n - \frac{\pi}{4}\right)$$

Now, since the output is a cosine with frequency $\pi/2$, we conclude that the input is a cosine with frequency $\pi/2$. Suppose the input is x[n] A $\cos(\frac{\pi}{2} n + \phi)$. Then we know

$$x[n] \rightarrow y[n] = |H(\tfrac{\pi}{2})|\, A \cos(\tfrac{\pi}{2} n + \phi + \text{angle}(H(\tfrac{\pi}{2})))$$

Since $H(\frac{\pi}{2}) = 2 + 2\,e^{-j\pi/2} = 2-2j = 2\sqrt{2}\, e^{-j\pi/4}$, we have $|H(\frac{\pi}{2})| = 2\sqrt{2}$ and angle angle$(H(\frac{\pi}{2})) = -\frac{\pi}{4}$.

Therefore, $y[n] = 2\sqrt{2}\, A\cos(\frac{\pi}{2} n + \phi - \frac{\pi}{4})$.

Since also $y[n] = 2\sqrt{2}\cos(\frac{\pi}{2} n - \frac{\pi}{4})$

we see that $A = 1$ and $\phi = 0$. Hence, $\mathbf{x[n] = \cos(\frac{\pi}{2} n)}$

$\boxed{\#2}$ $5 \rightarrow DC \Rightarrow C_0 \neq 0$

$$40\pi t \Rightarrow \omega = 40\pi \quad f = 20$$

$$60\pi t \Rightarrow \omega = 60\pi \quad f = 30$$

$$\Rightarrow f_0 = 10, \quad \omega_0 = 20\pi$$

$\cos(40\pi t - \pi/5) \Rightarrow$ second harmonic $k = 2$ $\Rightarrow C_2 \neq 0$, $C_{-2} \neq 0$

$\cos(60\pi t) \Rightarrow$ third harmonic $k=3$ $\Rightarrow C_3 \neq 0$, $C_{-3} \neq 0$

ANSWER: (c)

$\boxed{\#4}$ Poles closest to the unit circle are at $\hat{\omega} \simeq \pm \pi/3$

ANSWER: (b)

$\boxed{\#5}$

$\left.\begin{array}{l} k=3 \\ + \; k = 9 = 12-3 \end{array}\right\}$ correspond to cos at $\hat{\omega} = 2\pi \frac{k}{N} = 2\pi \cdot \frac{3}{12}$

$k=6$ } corresponds to $e^{j\hat{\omega}n}$ at $\hat{\omega} = 2\pi \frac{k}{N} = 2\pi \cdot \frac{6}{12}$

$\left.\begin{array}{l} k=3 \\ (k=9) \end{array}\right\}$ mag of DFT coeff. $= \frac{1}{2}$, phase of DFT coeff $= \pi/4$

FROM CLASS RESULTS } $x[n] = \frac{1}{2} \times 2\cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) + 6 \cdot e^{j\pi n} = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right) + 6 \cdot \cos(\pi n)$

ANSWER: (a)

$\boxed{\#6}$ $\max(5500, 500) = 5500 \Rightarrow f_s > 2 \times 5500 = 11000$

ANSWER: (c)

#10

(a) $H1(z) = \frac{z+1}{z}$

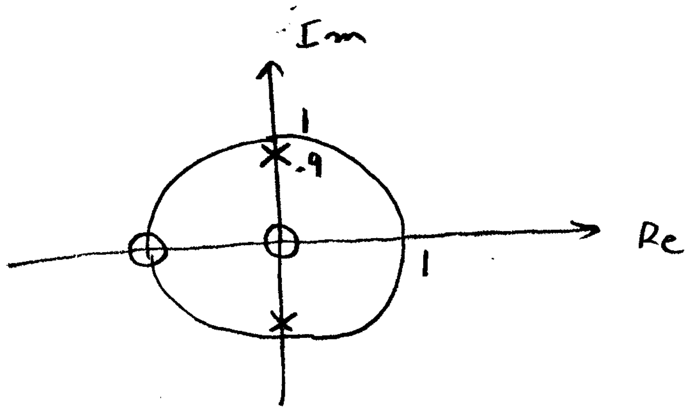
$$H2(z) = \frac{Y(z)}{S(z)} = \frac{5}{1+0.81z^{-2}} = \frac{5z^2}{z^2+0.81}$$

$$H(z) = \frac{Y(z)}{X(z)} = H1(z) \cdot H2(z) = \boxed{\frac{5z(z+1)}{z^2+0.81}}$$

(b) Zeros : $\boxed{z_{n1} = 0 \ , \ z_{n2} = -1}$

$$z^2 + 0.81 = (z + 0.9j)(z - 0.9j)$$

Poles : $\boxed{z_{p1} = -0.9j \ , \ z_{p2} = 0.9j}$



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#11

$a_1 = \frac{1}{3}$, $b_0 = 2$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 + \frac{1}{3} z^{-1}} = \frac{2z}{z + \frac{1}{3}}$$

(a) From tables, $\boxed{h(n) = 2 \times \left(-\frac{1}{3}\right)^n u[n]}$

(b) one pole at $z_{p1} = -\frac{1}{3}$ ⇒ inside unit circle.

⇒ FRF exists

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = \frac{2e^{j\hat{\omega}}}{e^{j\hat{\omega}} + \frac{1}{3}}$$

$$x[n] = \cos\left(\frac{\pi}{2} n\right) \quad \Rightarrow \hat{\omega}_0 = \pi/2$$

Sine in ⇒ Sine out

$$\Rightarrow y[n] = |\mathcal{H}(\pi/2)| \cdot \cos\left(\frac{\pi}{2} n + \angle \mathcal{H}(\pi/2)\right)$$

$$\mathcal{H}(\pi/2) = \frac{2e^{j\pi/2}}{e^{j\pi/2} + \frac{1}{3}} = \frac{2j}{j + 1/3} = \frac{6j}{1 + 3j} = \frac{6e^{j\pi/2}}{\sqrt{10}\, e^{j\tan^{-1}(3)}}$$

$$|\mathcal{H}(\pi/2)| = \frac{6}{\sqrt{10}} = 1.897$$

$$\angle \mathcal{H}(\pi/2) = \frac{\pi}{2} - \tan^{-1}(3) = 0.3218$$

$$\boxed{y[n] = \frac{6}{\sqrt{10}} \cdot \cos\left(\frac{\pi}{2} n + \frac{\pi}{2} - \tan^{-1}(3)\right)}$$

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