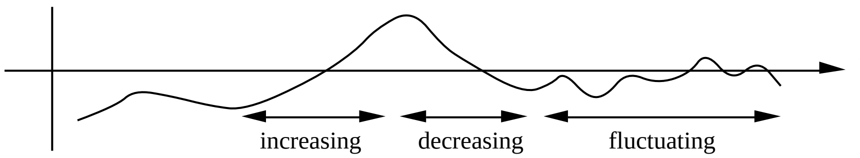
iii. Signal Shape Characteristics

In this section we consider signal characteristics related to what we loosely call signal "shape". Note that the signal value characteristics considered previously have nothing to do with signal shape, as one can see by noticing that very different signals can have the same signal value distribution, and consequently, the same min, max, average and mean-squared value. We will first focus on continuous-time signals and later comment briefly on the analogous characteristics for discrete-time signals.

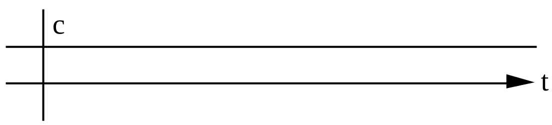
Local shape characteristics: When examining a signal $x(t)$, we often look at segments of it to see if it is *increasing*, *decreasing* or *fluctuating*, as illustrated in the example below.



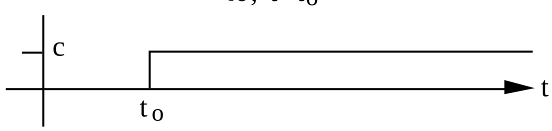
Common signal shapes:

The following is a listing of some common signal shapes. These can occur by themselves, or as segments of signals. That is, they may be thought of as local characteristics. The symbols b, c, d, t_o and t_1 represent parameters that need to be specified in order that the signals be completely determined.

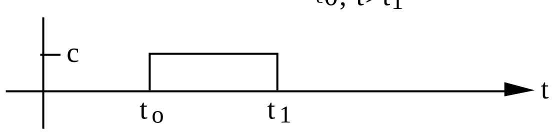
constant: $x(t) = c$



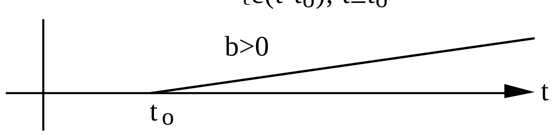
step[6]: $x(t) = \begin{cases} c, & t \ge t_o \\ 0, & t < t_o \end{cases}$



rectangular pulse[7]: $x(t) = \begin{cases} 0, & t < t_o \\ c, & t_o \le t \le t_1 \\ 0, & t > t_1 \end{cases}$



ramp: $x(t) = \begin{cases} 0, & t < t_o \\ c(t-t_o), & t \ge t_o \end{cases}$, increasing if $c>0$, decreasing if $c<0$



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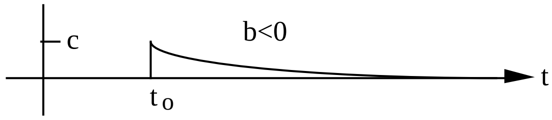
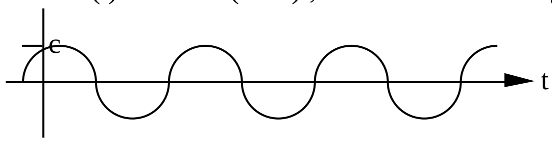
[6]Note that since the value of x at time t_o is c, strictly speaking, we should simply plot the value c at time t_o. Instead, we have drawn a vertical line, from 0 up to c. This emphasizes the change in x as it goes from $x(t) = 0$ for $t<t_o$ to $x(t) = c$ for $t>t_o$. This convention of drawing vertical lines where a function has a step change in value is quite common. We should also note that in the real world, there is no signal cam make a perfect instantaneous step from one value to another, as the formula for the step signal indicates. Instead, the signal value will rise rapidly from 0 to c in the vicinity of t_o. Thus a plot of a real world step signal will have a nearly vertical line rising from 0 to c at t_o. We may think of the vertical line shown in the figure above as a reminder that, in the real world, the signal can change rapidly, but cannnot actually have an ideal step change.

[7]Again notice the vertical lines, which are drawn for emphasis, and as a reminder of what happens in the real world.

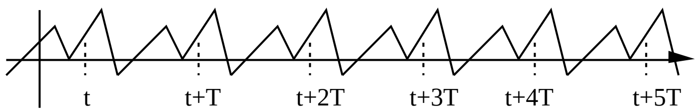
exponential: $x(t) = \begin{cases} 0, & t<t_o \\ ce^{b(t-t_o)}, & t \geq t_o \end{cases}$, increasing if $b>0$, decreasing if $b<0$, constant if $b=0$



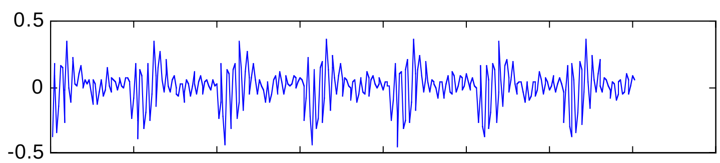
sinusoidal: $x(t) = c \sin(bt+d)$, fluctuating if $b \neq 0$



Periodicity: A continuous-time signal $x(t)$ is said to be *periodic with period* T if $x(t+T) = x(t)$ for all values of t. It is conventional to require the period T to be a positive number. For example the plot below shows a periodic signal. Its values are marked at a particular time t and also at times $t+T$, $t+2T$, ...



Many signals that appear in nature are periodic, or at least nearly so. For example, the following is a segment from a recording of someone speaking the vowel "ee".



Though many signals are *aperiodic*, i.e. not periodic, it turns out that periodic signals can play a key role in their analysis. Several important facts about periodic signals are given next.

1. A continuous-time signal $x(t)$ with period T is also periodic with period $2T$, because for any time t, $x(t+2T) = x((t+T)+T) = x(t+T) = x(t)$, where the last two inequalities follow from the definition of "periodic with period T". Indeed it is periodic with period nT for any positive integer n.

2. Though any periodic signals may be classified as having an infinitely many periods, there is always a unique smallest period, which is called the *fundamental period*, and often denoted T_o. That is, the fundamental period T_o of a signal $x(t)$ is the smallest positive number T such that $x(t+T) = x(t)$ for every value of t. The reciprocal of T_o is called the *fundamental frequency* f_o of the signal. That is, $f_o = 1/T_o$. It is the number of fundamental periods that occur per unit time. Warning: People often say "period" when they mean "fundamental period". So whenever you hear the word "period", you need to use the context to figure out if they really mean "fundamental period".

3. If $x(t)$ has fundamental period T_o, then $x(t)$ is periodic with period nT_o for every positive integer n. Conversely, these are the only periods of $x(t)$. That is, if $x(t)$ is periodic with period T, then $T = nT_o$ for some integer n.

Derivation of the converse statement[8]: Suppose $x(t)$ is periodic with fundamental period T_o and is also known to be periodic with period T. We must show that T is an integer multiple of T_o. We use proof by contradiction. Hypothetically suppose that T is not a

[8]This derivation is included for completeness. It is not expected that students can replicate this proof.

multiple of T. Then $T = nT_o + r$ where n is the integer part of T/T_o and r is the remainder, $0 < r < T_o$. Since $x(t)$ is periodic with period T_o, it must be that for any time t,

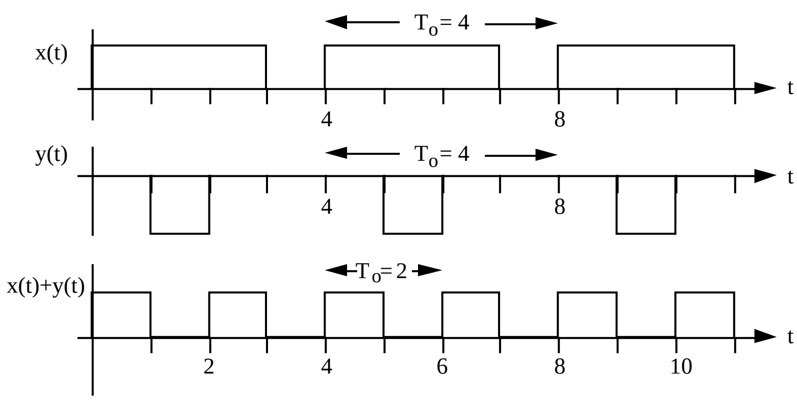
$x(t+r)$	$= x((t+r)+NT_o)$,	since $x(t)$ is periodic with period T_o, we have $x(t+NT_o) = x(t)$, which we apply with t replaced by $t+r$
	$= x(t+T)$	because $T = NT_o+r$
	$= x(t)$	because $x(t)$ is periodic with period T

Since $x(t+r) = x(t)$, we deduce that $x(t)$ is periodic with period r. But the fact that $r<T_o$ contradicts the fact that T_o is, by definition, the smallest period of $x(t)$. Therefore, our hypothetical assumption must be false. We conclude that T is a multiple of T_o.

4. A constant signal, e.g. $x(t) = 3$, is a special case. It satisfies $x(t+T) = x(t)$ for any choice of T. Thus it is periodic period T for every value of $T > 0$. However, it is conventionally defined to have fundamental period $T_o = \infty$ and fundamental frequency $f_o = 0$. This somewhat arbitrary definition turns out to be more useful than other definitions.

5. If signals $x(t)$ and $y(t)$ are both periodic with period T, then the sum of these two signals, $z(t) = x(t) + y(t)$ is also periodic with period T. This same property holds when one sums three or more signals. (The derivation of this will be given in class or given as a homework problem.)

6. The sum of two signals with fundamental period T_o is periodic with period T_o, but its fundamental period might be less than T_o, as the following example illustrates.



7. The sum of two signals with differing fundamental periods, T_1 and T_2, might or might not be periodic. They will be periodic when and only when the ratio of their fundamental periods equals the ratio of two integers. For example, if T_2/T_1 is 5/3, then the sum will be periodic. However, if $T_2/T_1 = \sqrt{2}$, then the sum will not be periodic.

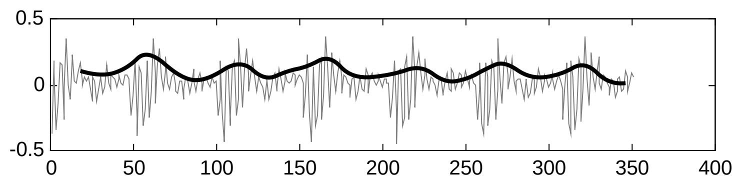
To see having an integer ratio makes a difference, consider two signals: $x(t)$ with fundamental period T_1, and $y(t)$ with fundamental period T_2. Suppose that $T_2/T_1 = m/n$, where m and n are integers. Then $nT_2 = mT_1$. Letting $T = nT_2 = mT_1$, we see that

$$\begin{aligned} x(t+T) + y(t+T) &= x(t+mT_1) + y(t+nT_2) \\ &= x(t) + y(t) \end{aligned}$$

because x has period T_1 and y has period T_2. This shows that $x(t)+y(t)$ is periodic with period T. To complete our discussion, we should also show that if T_2/T_1 is not the ratio of integers, then $x(t)+y(t)$ is not periodic. However, the proof of this is beyond the scope of the course and will not be given here.

In the case where T_2/T_1 is the ratio of two integers, the fundamental period of the sum signal can be found by finding the smallest integers m and n such that $nT_2 = mT_1$. In other words, the fundamental period is the least common multiple of T_2 and T_1. Correspondingly, the fundamental frequency is the greatest common divisor of the fundamental frequencies f_2 and f_1 of the two signals.

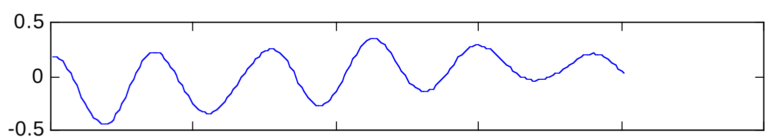
Envelope: This is best introduced with an example. The thick black line overlaying the signal shown below is the *envelope* of the signal. That is, for a rapidly fluctuating signal $x(t)$, the envelope is a smooth curve that approximately follows the peaks of the signal. Admittedly this is not a very precise definition, and there is no universally accepted definition that can make it precise. Nevertheless, the envelope is often a useful concept.



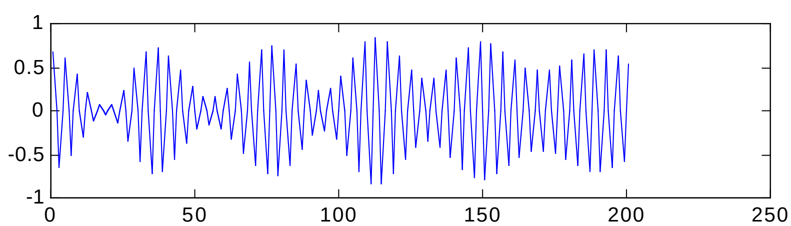
As an example, an AM radio station transmits an audio signal by embedding it in the envelope of a high frequency signal. Specifically, suppose $m(t)$ is the audio signal to be transmitted. Then the radio station assigned to frequency f_o transmits a signal of the form

$$s(t) = (m(t)+c) \cos(2\pi f_o t)$$

where c is a parameter chosen so that $m(t)+c \geq 0$ for all, or at least most, times t. Typically, f_o is a frequency much higher than the rate of fluctuation of $m(t)$. For example, if $m(t)$ is the audio signal shown below,



then the transmitted signal $s(t)\ (m(t)+.5) \cos(2\pi f_o t)$ is



Can you see the audio signal $m(t)$ embedded in the envelope of the transmitted signal $s(t)$? Can you think of a way of recovering $m(t)$ from $s(t)$?

Spectrum: The spectrum of a signal is terrifically important signal-shape-related characteristic. It is so important that we will not discuss it here. Rather, beginning with Chapter 3, it will be a focus of much of the remainder part of the class.

Signal Shape Characteristics of Discrete-Time Signals

Discrete-time signals can have all the same shape characteristics as continuous-time signals. For example, they can be increasing, decreasing or fluctuating. Common signal shapes include all of those mentioned previously: constant, step, rectangular pulse, ramp exponential and sinusoidal. Envelope is again a useful concept, as is periodicity. Because periodicity is such an important concept, we repeat the discussion of it here, this time for discrete-time signals.

Periodicity of discrete-time signals: A discrete-time signal x[n] is said to be *periodic with period* N if x[n+N] = x[n] for all integers n. This definition is the same as the definition for continuous-time signals, except that instead of the equality holding for all continuous times t, it holds for all integer times n. It is conventionally required that N > 0. We now reprise the various facts about periodicity. They are essential identical to the corresponding facts for continuous

1. A discrete-time signal with period N is also periodic with period mN for any positive integer m.

2. The *fundamental period*, denoted N_o, is the smallest positive integer N such that x[n+N] = x[n] for all integers n. The reciprocal of N_o is called the *fundamental frequency* f_o of the signal. That is, $f_o = 1/N_o$. It is the number of fundamental periods occurring per sample. (It is always less than or equal to one.) Warning: People often say "period" when they mean "fundamental period".

3. If x[n] has fundamental period N_o, then x[n] is periodic with period mN_o for every positive integer m. Conversely, these are the only periods of x[n]. That is, if x[n] is periodic with period N_o, then $N = mN_o$ for some integer m.

4. A constant signal, e.g. x[n] = 3, is a special case. It satisfies x[n+N] = x[n] for any choice of N. Thus it is periodic period N for every value of N > 0. However, it is conventionally defined to have fundamental period $N_o = \infty$ and fundamental frequency $f_o = 0$. This somewhat arbitrary definition turns out to be more useful than other definitions.

5. If signals x[n] and y[n] are both periodic with period N, then the sum of these two signals, z[n] = x[n] + y[n] is also periodic with period N. This same property holds when one sums three or more signals.

6. The sum of two signals with fundamental period N_o is periodic with period N_o, but its fundamental period might be less than N_o.

7. The sum of two signals with differing fundamental periods, N_1 and N_2, is periodic with fundamental equal to the least common multiple of N_1 and N_2 and fundamental frequency equal to the greatest common divisor of their fundamental frequencies f_1 and f_2. Note that unlike continuous-time case, the ratio of the fundamental periods of discrete-time periodic signals is always the ratio of two integers. Therefore, the sum is always periodic.

C. Two-Dimensional Signals

A picture or *image*, as we will usually say, can also be modeled as a signal. However, in this case, it must be modeled as a *two-dimensional* signal x(t,s). That is, instead of single independent parameter t representing time, there are two independent parameters t and s, representing vertical and horizontal position respectively. That is, x(t,s) represents the intensity or brightness of the image at the position specified by horizontal position t and vertical position s, relative to some coordinates. All of the previously mentioned concepts and characteristics can be extended to apply to two-dimensional signals. But we won't discuss them here. However, we do wish to mention that two-dimensional images can be discrete-time as well as continuous-time (discrete-space and continuous-space are better terms). In this case, the signal is x[m,n] where m and n are integers representing vertical and horizontal positions, respectively.

II. Elementary Signal Operations

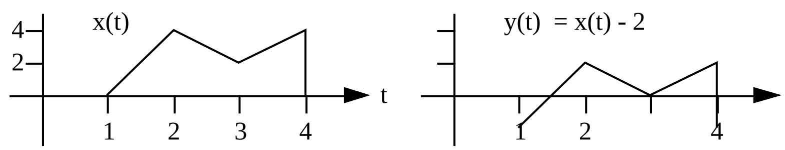
A. Elementary operations one a single signal.

In our discussions to come of signals and systems, we will routinely use a number of elementary *operations* that, when applied to one signal, result in another closely related signal. In the following we introduce these using continuous-time notation. With one exception to be noted, they apply equally to discrete-time signals, as well.

Adding a constant: This is the operation of adding a constant to the signal. More specifically, there is a number c that is added to the signal value at every time t. If the original signal is x(t), then the result is a new signal

$$y(t) = x(t) + c \ .$$

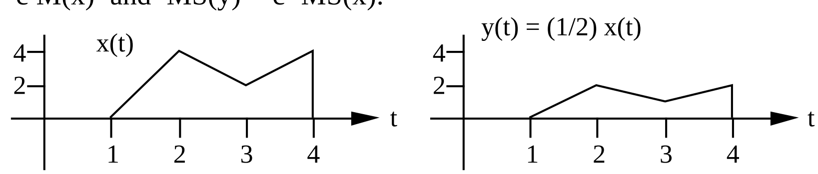
It should be easy to see that this has the effect of increasing the average value of x by c. That is, M(y) = M(x) + c.



Amplitude scaling: *Amplitude scaling* is the operation of multiplying a signal by a constant. That is, there is a constant c, called a *scale factor* or *gain*, the value of the signal at every time t is multiplied by c. If the signal being scaled is x(t), then the result of the scaling is

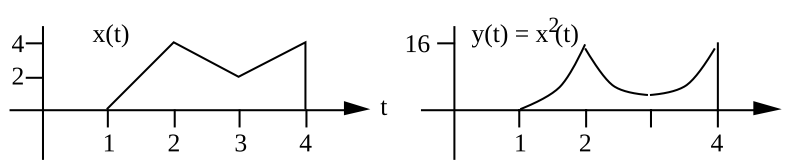
$$y(t) = c\ x(t) \ .$$

This has the effect of scaling both the average and the mean-squared values. Specifically, M(y) = c M(x) and MS(y) = c^2 MS(x).



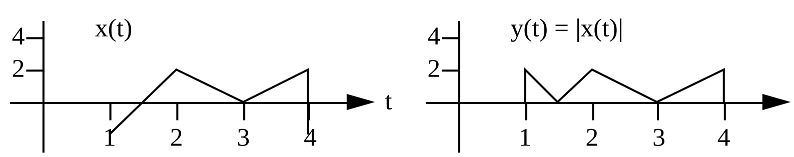
Squaring: Here we simply square the value of the signal at each time, yielding

$$y(t) = x^2(t) \ .$$



Absolute value: As the name suggests,

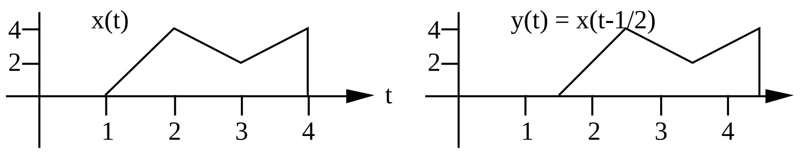
$$y(t) = |x(t)| \ .$$



Time shifting: If $x(t)$ is a signal and T is some number, then the signal

$$y(t) = x(t-T)$$

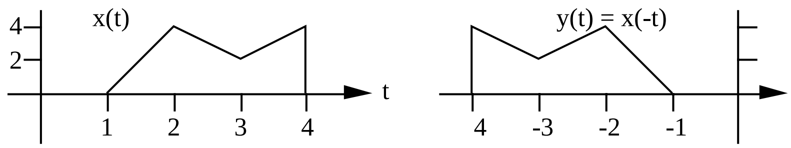
is a *time-shifted* version of $x(t)$. That is, the value of y at time t is precisely the value of x at time $t-T$. This means that if $T > 0$, then as illustrated below, anything that "happens" in the signal x also happens in the signal y, but it happens T time units later in y than in x. Similarly, if $T < 0$, it happens T time units earlier in y. It is useful to remember the rule that a positive value of T leads to a right shift of the plot of $x(t)$ and a negative value of T leads to a left shift.



Time reflection/reversal: The time reflected or time reversed version of a signal $x(t)$ is

$$y(t) = x(-t).$$

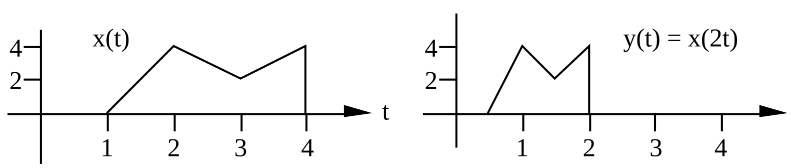
That is, whatever happens in x also happens in y, but at the negative of the time it happens in x.



Time scaling: The operation of *time-scaling* a signal $x(t)$ produces a signal

$$y(t) = x(ct)$$

where c is some positive constant. If $c > 1$, this has the effect of "speeding up time" in the sense that the value of y at time t is the value of x at time ct, which is a later time. Alternatively, whatever happens in x in the time interval $[t_1,t_2]$ now happens in y in the earlier and shorter time interval $[t_1/c, t_2/c]$.



This is the one property that for which the discrete-time case includes an extra wrinkle. Specifically, in discrete-time, the time values must be an integer. Therefore, if take

$$y[n] = x[cn],$$

then c needs to be an integer.

Combinations of the above operations: In the future we will frequently encounter signals obtained by combining several of the operations introduced above, for example,

$$y(t) = 3\ x(-2(t-1))\ .$$

To figure out what signal this is, it is useful to introduce some intermediate signals. For example, in the above, we might start by plotting