Part. 3a: Spectra of Continuous-Time Signals

Outline

A. Definition of spectrum
B. Spectra of signals that are *sums of sinusoids*
  ○ AM radio
C. Spectra of *periodic signals*
  ○ Fourier series analysis and synthesis
D. Spectra of *segments of signals* and *aperiodic signals*
E. Bandwidth

Reading

- "Part 3a" lecture notes
- Ch. 3 of text
- 3.4.5 supplement
- Wakefield Fourier series "quick primer"

Principal questions to be addressed

- What, in a general sense, is the **spectrum** of a signal?
- Why are we interested in spectra? ("spectra" = plural of "spectrum")
- How does one assess the spectrum of a given signal?

Here we consider continuous-time signals. The next part of the course discusses spectra of discrete-time signals.

Notes

The spectra has two important roles:

- Analysis and design. The spectra is a theoretical tool that enables one to understand, analyze and design signals and systems.
- System component: The computation and manipulation of spectra is a component of many important systems.

Spectra of Continuous-Time Signals

- Our coverage of spectra goes significantly beyond the coverage in Chapter 3.
- See the list of errata for Chapter 3.

A. Rough definition of spectrum and motivation for studying spectra

A.1. Introduction to the concept of "spectrum"

Definition

Roughly speaking, a "spectrum of a signal" is a description of the signal as a *sum* of *sinusoids*.

This definition involves (at least) two key *choices*:
- We choose **sinusoids** as our elementary components, because of the reasons described in the previous part.
- We choose to **sum** the sinusoids because that is simpler than other ways of combining.

A spectrum describes the frequencies, amplitudes and phases of the sinusoids that "sum" to yield the signal.

The individual sinusoids that sum to give the signal are called **sinusoidal components**.

Alternatively, the spectrum describes the distributions of amplitude and phase versus frequency of the sinusoidal components.

Since each sinusoid can be decomposed into the sum of two complex exponentials, the spectrum equivalently indicates how the signal may be thought of as being composed of **complex exponentials**.

It describes the frequencies, amplitudes and phases of the complex exponentials that "sum" to yield the signal.

The individual complex exponentials that sum to give the signal are called **complex exponential components**.

Sinusoidal and complex exponential components are also called **spectral components** or **frequency components**.

The "descriptions"

A signal that is a sum of sinusoids can be *described* in at least 4 distinct ways.
- Descriptions in the **time domain**:
  - Mathematical formulas

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

  - A plot of $x(t)$ versus time
- Descriptions in the **frequency domain**:
  - A **list** of the amplitudes, phases, and frequencies (and how many there are) such as

$$\{(A_0), (A_1, \phi_1, f_1), \ldots, (A_N, \phi_N, f_N)\}$$

  - As a plot of those amplitudes and phases as a function of frequency!

The "frequency domain" descriptions are examples of what is meant by spectra.

Plotting the spectra

To understand signals, we like to plot and visualize their spectra. We plot "spectral" lines at the frequencies of the exponential components (at both positive and negative frequencies). The height of the line is the magnitude of the component. We label the line with the complex amplitude of the component, *e.g.*, with $2e^{\jmath\pi/3}$.

Alternatively, we might make two line plots, one showing the magnitudes of the components and the other showing the phases. These are called the **magnitude spectrum** and **phase spectrum**, respectively.

Important note: Why a "rough" definition?

"Spectrum" is a broad collective noun, like "economy" or "health," for which there is no universal mathematically precise definition. Rather as with economy and health, there are a variety of specific ways to assess the spectrum of a signal.

For example, to assess the economy, one can measure GNP, average income, unemployment rate, poverty rate, DJIA, NASDAQ, money supply, ...

For example, to assess health one can measure body temperature, heart rate, blood pressure, blood chemistry, weight, etc.

Similarly, there are a variety of ways to assess the spectrum of a signal. A limited set will be discussed in this course: principally, Fourier series (FS) for periodic continuous-time signals, and discrete Fourier transform (DFT) for periodic discrete-time signals. But there will also be some discussion and use (mainly in the labs) of FS and DFT to assess the spectra of finite segments of signals. The (continuous-time) Fourier transform, which is another important method of assessing the spectrum of continuous-time signals, will be discussed in EECS 306.

Reasons for decomposing into sinusoids.

It's mainly that sinusoids into linear, time-invariant systems lead to sinusoidal output signals.
(No other class of signals has this property.)

This causes the input-output relationship for linear systems to be particularly simple for sinusoidal signals.

So representing signals with sinusoids simplifies analysis greatly.

Because analysis is simplified, efficient design methods can be developed.

A.2. Why are we interested in spectra?

Here are some reasons:

- **Preventing signal interference**, *e.g.*, AM radio

  Signals with non-overlapping spectrum do not interfere with one another. Thus many information carrying signals can be transmitted over a single communication medium (wire, fiber, cable, atmosphere, water, etc.).

  To design such systems, we need to be able to quantitatively determine the spectrum of signals to be able to assess whether or not they overlap, and if they do, by how much. Also, we need to able to develop systems (*e.g.*, filters) that select one signal over another, based on its spectrum.

- **Signal recognition**

  Some signals can be recognized based on their spectra, *e.g.*, vowels (Labs 8,9), touchtone telephone key presses, musical notes and chords, bird songs, whale sounds, mechanical vibration analysis, atomic/molecular makeup of sun and other stars, etc. To build systems that automatically recognize such signals, we need to able to quantitatively determine the spectrum of a signal.

- **Signal propagation**

  Communication media, *e.g.*, the atmosphere, the ocean, a wire, an optical fiber, often limit propagation to signals with components only in a certain frequency range (atmosphere is high frequency, ocean is low frequency, wire is low frequency, optical fiber is high frequency, but what is considered "high" or "low" depends on the media. We need to be able to assess the spectrum of a signal to see if it will propagate. We need to be able to design signals to have appropriate spectra for appropriate media.

- **System design**

  In many situations, the behavior of many natural and man-made linear systems is best analyzed in the "frequency domain", *i.e.*, one determines the behavior in response to sinusoids (or complex exponentials) at various frequencies, and from this one can deduce the response to other signals. The previous bullet is a special case of this.

- **Noise removal**

  In many situations, an undesired signal interferes with a desired signal, *e.g.*, the desired signal might correspond to someone speaking and the undesired signal might be background noise. We wish to reduce or eliminate the background signal. In order to be able to reduce or eliminate the background signal it must have some characteristic that is distinctly different than the desired signal. Often it happens that the desired and undesired signals have distinctly different spectra (*e.g.*, the noise has mostly high frequency components). In such cases, one can design systems, called "filters", that selectively reduce certain frequency components. These can be used to reduce the noise while having little effect on the desired signal.

- **Information hiding**

  Watermarking, etc.

- Many other signals and systems methods are based on spectra: *e.g.*, control engineering, data compression, voice recognition, music processing.

  <u>Example</u>. The bass and treble controls on an audio amplifier have been designed to affect the *frequency components* of a signal. "Turning up the bass" means amplifying the low frequency components. To describe this quantitatively, and to design such systems, one must understand spectra thoroughly.

- And....

A.3. How does one assess the spectrum of a given signal?

The remainder of these notes are intended to make progress on this question, with occasional references to questions 1 and 2.

There is no single answer.

The answer/answers do not fit into one course.

We address this question in spiral fashion in EECS 206. The answer continues in EECS 306 and beyond. (Just like you don't learn all there is to know about the economics in Econ. 101.)

We will develop several methods for continuous-time signals, several methods for discrete-time signals.

There is no single universal spectral concept in wide use.

We use different measures of the spectrum for different types of signals.

We will discuss mainly:

B. spectra of a sum of sinusoids (with support $(-\infty, \infty)$)

C. spectra of periodic signals (with support $(-\infty, \infty)$) via Fourier series

and briefly discuss

- spectra of a segment of a signal via Fourier series, which leads to:
  - the spectra of signal with finite support
  - the spectra of signal with infinite support via Fourier series applied to successive segments

We won't discuss:

- spectra of a signal with infinite support and finite energy via Fourier transform.
  This will be discussed in EECS 306.

We will have a similar discussion of spectra for discrete-time signals in the next part of the course.

We won't get too rigorous in our treatment of Fourier series. We'll leave that to future courses such as EECS 306.

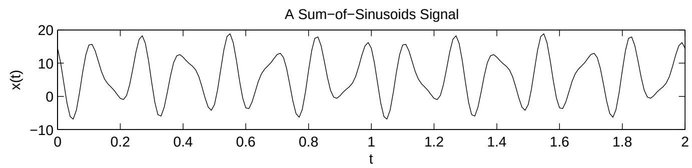
B. The spectrum of a finite sum of sinusoids

As in the text Section 3.1, we begin the discussion of how to assess a spectrum by considering signals that are finite sums of sinusoids.

To illustrate the main ideas, we start with an example.

Example. Consider the following sum-of-sinusoids signal (in standard form):

$$x(t) = 6 + 9\cos(2\pi 7t + \pi/3) + 4\cos(2\pi 11t - 0.1).$$



From this graph, would you expect this signal to sound pleasing to the ear? ("Harmonic") Who knows!
Can you tell from the graph how many sinusoids there are, or even for sure if they are sinusoids? Doubtful!

First, we emphasize that this is sum of sinusoids *of different frequencies* so the "simplifications" developed in the previous part are inapplicable.

Second, it is going to be more convenient later to express spectra in terms of complex exponential signal components instead of sinusoidal signal components. So our next step is to rewrite $x(t)$ using the following inverse Euler identity:

$$A\cos(2\pi f_0 t + \phi) = A\frac{e^{\jmath(2\pi f_0 t+\phi)} + e^{-\jmath(2\pi f_0 t+\phi)}}{2} = \left(\frac{A}{2}e^{\jmath\phi}\right)e^{\jmath 2\pi f_0 t} + \left(\frac{A}{2}e^{-\jmath\phi}\right)e^{-\jmath 2\pi f_0 t}.$$

Applying that identity to each term in our signal yields the following **formula**:

$$x(t) = 6 + \left(\frac{9}{2}e^{\jmath\pi/3}\right)e^{\jmath 2\pi 7t} + \left(\frac{9}{2}e^{-\jmath\pi/3}\right)e^{-\jmath 2\pi 7t} + \left(2e^{-\jmath 0.1}\right)e^{\jmath 2\pi 11t} + \left(2e^{\jmath 0.1}\right)e^{-\jmath 2\pi 11t}.$$

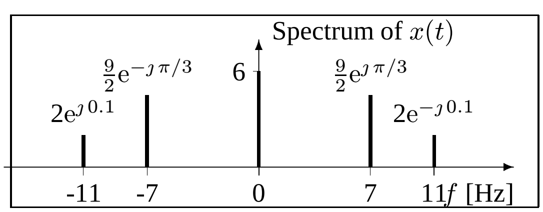
Recall that the terms in parentheses are called **phasors**.

One way to *describe* this signal would be to use a **list** of the (phasor, frequency) pairs, as follows:

$$\left\{(2e^{\jmath 0.1}, -11),\ (\frac{9}{2}e^{-\jmath\pi/3}, -7),\ (6, 0),\ (\frac{9}{2}e^{\jmath\pi/3}, 7),\ (2e^{-\jmath 0.1}, 11)\right\}.$$

This is a *complete* description of the signal in the sense that if I give you this list then you know what the signal $x(t)$ is, and could write down its formula, or plot it, or compute the value of $x(t)$ at some time of interest, etc.

However, our visual system is much better at understanding patterns shown graphically than it is at understanding a list of numbers. So we visualize the spectrum using the following **plot**:



Compare the plot of the spectrum to the plot of the signal. The spectrum plot is "simpler, more compact and more intuitively informative" than the plot of $x(t)$. This illustrates what we mean by "the spectrum is a compact representation of the signal."

Would this sound harmonic? Probably not. The ratio 11/7 is not a ratio of powers of 3 and 2 (the **Pythagorean intervals**).

General case for sum-of-sinusoids

Consider a **sum-of-sinusoids** signal of the form

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k) \\ &= A_0 + A_1 \cos(2\pi f_1 t + \phi_1) + A_2 \cos(2\pi f_2 t + \phi_2) + \cdots + A_N \cos(2\pi f_N t + \phi_N), \end{aligned}$$

where $N, A_0, A_1, \phi_1, f_1, \ldots, A_N, \phi_N, f_N$, are parameters that specify the signal $x(t)$. We derive its spectrum.

Using Euler's formula, we can rewrite $x(t)$ as

$$x(t) = X_0 + \sum_{k=1}^{N} \mathrm{Re}\left\{X_k e^{\jmath 2\pi f_k t}\right\}$$

where

$$X_0 = A_0, \quad \text{and} \quad X_k = A_k e^{\jmath \phi_k}, k = 1, \ldots, N$$

is the phasor corresponding to $A_k \cos(2\pi f_k t + \phi_k)$. (The phasor is a complex number.)

Using the fact that $\mathrm{Re}\{z\} = \frac{z+z^\star}{2}$, we further rewrite this as

$$\begin{aligned} x(t) &= X_0 + \sum_{k=1}^{N} \left[\frac{X_k}{2} e^{\jmath 2\pi f_k t} + \frac{X_k^\star}{2} e^{-\jmath 2\pi f_k t}\right] \\ &= \left(\frac{X_N^\star}{2} e^{-\jmath 2\pi f_N t} + \cdots + \frac{X_1^\star}{2} e^{-\jmath 2\pi f_1 t}\right) + X_0 + \left(\frac{X_1}{2} e^{\jmath 2\pi f_1 t} + \cdots + \frac{X_N}{2} e^{\jmath 2\pi f_N t}\right). \end{aligned}$$

To make this expression more compact, we rewrite it as the following single **sum-of-complex-exponentials**:

$$\boxed{x(t) = \sum_{k=-N}^{N} \alpha_k e^{\jmath 2\pi f_k t}}$$

$$= \left(\alpha_{-N} e^{\jmath 2\pi f_{-N} t} + \cdots + \alpha_{-1} e^{\jmath 2\pi f_{-1} t}\right) + \alpha_0 + \left(\alpha_1 e^{\jmath 2\pi f_1 t} + \cdots + \alpha_N e^{\jmath 2\pi f_N t}\right).$$

Two make these two forms match, the **DC term** (corresponding to zero frequency, a constant signal) is given by

$$\boxed{\alpha_0 = X_0 = A_0 = \mathrm{M}(x),}$$

and the coefficients for positive frequencies are given by

$$\boxed{\alpha_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{\jmath \phi_k},} \qquad k = 1, \ldots, N,$$

and the coefficients for negative frequencies are given by the following **conjugate symmetry** relationship:

$$\boxed{\alpha_{-k} = \alpha_k^\star,} \qquad k = 1, \ldots, N,$$

and where we define the negative frequencies by

$$f_{-k} = -f_k, \qquad k = 1, \ldots, N.$$

(And $f_0 = 0$.)

Using the above sum-of-complex-exponentials form for $x(t)$, we make the following definition.

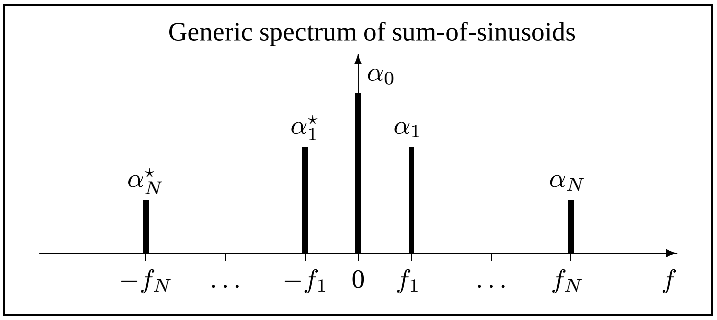
Definition: The (two-sided) spectrum of this signal is the list of pairs

$$\{(\alpha_{-N}, f_{-N}),\ \ldots,\ (\alpha_{-1}, f_{-1}),\ (\alpha_0, 0),\ (\alpha_1, f_1),\ \ldots,\ (\alpha_N, f_N)\}$$

or equivalently,

$$\left\{(\alpha_N^\star, -f_N),\ \ldots,\ (\alpha_{-1}^\star, -f_1),\ (\alpha_0, 0),\ (\alpha_1, f_1),\ \ldots,\ (\alpha_N, f_N)\right\}.$$

A picture of a generic spectrum for a sum-of-sinusoids signal is the following.



Notes

- The spectrum, *i.e.*, this list, is considered to be a "compact" representation of the signal $x(t)$, *i.e.*, just a few numbers.
  Example. This compactness is the essence of how MP3 audio compression can shrink an entire hour of music from an audio recoding into a modest number of bits!
- The "spectrum" is also called the **frequency-domain representation** of the signal.
  In contrast $x(t)$ is the **time-domain representation** of the signal.
- The term $A_k \cos(2\pi f_k t + \phi_k)$ is called the **sinusoidal component** of $x(t)$ at frequency f_k.
- The term α_k is called the **complex exponential component** or **spectral component** of $x(t)$ at frequency f_k.
- It is equally valid to express the spectrum with frequencies in rad/sec or Hz. However, Hz, kHz, and MHz, etc., are more typical in engineering practice, as opposed to in engineering textbooks...
- To obtain a useful visualization, we often plot the spectrum by drawing, for each k, a **spectral line** at frequency f_k with height equal to $|\alpha_k|$ and labelling the line with the (usually complex) value of α_k.
- Alternatively, we sometimes separate the spectrum into magnitude and phase parts:
  - The **magnitude spectrum**

$$\{(|\alpha_{-N}|, f_{-N}),\ \ldots,\ (|\alpha_{-1}|, f_{-1}),\ (\alpha_0, 0),\ (|\alpha_1|, f_1),\ \ldots,\ (|\alpha_N|, f_N)\}$$
$$= \{(|\alpha_N|, -f_N),\ \ldots,\ (|\alpha_k|, -f_1),\ (\alpha_0, 0),\ (|\alpha_1|, f_1),\ \ldots,\ (|\alpha_N|, f_N)\}.$$

  - The **phase spectrum**

$$\{(\angle\alpha_{-N}, f_{-N}),\ \ldots,\ (\angle\alpha_{-1}, f_{-1}),\ (\alpha_0, 0),\ (\angle\alpha_1, f_1),\ \ldots,\ (\angle\alpha_N, f_N)\}$$
$$= \{(-\angle\alpha_N, -f_N),\ \ldots(-\angle\alpha_k, -f_1),\ (\alpha_0, 0),\ (\angle\alpha_1, f_1),\ \ldots,\ (\angle\alpha_N, f_N)\}.$$

  In particular, we often make separate plots of magnitude and phase. That is, for each k, the magnitude plot has a line of height $|\alpha_k|$ at frequency f_k, and the phase plot has a line of height $\angle\alpha_k$ at frequency f_k.
- Often, but certainly not always, we are more interested in the "magnitude spectrum", *i.e.*, the magnitude of the α_k's, than the "phase spectrum".
- Alternatively, people sometimes focus on the **one-sided spectrum** (rather than our "two-sided" spectrum), which in the case of a finite sum of sinusoids is

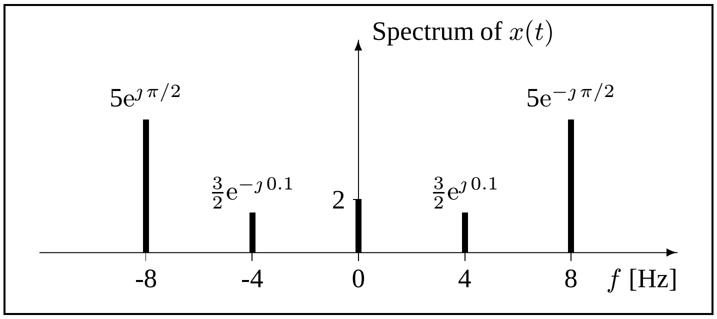
$$\{(\alpha_0, 0),\ (\alpha_1, f_1),\ \ldots,\ (\alpha_N, f_N)\}$$

Example. Problem: Assess the spectrum of the following sum-of-sinusoids signal:

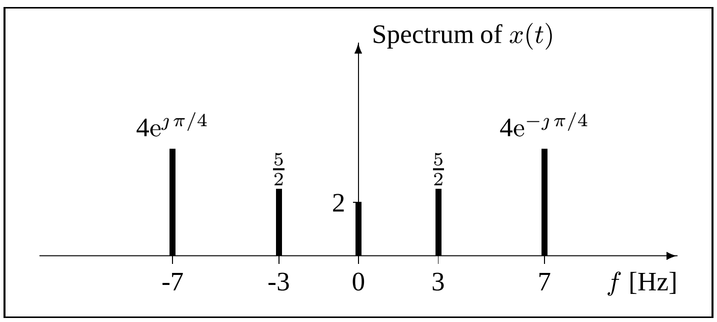
$$x(t) = 2 + 3\cos(2\pi 4t + .1) + 10\sin(2\pi 8t).$$

Since $\sin(2\pi 8t) = \cos(2\pi 8t - \pi/2)$, the spectrum is:

$$\left\{(5e^{\jmath\pi/2}, -8),\ (\frac{3}{2}e^{-\jmath 0.1}, -4),\ (2, 0),\ (\frac{3}{2}e^{\jmath 0.1}, 4),\ (5e^{-\jmath\pi/2}, 8)\right\}$$



Example: Given the spectrum of the signal $x(t)$ shown below, find $y(t) = 2x(3t - 1/4)$.



First we find $x(t)$ by "reading off" the components:

$$\begin{aligned} x(t) &= 2 + \frac{5}{2}e^{\jmath 2\pi 3t} + \frac{5}{2}e^{-\jmath 2\pi 3t} + 4e^{-\jmath\pi/4}e^{\jmath 2\pi 7t} + 4e^{\jmath\pi/4}e^{-\jmath 2\pi 7t} \\ &= 2 + 5\cos(2\pi 3t) + 8\cos(2\pi 7t - \pi/4). \end{aligned}$$

Alternatively, we could jump right to that second expression as long as one remembers that $A_k = 2|\alpha_k|$ for $k \neq 0$.

Thus we find $y(t)$ by substituting and simplifying (watch the phases!):

$$\begin{aligned} y(t) &= 2x(3t - 1/4) \\ &= 2\left[2 + 5\cos(2\pi 3(3t - 1/4)) + 8\cos(2\pi 7(3t - 1/4) - \pi/4)\right] \\ &= 4 + 10\cos(2\pi 9t - 3\pi/2) + 16\cos(2\pi 21t - 7\pi/2 - \pi/4) \\ &= 4 + 10\cos(2\pi 9t + \pi/2) + 16\cos(2\pi 21t + \pi/4). \end{aligned}$$

Effects of time shift/scale and amplitude shift/scale on spectra

At this point we have considering the spectra only of sums-of-sinusoids signals: *i.e.*,

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k).$$

If we apply some of our simple signal operations, *e.g.*,

$$y(t) = a + bx(ct + d)\,,$$

then how does the spectrum of $y(t)$ relate to that of $x(t)$?

First note that

$$\cos(2\pi f_k(ct + d) + \phi_k) = \cos(2\pi(cf_k)t + \phi_k + 2\pi f_k d).$$

Simply substituting in then we see

$$y(t) = (a + A_0) + \sum_{k=1}^{N} (bA_k)\cos(2\pi(cf_k)t + [\phi_k + 2\pi f_k d])$$

i.e.,

$$y(t) = \underbrace{(a + A_0)}_{\text{new DC}} + \sum_{k=1}^{N} \underbrace{(bA_k)}_{\text{new amp.}} \cos(2\pi \underbrace{(cf_k)}_{\text{new freq.}} t + \underbrace{\phi_k + 2\pi f_k d}_{\text{new phase}}).$$

From this expression we could read off the coefficients to plot the spectrum.

The most interesting is perhaps the time shift and time scale effects. Visualizing these is left as an exercise.

It is also useful to examine these effects in the compact sum-of-complex-exponentials form:

$$x(t) = \sum_{k=-N}^{N} \alpha_k e^{\jmath 2\pi f_k t}.$$

Then

$$\begin{aligned} y(t) &= a + bx(ct + d) \\ &= a + b \sum_{k=-N}^{N} \alpha_k e^{\jmath 2\pi f_k (ct+d)} \\ &= a + \sum_{k=-N}^{N} b\alpha_k e^{\jmath 2\pi f_k d} e^{\jmath 2\pi f_k ct} \\ &= \sum_{k=-N}^{N} \beta_k e^{\jmath 2\pi (cf_k) t}, \end{aligned}$$

where the coefficients of $y(t)$ are related to the coefficients of $x(t)$ as follows:

$$\beta_k = \begin{cases} a + b\alpha_0, & k = 0 \\ be^{\jmath 2\pi f_k d}\alpha_k, & k \neq 0. \end{cases}$$

So the DC term is scaled and amplitude shifted, whereas the other terms are scaled and phase shifted by the $e^{\jmath 2\pi f_k d}$ term.

Amplitude Modulation (AM)

As an example to illustrate the utility of the concept of spectra, consider the form of a signal transmitted by an AM radio station

$$x(t) = (v(t) + d)\cos(2\pi f_c t),$$

where $v(t)$ is the audio signal, and $\cos(2\pi f_c t)$ is the **carrier signal**.

We assume that $v(t)$ is scaled so that $v(t) \geq -d$ for all t, so that the audio information is encoded in the **envelope** of $x(t)$ as discussed previously.

The **carrier frequency** f_c is usually a high frequency, *e.g.*, 660 kHz. This is the frequency that you to tune your radio to.

A block diagram of the transmission process is:

$$v(t) \rightarrow \underset{\substack{\uparrow \\ d}}{\bigoplus} \rightarrow \underset{\substack{\uparrow \\ \cos(2\pi f_c t)}}{\bigotimes} \rightarrow \boxed{\text{amplifier}} \rightarrow \text{antenna}$$

Motivation:

Our audio signal is low frequency typically 0 to 5 kHz.
Low frequencies do not propagate through the atmosphere.

Need to generate a high frequency signal that "carries" the audio signal.

The carrier signal $\cos(2\pi f_c t)$ has high frequency, so it can propagate.

$x(t)$ is obtained by "modulating" the carrier signal by the audio signal.

$v(t)$ becomes the envelope of $x(t)$. (adding the constant d insures this)

Example. Suppose a single "audio test tone" is to be transmitted. Specifically, we assume:

$$v(t) = A\cos(2\pi f_v t),$$

for $0 < A \leq d$.

Problem: find and plot the spectrum of $x(t)$.

(A real radio station is not usually interested in transmitting a sinusoidal audio signal. The sinusoidal $v(t)$ is just a stand-in for a genuine audio signal. We're assuming this choice of $v(t)$, because so far it is about all that we can analyze.)

Solution:

$$x(t) = [d + A\cos(2\pi f_v t)]\cos(2\pi f_c t) = d\cos(2\pi f_c t) + \frac{A}{2}\cos(2\pi(f_c + f_v)t) + \frac{A}{2}\cos(2\pi(f_c - f_v)t).$$

The spectrum has components at frequencies $\{\pm(f_c - f_v), \pm f_c, \pm(f_c + f_v)\}$. Specifically, the spectrum is:

$$\{(A/4, -(f_c + f_v)),\ (d/2, -f_c),\ (A/4, -(f_c - f_v)),\ (A/4, f_c - f_v),\ (d/2, f_c),\ (A/4, f_c + f_v)\}.$$

(Picture) of spectrum.

Discuss how it depends on f_c, f_v, and d. Mention the **bandwidth.**

What values of d would be preferable?

Note: This example is intended as a simple example of using the concept of "spectrum" to do an "analysis". Usually we must *analyze* before we can attempt to *design*.

Design problem: AM radio station carrier frequency spacing

How closely can one space the carrier frequencies of AM radio broadcasters?

Example. Frequency multiplexing of AM signals
(This example uses spectra to design a frequency multiplexing parameter.)

Suppose:
Radio station 1 wants to transmit audio signal $v_1(t)$ at carrier frequency c_1
Radio station 2 wants to transmit audio signal $v_2(t)$ at carrier frequency $c_2 > c_1$.

Question. How far apart must c_1 and c_2 be to prevent interference of the two transmitted signals?

For concreteness assume: $v_1(t) = \cos(2\pi a_1 t), v_2(t) = \cos(2\pi a_2 t)$.

Then:
Radio station 1 transmits: $x_1(t) = (1 + v_1(t))\cos(2\pi c_1 t)$
Radio station 2 transmits: $x_2(t) = (1 + v_2(t))\cos(2\pi c_2 t)$

Solution:
The spectrum of $x_1(t)$ has components at frequencies $c_1 - a_1, c_1, c_1 + a_1$ ***(Picture)***

The spectrum of $x_2(t)$ has components at frequencies $c_2 - a_2, c_2, c_2 + a_2$ ***(Picture)***

To prevent overlap of the spectra, we need to choose c_2 and c_1 so that

$$c_1 + a_1 < c_2 - a_2$$

i.e., so that

$$c_2 > c_1 + a_1 + a_2.$$

In a practical AM system, the audio signal has spectrum ranging from 0 kHz to +5 kHz. In fact they limit the audio signals to this range. So the AM radio signal has **bandwidth** about 10kHz, from $f_c - 5\text{kHz}$ to $f_c + 5\text{kHz}$. Because of this, AM radio stations are assigned frequencies in increments of 10 kHz. And the FCC avoids having two stations in the same area being separated only by 10 kHz. This is because the spectra of real audio signals do not quite fit exactly between 0 and +5 kHz. (See EECS 306 for more details.) And because even if they did, a radio receiver cannot pick out the signal components in the range $f_c - 5\text{kHz}$ to $f_c + 5\text{kHz}$ without also accepting at least some signal components outside this band. Sometimes you can hear two AM radio stations at once, especially if you've tuned to a weak one while a powerful one is transmitting at a frequency only 10kHz away, especially if you have an old/cheap radio receiver.

Note: This example is intended to be a concrete example of the practical use of the concept of spectrum to do a simple design task.

Other examples of spectra

Example. Modern digital oscilloscopes usually have "spectrum" option that you can select to examine the spectrum of the input signal, instead of seeing just the usual time-domain display of the signal.

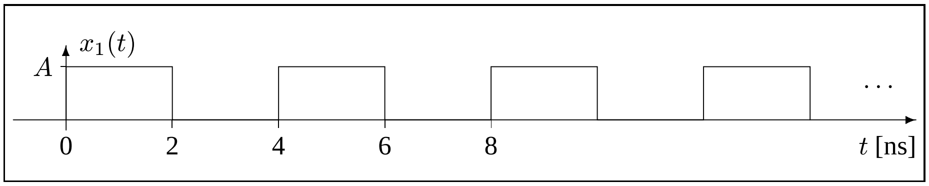
Example. Some adventurous musicians use "pitch trackers" that (in real time) determine which note is being played on the instrument (by analyzing the spectrum of the signal measured by a microphone or some other electrical input), so that an electronic instrument (usually a digital synthesizer) can be synchronized to play the same note (or related notes).

We will better understand these examples *after* we describe how to compute a spectrum from sampled data.

C. The spectrum of a periodic signal

We have seen how to assess and plot the spectrum of a sum-of-sinusoids signal. You might say that those plots are fairly "obvious" since you can just read off the amplitudes, frequencies, and phases from the sum-of-sinusoids expression. So at this point it might not seem like the spectrum offers much "value added" over the time-domain formula.

But what about a signal like the following square wave?

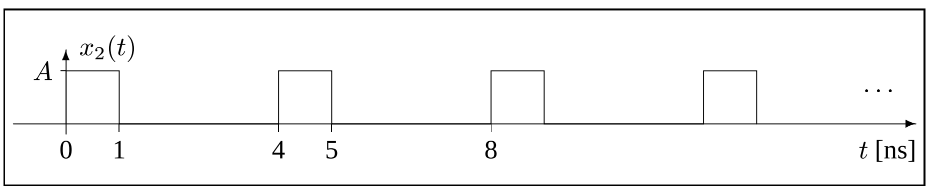


Since t is in nanoseconds, the fundamental period is $T_0 = 4$ns.
So the fundamental frequency of this periodic signal is $f_0 = 1/T_0 = 250$MHz.
But what is the **spectrum** of this signal?
You might wonder why we should care, since this signal appears perfectly easy to "understand" in the time domain.

Example.

Here is a (simplified) example of an engineering design problem where we would need to know the spectrum of the above square wave. Suppose you are designing a very high-speed digital system (*e.g.*, computer motherboard) and you need to have a common clock signal to synchronize different subunits of that system. The conductors (printed circuit board paths) that connect the different subunits will attenuate frequencies that are "too high," due to parasitic capacitances and resistances. For simplicity, we assume here that these interconnects completely attenuate all frequency components above 5GHz, while passing unchanged all frequency components below that cutoff[1]. Notice that this is a frequency-domain description of the conductors. Such descriptions are commonplace in engineering systems, for example, plain old telephone service has a maximum frequency of about 3kHz.

You are debating between using the square wave given above or instead the pulse train given below as the clock signal.



Both signals will be degraded by the attenuating properties of the interconnects.

Which signal do you use?

One way to make the choice would be to use whichever signal is degraded *less*, *i.e.*, whichever signal is more immune to the imperfections of the interconnects. (This is not the whole picture by any means, but it is one reasonable way to start thinking about such problems.)

We cannot solve even this elementary design problem with the tools discussed so far! We will revisit it after laying more foundation.

Although the signals above appear easy to "understand" in the time domain, for all the reasons enumerated previously, it is also very important to understand what are the properties of this signal, and other periodic signals, in the **frequency domain**.

But how can we do that since all we have above is a picture!? There is no "sum-of-sinusoids" anywhere in view!

Fortunately for us, a brilliant French mathematician and Egyptologist named Joseph Fourier (1768-1830)[2] proved in 1807 the following amazing fact: **any periodic signal can be expressed as a sum-of-sinusoids!**

This result is so surprising that it was quite controversial when Fourier first discovered it, and some mathematicians and scientists of the day did not believe it!

[1] We will see later in the course that in reality there is a shoulder region where some frequency components are partially attenuated, but at this point in the course we stick with this "all or nothing" model for simplicity.

[2] See Oppenheim & Willsky for biosketch.

C.1 Fourier Series

The main point of this section is the following theorem, which we will not prove, but which we will illustrate and use.

Fourier Series Theorem

Any periodic signal[3] $x(t)$ with period T can be written as a (usually infinite) sum of sinusoids, all of which have frequencies that are integer multiples of $1/T$. That is, there is a set of amplitudes and phases $\{(A_k, 0),\ (A_1, \phi_1),\ (A_2, \phi_2),\ \ldots\}$ and corresponding frequencies $\{0, 1/T, 2/T, \ldots\}$, such that

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{\infty} A_k \cos\left(2\pi \frac{k}{T} t + \phi_k\right) \\ &= A_0 + A_1 \cos\left(2\pi \frac{1}{T} t + \phi_1\right) + A_2 \cos\left(2\pi \frac{2}{T} t + \phi_2\right) + \cdots . \end{aligned}$$

Notice that the frequency of the kth sinusoid in this expression is

$$f_k \triangleq \frac{k}{T},$$

which is a multiple of the **fundamental frequency** $f_1 = 1/T$. (These frequencies are called **harmonic frequencies**.)

This form of the Fourier series is called the **sinusoidal Fourier series**, since it is a sum-of-sinusoids form.

As we have stated repeatedly, it will be more convenient to use inverse Euler identities to rewrite this expression in terms of complex exponential signals, instead of sinusoidal signals, recalling that

$$\begin{aligned} A_k \cos(2\pi f_k t + \phi_k) &= \frac{1}{2}\left(A_k e^{\jmath \phi_k}\right) e^{\jmath 2\pi f_k t} + \frac{1}{2}\left(A_k e^{-\jmath \phi_k}\right) e^{-\jmath 2\pi f_k t} \\ &= \frac{1}{2} X_k e^{\jmath 2\pi f_k t} + \frac{1}{2} X_k^\star e^{-\jmath 2\pi f_k t}, \end{aligned}$$

where, as usual, $X_k = A_k e^{\jmath \phi_k}$ denotes the **phasor** associated with the component $A_k \cos(2\pi f_k t + \phi_k)$.

Substituting in this equality yields

$$\begin{aligned} x(t) &= A_0 + \sum_{k=1}^{\infty} \left[\frac{1}{2}\left(A_k e^{\jmath \phi_k}\right) e^{\jmath 2\pi f_k t} + \frac{1}{2}\left(A_k e^{-\jmath \phi_k}\right) e^{-\jmath 2\pi f_k t}\right] \\ &= X_0 + \sum_{k=1}^{\infty} \left[\frac{1}{2} X_k e^{\jmath 2\pi f_k t} + \frac{1}{2} X_k^\star e^{-\jmath 2\pi f_k t}\right] = X_0 + \sum_{k=1}^{\infty} \mathrm{Re}\left\{X_k e^{\jmath 2\pi f_k t}\right\}. \end{aligned}$$

With some judicious renaming of things, we can simplify the preceding form to the following **sum-of-complex-exponentials** form:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t},} \qquad \textbf{synthesis formula}$$

where we express the **Fourier coefficients** $\{\alpha_k\}$ in terms of the phasors as follows:

$$\boxed{\alpha_0 = A_0 = X_0 = \mathrm{M}(x),}$$

$$\boxed{\alpha_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{\jmath \phi_k} \quad k = 1, 2, \ldots,}$$

$$\boxed{\alpha_{-k} = \alpha_k^\star \quad k = 1, 2, \ldots .}$$

This form is called the **exponential Fourier series** and we will focus on it throughout 206.

[3] Well, almost any periodic signal. Any periodic signal of any practical interest is covered by the conditions of the theorem. There are pathological periodic functions for which Fourier series would not work, but they have no practical relevance. But they do necessitate footnotes like this.

Notes

- The proof of the theorem is beyond the scope of this class, and EECS 306, too.
- The theorem says that *any* periodic signal can be represented as a sum-of-sinusoids. But it *may* take an infinite number of them, and often will!
- $A_k \cos(2\pi \frac{k}{T} t + \phi_k)$ is the sinusoidal component of $x(t)$ at frequency k/T.
- All sinusoids in the above formulae have frequencies that are multiples of $1/T$.
- Usually we choose T to be the **fundamental period** of $x(t)$, but any period will do.
- The theorem also says that *any* periodic signal can be represented as a sum of complex exponentials. (It may take an infinite number.)
- α_k is the complex exponential component (equivalently, the spectral component) of $x(t)$ at frequency k/T.
- It follows from the theorem that the spectrum of a periodic signal with period T is concentrated at frequencies

  $$0, \pm\frac{1}{T}, \pm\frac{2}{T}, \pm\frac{3}{T}, \ldots,$$

  or some subset thereof, *i.e.*, $x(t)$ has spectral components *only* at these frequencies.
  So by examining the spectrum of a signal, it is easy to see whether it is periodic.
- The three summation expressions for $x(t)$ given in the theorem are considered to be three forms of the "Fourier series". (A "series" is an infinite sum.)
  The first is called the "sinusoidal Fourier series"; the third is called the "exponential Fourier series".
  The book introduces the first two forms in Section 3.4 (equation (3.4.1)
  It is most common to use the third form (the exponential Fourier series), because it is easier to work with. We'll primarily use the third form.
  There is another form called the **trigonometric** Fourier series that looks like

  $$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(2\pi \frac{k}{T} t) + b_k \sin(2\pi \frac{k}{T} t) \right].$$

  This form is the least convenient for the purposes of signals and systems.
- The A_k's, ϕ_k's, X_k's, and α_k's are called **Fourier series coefficients** or just **Fourier coefficients**.
- The summing of sinusoids to obtain an arbitrary signal is very well illustrated with the `sinsum` demo program in the 206 Lab and with the Matlab demo program called `xfourier.m`.
- The Fourier series coefficients are **unique**: there is one and only one set of α_k's that will reproduce a given periodic signal $x(t)$ for a given period T.

The analysis formula

Fourier proved more than what we have stated so far! Not only did he prove that any periodic signal can be expressed as a sum of harmonic sinusoids with certain amplitudes and phases, he also *derived a simple formula* for finding those coefficients:

$$\alpha_k = \frac{1}{T} \int_0^T x(t) \, e^{-\jmath 2\pi \frac{k}{T} t} \, dt.$$

This is called the **analysis formula**.
The derivation of the analysis formula is presented well in the new section 3.4.5. Reading it is strongly recommended.

This formula is quite remarkable. It tells us that we can start with a *picture* or some other description of $x(t)$ that appears to have nothing to do with sinusoids, and then we can use the analysis formula to find the coefficients $\{\alpha_k\}$, and then we can insert those coefficients into the synthesis formula, and *voila*, we have a sum-of-sinusoids expression for $x(t)$! And of course, once we have that type of expression, we can display the spectrum of $x(t)$.

Summary of Fourier series

Synthesis formula: shows how $x(t)$ is a sum of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t}$$

Analysis formula: shows how to compute the α_k's, *i.e.*, the Fourier coefficients

$$\alpha_k = \frac{1}{T} \int_0^T x(t)\, e^{-\jmath 2\pi \frac{k}{T} t} \, dt.$$

(Alternatively, we could integrate over any T-second interval since both $x(t)$ and the complex exponential are T-periodic.)

Definition: The (two-sided) spectrum of a periodic signal with period T is

$$\{\ldots, (\alpha_{-2}, -2/T), (\alpha_{-1}, -1/T), (\alpha_0, 0), (\alpha_1, 1/T), (\alpha_2, 2/T), \ldots\}.$$

More Notes:

- Finding the spectrum of a periodic signal involves finding the period T and the Fourier coefficients $\{\alpha_k\}$.
- Finding the α_k's is often called "taking the Fourier series".
- To aid the understanding of the synthesis formula, it can be useful to view it in long form:

$$x(t) = \ldots + \alpha_{-2} e^{-\jmath 2\pi \frac{2}{T} t} + \alpha_{-1} e^{-\jmath 2\pi \frac{1}{T} t} + \alpha_0 + \alpha_1 e^{\jmath 2\pi \frac{1}{T} t} + \alpha_2 e^{\jmath 2\pi \frac{2}{T} t} + \ldots$$

- The frequency $1/T$ (usually in Hz) is called the **fundamental** or **first harmonic** frequency.
  The frequency k/T is called the k**th-harmonic frequency**.
  Likewise, the component at frequency $1/T$ is called the **fundamental** or **first harmonic** component, and the component at frequency k/T is called the **kth-harmonic** component.
- If a signal has period T, then it also has period $2T$. So when applying Fourier analysis, we have a choice as to T. Often, but certainly not always, we choose T to equal the fundamental period.
  When we want to explicitly specify the value of T used, we will say "the T-second Fourier series".
- If you wish to find the other forms of the Fourier series, use the formulas:

$$A_0 = \alpha_0, \qquad A_k = 2\,|\alpha_k|, \qquad \phi_k = \angle \alpha_k, \qquad k = 1, 2, \ldots$$

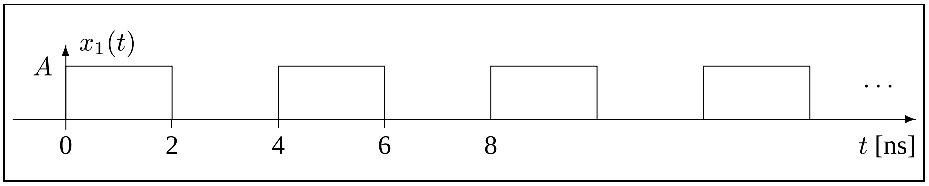
$$X_0 = \alpha_0, \qquad X_k = 2\alpha_k, \qquad k = 1, 2, \ldots$$

- The Fourier series Theorem applies to complex signals as well as to real signals.
- Notice that α_k is the correlation of $x(t)$ with $e^{\jmath 2\pi \frac{k}{T} t}$ normalized by $1/T$, which is the energy of one period of the exponential.
- Suggested reading. The discussion of "signal components" at the end of Section III.B of "Introduction to Signals" by DLN. This will show help one to understand why the analysis formula has the form that it has.
  In the terminology of that discussion:
  $\alpha_k e^{\jmath 2\pi \frac{k}{T} t}$ is the component of $x(t)$ that is like $e^{\jmath 2\pi \frac{k}{T} t}$,
  α_k measures the similarity of $x(t)$ to the exponential.
  There is a similar interpretation that $A_k \cos(2\pi \frac{k}{T} t + \phi_k)$ is the component of $x(t)$ that is like a cosine at frequency k/T.

At first glance the analysis and synthesis formulae for the Fourier series might seem "circular," since it appears that $x(t)$ depends on the α_k's yet the α_k's depend on $x(t)$. When working with Fourier series, the usual "chain of events" is the following.

- We start with some simple description of $x(t)$, usually a picture, for which there are no sinusoids in sight.
- We compute the α_k coefficients using the analysis formula.
- We substitute those α_k's into the synthesis formula.
  - Having made that substitution, we can readily display the spectrum of $x(t)$.
  - Or we can compute $x(t)$ (approximately) for any values of t of interest (using a finite number of terms).
  - Or we can can build a device (called a **synthesizer**) that generates $x(t)$ (approximately) by connecting together several sin-wave generators with appropriate amplitudes and phases.
    (This is how the additive synthesis worked in the early days of electronic music.)

Example. Find the spectrum of the following squarewave signal.



Since t is in nanoseconds, the fundamental period is $T_0 = 4$ns.
So the fundamental frequency of this periodic signal is $f_0 = 1/T_0 = 250$MHz.

For this signal, the spectrum cannot be computed by inspection! We very much need the analysis formula.

First we find the DC term:

$$\alpha_0 = \mathrm{M}(x) = A/2.$$

Using the analysis formula, the Fourier coefficients are given by

$$\alpha_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-\jmath 2\pi \frac{k}{T_0} t}\, dt.$$

For convenience, we integrate using nanoseconds units for t:

$$\begin{aligned} \alpha_k &= \frac{1}{4} \int_0^4 x(t)\, e^{-\jmath 2\pi \frac{k}{4} t}\, dt = \frac{1}{4} \int_0^2 A e^{-\jmath 2\pi \frac{k}{4} t}\, dt = \frac{A}{4} \left. \frac{1}{-\jmath 2\pi (k/4)} e^{-\jmath 2\pi \frac{k}{4} t} \right|_0^2 \\ &= \frac{A}{-\jmath 2\pi k} \left[e^{-\jmath 2\pi \frac{k}{4} 2} - 1 \right] = \frac{A}{\jmath 2\pi k} \left[1 - e^{-\jmath \pi k} \right] = \frac{A}{\jmath 2\pi k} \left[1 - (-1)^k \right]. \end{aligned}$$

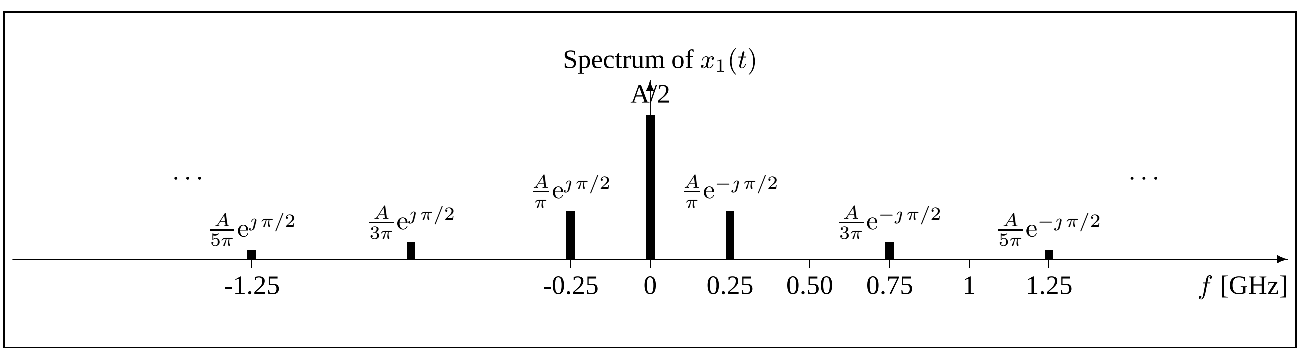
This formula is valid *only* for $k \neq 0$ since otherwise we would have divided by zero. To simplify, note that

$$\frac{1 - (-1)^k}{2} = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd.} \end{cases}$$

Thus we have

$$\alpha_k = \begin{cases} A/2, & k = 0 \\ \frac{A}{\jmath \pi k}, & k \text{ odd} \\ 0, & k \neq 0,\ k \text{ even.} \end{cases} = \begin{cases} A/2, & k = 0 \\ \frac{A}{\pi k} e^{-\jmath \pi/2}, & k > 0 \text{ odd} \\ \frac{A}{\pi |k|} e^{\jmath \pi/2}, & k < 0 \text{ odd} \\ 0, & k \neq 0,\ k \text{ even.} \end{cases} \tag{3a-1}$$

Thus, the spectrum of this square wave $x(t)$ has the following plot.



The sinusoidal Fourier series representation for this signal is (for t in ns):

$$x_1(t) = \frac{A}{2} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{A}{\pi k} \cos\left(2\pi \frac{k}{4} t - \pi/2 \right).$$

Coefficient matching

In the preceding example, we found the Fourier coefficients by integration. Integration is the usual approach, but of course we would like to avoid integration when possible. One case where integration is avoidable is when we can use some simple manipulations to express the signal directly in a sum-of-complex-exponentials form. For such signals, we can determine the Fourier coefficients *by inspection*, as the following example illustrates.

Example. Find the Fourier coefficients and plot the spectrum of the signal $x(t) = \cos^2(\pi t - \pi/3)$.

First, what is the fundamental period of $x(t)$?
Since $\cos(\pi t - \pi/3) = \cos(2\pi \frac{1}{2} t - \pi/3)$, a period of this signal is $T = 2$. We will start with that for now even though it is not the fundamental period.

The *hard way* to find the Fourier coefficients would be to use the analysis equation:

$$\alpha_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, \mathrm{e}^{-\jmath 2\pi \frac{k}{T_0} t}\, \mathrm{d}t = \frac{1}{2} \int_0^2 \cos^2(\pi t - \pi/3) \mathrm{e}^{-\jmath 2\pi \frac{k}{2} t}\, \mathrm{d}t = \int_0^1 \cos^2(2\pi t - \pi/3) \mathrm{e}^{-\jmath 2\pi k t}\, \mathrm{d}t.$$

This integral can be done by using the inverse Euler identity for $\cos(\cdot)$. You should try doing it as an exercise. Chances are very high that you will do it incorrectly because you will divide by zero at some point.

The easier way is to express the signal directly in a sum-of-complex-exponentials form by using the inverse Euler identity in the first place.

$$x(t) = \cos^2(\pi t - \pi/3) = \frac{1}{2} + \frac{1}{2}\cos(2\pi t - 2\pi/3) = \frac{1}{2} + \frac{1}{4}\mathrm{e}^{-\jmath 2\pi/3}\mathrm{e}^{\jmath 2\pi t} + \frac{1}{4}\mathrm{e}^{\jmath 2\pi/3}\mathrm{e}^{-\jmath 2\pi t}.$$

Now we see that the fundamental period is $T_0 = 1$.

Compare the above expression to the sum-of-complex-exponentials synthesis formula expanded out:

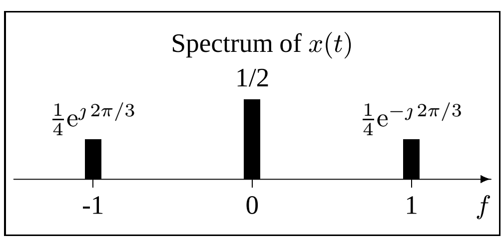
$$x(t) = \cdots + \alpha_{-2}\mathrm{e}^{-\jmath 2\pi \frac{2}{T} t} + \alpha_{-1}\mathrm{e}^{-\jmath 2\pi \frac{1}{T} t} + \alpha_0 + \alpha_1 \mathrm{e}^{\jmath 2\pi \frac{1}{T} t} + \alpha_N \mathrm{e}^{\jmath 2\pi \frac{2}{T} t} + \cdots.$$

By matching the corresponding coefficients we see that for this signal:

$$\alpha_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{4}\mathrm{e}^{-\jmath 2\pi/3}, & k = 2 \\ \frac{1}{4}\mathrm{e}^{\jmath 2\pi/3}, & k = -2 \\ 0, & \text{otherwise.} \end{cases}$$

Notice that there are only a finite number of nonzero α_k's. This is always the case when $x(t)$ is finite sum-of-sinusoids.

The spectrum of this $x(t)$ is simply the following.



In this example we found the spectrum (*i.e.*, the Fourier series coefficients) by inspection, just as we did in the section on finite sums of sinusoids. Since there is a one-to-one relation between Fourier coefficients and periodic signals, the coefficients we obtain by inspection are the Fourier series coefficients.

Example. Show a real-world nearly periodic signal, like a vowel.

Show its spectrum, as computed by a computer.

C.2 Properties of Fourier series

This section lists several useful properties of Fourier series. These properties are important both in terms of understanding the concepts and because often one can use these properties to avoid integration!

P1. Uniqueness
There is a one-to-one relationship between periodic signals with period T and sequences of Fourier coefficients. Specifically for any given signal $x(t)$, the analysis formula gives the unique set of coefficients from which the synthesis formula yields $x(t)$.

This implies that the Fourier coefficients can sometimes by found by means other the analysis formula, *e.g.* by inspection. That is, if by some means you find a collection $\{\alpha_k\}$ such that

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t},$$

then those α_k's are necessarily the Fourier coefficients that would be computed by the analysis formula.

Similarly, for any given set of coefficients $\{\alpha_k\}$, the synthesis formula gives the unique signal $x(t)$ with period T from which the analysis formula yields these α_k's. That is, if by some means you find a signal $x(t)$ such that

$$\alpha_k = \frac{1}{T} \int_0^T x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt, \qquad k \in \mathbb{Z},$$

then $x(t)$ is the one and only signal that has these α_k's as its Fourier coefficients.

Another statement of the one-to-oneness is the following. If $x_1(t)$ and $x_2(t)$ are distinct signals[4], and each is T-periodic, then for at least one k, α_k for $x_1(t)$ does not equal α_k for $x_2(t)$.

P2. Mean value (important)
α_0 is the mean or DC value of $x(t)$

This is because (putting $k = 0$ in the analysis formula):

$$\alpha_0 = \frac{1}{T} \int_0^T x(t)\, e^{-\jmath 2\pi \frac{0}{T} t}\, dt = \frac{1}{T} \int_0^T x(t)\; dt = \mathrm{M}(x)\,.$$

P3. Integration limits
One can compute the Fourier coefficients by integrating over any time interval of length T.

$$\begin{aligned}
\alpha_k &= \frac{1}{T} \int_0^T x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \\
&= \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \qquad \text{for any value of } t_0 \text{ (important)} \\
&= \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \qquad \text{(shorthand for “some interval of length } T\text{”)}
\end{aligned}$$

P4. Conjugate symmetry (important)
If one knows α_k for $k \geq 0$, then one can easily find the remaining α_k's using:

$$\alpha_{-k} = \alpha_k^\star, \qquad \text{for real signals.}$$

(This property does not apply to complex signals.)

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[4]Here, “distinct” means that their difference has nonzero power, *i.e.*, $\mathrm{MSD}(x_1, x_2) \neq 0$.

Derivation:

$$\begin{aligned}
\alpha_k^\star &= \left[\frac{1}{T}\int_{\langle T\rangle} x(t)\,\mathrm{e}^{-\jmath 2\pi \frac{k}{T} t}\,\mathrm{d}t\right]^\star = \frac{1}{T}\int_{\langle T\rangle} x(t)^\star\,\mathrm{e}^{\jmath 2\pi \frac{k}{T} t}\,\mathrm{d}t \\
&= \frac{1}{T}\int_{\langle T\rangle} x(t)\,\mathrm{e}^{\jmath 2\pi \frac{k}{T} t}\,\mathrm{d}t \qquad \text{because } x(t) \text{ is real, so } x(t)^\star = x(t) \\
&= \frac{1}{T}\int_{\langle T\rangle} x(t)\,\mathrm{e}^{-\jmath 2\pi (\frac{-k}{T}) t}\,\mathrm{d}t = \alpha_{-k}.
\end{aligned}$$

The following properties follow.

- $|\alpha_{-k}| = |\alpha_k|$ so the magnitude spectrum is has **even symmetry**.
- $\angle\alpha_{-k} = -\angle\alpha_k|$ so the phase spectrum is has **odd symmetry**.

P5. Sinusoids

Each conjugate pair of coefficients synthesizes a sinusoid (this is also important):

$$\alpha_k \mathrm{e}^{\jmath 2\pi \frac{k}{T} t} + \alpha_{-k}\mathrm{e}^{-\jmath 2\pi \frac{k}{T} t} = 2|\alpha_k| \cos\left(2\pi \frac{k}{T} t + \angle\alpha_k\right).$$

Thus, when looking at a spectrum, one should "see" the sinusoidal terms in the signal, one for every conjugate pair of coefficients.

Derivation:

$$\begin{aligned}
\alpha_k \mathrm{e}^{\jmath 2\pi \frac{k}{T} t} + \alpha_{-k}\mathrm{e}^{-\jmath 2\pi \frac{k}{T} t} &= \alpha_k \mathrm{e}^{\jmath 2\pi \frac{k}{T} t} + \alpha_k^\star \mathrm{e}^{-\jmath 2\pi \frac{k}{T} t} \qquad \text{by the previous property} \\
&= \alpha_k \mathrm{e}^{\jmath 2\pi \frac{k}{T} t} + \left[\alpha_k \mathrm{e}^{\jmath 2\pi \frac{k}{T} t}\right]^\star \\
&= 2\mathrm{Re}\left\{\alpha_k \mathrm{e}^{\jmath 2\pi \frac{k}{T} t}\right\} = 2|\alpha_k| \cos\left(2\pi \frac{k}{T} t + \angle\alpha_k\right).
\end{aligned}$$

In particular, the sinusoidal Fourier series is as follows:

$$x(t) = \alpha_0 + \sum_{k=1}^{\infty} 2|\alpha_k| \cos\left(2\pi \frac{k}{T} t + \angle\alpha_k\right).$$

P6. Linearity

Suppose $x(t)$ and $y(t)$ are periodic with period T and with α_k and β_k as their T-second Fourier coefficients, respectively. Then the T-second Fourier coefficients of $x(t) + y(t)$ are given by $\alpha_k + \beta_k$.

This property is useful for "recycling" previously computed Fourier series.

Similarly, if α_k and β_k are sequences of Fourier coefficients, then the signal whose Fourier coefficients are $\alpha_k + \beta_k$ is the sum of the signals corresponding to α_k and β_k.

P7. Harmonic complex exponentials are uncorrelated

A key step in many derivations involving Fourier series is the following very useful property of the integral of complex exponentials:

$$\frac{1}{T}\int_{\langle T\rangle} \mathrm{e}^{\jmath 2\pi \frac{m}{T} t}\,\mathrm{d}t = \begin{cases} 1, & m = 0 \\ 0, & m = \pm 1, \pm 2, \ldots. \end{cases}$$

Because of this property, different complex exponential signals with harmonically related frequencies are uncorrelated. Define $\psi_k(t) = \mathrm{e}^{\jmath 2\pi \frac{k}{T} t}$ for $k \in \mathbb{Z}$. Then

$$C(\psi_k, \psi_l) = \int_{\langle T\rangle} \psi_k(t)\psi_l^\star(t)\,\mathrm{d}t = \int_{\langle T\rangle} \mathrm{e}^{\jmath 2\pi \frac{k}{T} t}\left[\mathrm{e}^{\jmath 2\pi \frac{l}{T} t}\right]^\star \mathrm{d}t = \int_{\langle T\rangle} \mathrm{e}^{\jmath 2\pi \frac{k}{T} t}\mathrm{e}^{-\jmath 2\pi \frac{l}{T} t}\,\mathrm{d}t = \int_{\langle T\rangle} \mathrm{e}^{\jmath 2\pi \frac{k-l}{T} t}\,\mathrm{d}t = \begin{cases} T, & k = l \\ 0, & k \neq l. \end{cases}$$

This property is useful in proving the next theorem.

P8. Parseval's theorem

The **average power** of a periodic signal can be computed in the time domain *or* in the frequency domain as follows:

$$\boxed{\mathrm{MS}(x) = \frac{1}{T}\int_{\langle T \rangle} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |\alpha_k|^2.}$$

(Recall that for periodic signals, we compute the average power over a period of the signal.)

Derivation:

$$\begin{aligned}
\mathrm{MS}(x) &= \frac{1}{T}\int_{\langle T \rangle} |x(t)|^2 \, dt = \frac{1}{T}\int_{\langle T \rangle} x(t)\, x^*(t) \, dt \\
&= \frac{1}{T}\int_{\langle T \rangle} \left[\sum_{k=-\infty}^{\infty} \alpha_k e^{j 2\pi \frac{k}{T} t}\right] \left[\sum_{l=-\infty}^{\infty} \alpha_l e^{j 2\pi \frac{l}{T} t}\right]^\star dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \alpha_k \alpha_l^\star C(\psi_k, \psi_l) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} \alpha_k \alpha_k^\star T \qquad \text{by preceding property} \\
&= \sum_{k=-\infty}^{\infty} |\alpha_k|^2.
\end{aligned}$$

(You should think about why we used l rather than k for the second summation.)

Interpretation.

The kth frequency component in the spectrum contributes an amount $|\alpha_k|^2$ to the overall average power of the signal. So the magnitude (squared) spectrum directly reveals the relative power in each frequency component. For example, one can see easily which frequency ranges have the greatest fraction of the power.

Though useful, the following properties will be emphasized less in this class. They are studied in more detail in EECS 306.

P9. Choice of period

Suppose $x(t)$ is periodic with period T, and suppose $\{\alpha_k\}$ are the T-second Fourier coefficients of $x(t)$ and suppose $\{\beta_k\}$ are the $2T$-second Fourier coefficients of $x(t)$. Then,

$$\beta_k = \begin{cases} \alpha_{k/2}, & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

This means that the spectrum based on the $2T$-second Fourier series is the same as that based on the T-second Fourier series. That is,

$$\begin{aligned}
&\left\{\ldots, (\alpha_{-2}, \frac{-2}{T}),\ (\alpha_{-1}, \frac{-1}{T}),\ (\alpha_0, 0),\ (\alpha_1, \frac{1}{T}),\ (\alpha_2, \frac{2}{T}), \ldots\right\} \\
= &\left\{\ldots, (\alpha_{-2}, \frac{-4}{2T}),\ (0, \frac{-3}{2T}),\ (\alpha_{-1}, \frac{-2}{2T}),\ (0, \frac{-1}{2T}),\ (\alpha_0, 0),\ (0, \frac{1}{2T}),\ (\alpha_1, \frac{2}{2T}),\ (0, \frac{3}{2T}),\ (\alpha_2, \frac{4}{2T}), \ldots\right\}
\end{aligned}$$

P10. Finite approximation

In most practical situations, if we want to calculate the values of a signal from the Fourier synthesis formula, we must approximate the signal using a finite number of terms. Usually we use the lower frequency terms:

$$x(t) \approx x_K(t) \triangleq \sum_{k=-K}^{K} \alpha_k e^{j 2\pi \frac{k}{T} t}.$$

How good is this approximation? One way to answer this is to look at the **mean-squared difference** between $x(t)$ and the approximation $x_K(t)$, or equivalently to look at the average power of the difference signal

$$e_K(t) \triangleq x(t) - x_K(t)\,.$$

One can show that the average power of this difference signal is

$$\mathrm{MS}(e_K) = \sum_{k=K+1}^{\infty} \left[|\alpha_k|^2 + |\alpha_{-k}|^2 \right].$$

Furthermore, this mean-squared difference goes to zero as K increases.

Indeed, the proof of this fact is essentially the proof of the Fourier Series Theorem. If possible, we choose K large so that the error is small.

P11. Time shift
If $x(t)$ has Fourier coefficients $\{\alpha_k\}$, then $y(t) = x(t - t_0)$ has Fourier coefficients

$$\beta_k = \alpha_k e^{-\jmath 2\pi \frac{k}{T} t_0}.$$

This shows, not surprisingly, that a time shift causes a phase shift of each spectral component, where the phase shift is proportional to the frequency of the component. The derivation is left as an exercise.

P12. Frequency shifting
If $x(t)$ has Fourier coefficients $\{\alpha_k\}$, then $y(t) = x(t)\, e^{\jmath 2\pi \frac{m}{T} t}$ has Fourier coefficients

$$\beta_k = \alpha_{k-m}.$$

This shows that multiplying a signal by a complex exponential has the effect of shifting the spectrum of the signal. The derivation is left as an exercise.

P13. Time scaling
Let $a > 0$. If $x(t)$ is T-periodic with T-second Fourier coefficients $\{\alpha_k\}$, then $y(t) = x(at)$ is T/a-periodic and has T/a-second Fourier coefficients given by

$$\beta_k = \alpha_k.$$

This shows that the Fourier coefficients are not affected by time scaling. However, time scaling does affect the spectrum. Specifically, the Fourier coefficients of $x(t)$ are spaced at intervals of $1/T$ Hz, whereas the Fourier coefficients of $y(t)$ are spaced at intervals of a/T Hz. For example, if $a > 1$, then the Fourier coefficients are more widely spaced, and consequently, the spectrum of $y(t)$ is expanded towards higher frequencies. This is consistent with the fact that using $a > 1$ means that $y(t)$ fluctuates more rapidly than $x(t)$. The derivation is left as an exercise.

P14. Technicalities
(mostly a warning that there are such)

For the integral in the analysis formula to be well defined and for the synthesis formula to hold, one needs to assume

$$\int_{\langle T \rangle} |x(t)|\, dt < \infty \quad \text{and/or} \quad \int_{\langle T \rangle} |x(t)|^2\, dt < \infty.$$

These are very mild conditions from a practical perspective; any signal of practical interest will satisfy both of these conditions.

When mathematicians prove that

$$x(t) = \hat{x}(t) \triangleq \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t},$$

what they really show is that the average power of the difference signal is zero, *i.e.*,

$$0 = \mathrm{MS}(x(t) - \hat{x}(t)),$$

assuming that $\int_{\langle T \rangle} |x(t)|^2\, dt < \infty$. So $x(t)$ and $\hat{x}(t)$ can differ at *isolated points*. Such differences have no practical engineering importance.

Moreover, assuming $\int_{\langle T \rangle} |x(t)|\, dt < \infty$ and the so-called **Dirichlet conditions**[5], the only points at which they can differ are points of discontinuity in $x(t)$. Specifically,

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t} = \begin{cases} x(t), & \text{if } x(t) \text{ is } \textbf{continuous} \text{ at } t \\ \frac{1}{2}\left[x(t^+) + x(t^-)\right], & \text{if } x(t) \text{ is } \textbf{discontinuous} \text{ at } t. \end{cases}$$

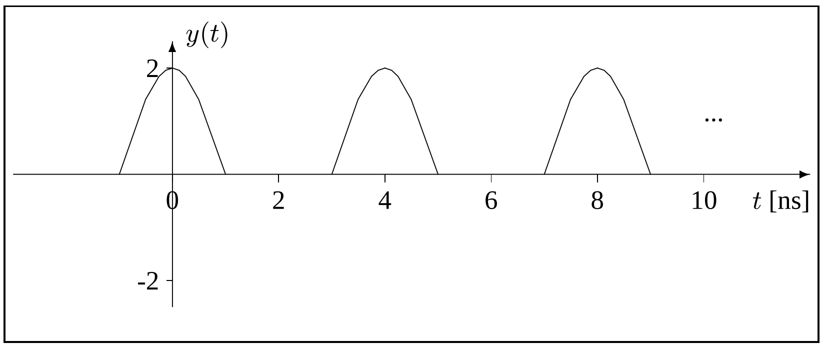
[5] Dirichlet conditions. In addition to $\int_{\langle T \rangle} |x(t)|\, dt < \infty$, in any one period $x(t)$ has only a finite number of maxima and minimum and only a finite number of discontinuities.

There's more discussion of "technicalities" in EECS 306.

P15. Self consistency
The following argument shows that the Fourier series analysis and synthesis formulae are self consistent, but this is *not* by itself a rigorous proof of correctness:

$$\begin{aligned}\alpha_l &= \frac{1}{T}\int_{\langle T\rangle} x(t)\, e^{-\jmath 2\pi \frac{l}{T} t}\, dt = \frac{1}{T}\int_{\langle T\rangle} \left[\sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t}\right] e^{-\jmath 2\pi \frac{l}{T} t}\, dt \\ &= \sum_{k=-\infty}^{\infty} \alpha_k \left[\frac{1}{T}\int_{\langle T\rangle} e^{\jmath 2\pi \frac{k}{T} t} e^{-\jmath 2\pi \frac{l}{T} t}\, dt\right] = \sum_{k=-\infty}^{\infty} \alpha_k \underbrace{\left[\frac{1}{T}\int_{\langle T\rangle} e^{\jmath 2\pi \frac{k-l}{T} t}\, dt\right]}_{1 \text{ if } k=l, \text{ else } 0} = \alpha_l.\end{aligned}$$

Example. Find the Fourier series of the following signal.



One way to solve this problem would be to use the analysis formula. The integral is a bit messy. An alternative approach is to use properties of the Fourier series since we can recognize that the above signal is related to our earlier $x_1(t)$ as follows:

$$y(t) = x_1(t-3)\cos\left(2\pi \frac{1}{4} t\right) = x_1(t-3)\,\frac{1}{2}\left[e^{\jmath 2\pi \frac{1}{4} t} + e^{-\jmath 2\pi \frac{1}{4} t}\right] = \frac{1}{2} x_1(t-3)\, e^{\jmath 2\pi \frac{1}{4} t} + \frac{1}{2} x_1(t-3)\, e^{-\jmath 2\pi \frac{1}{4} t}.$$

To express the Fourier series coefficients $\{\beta_k\}$ of $y(t)$ in terms of the coefficients of $x_1(t)$, denoted $\{\alpha_k\}$, we apply the linearity property, the time-shift property and the frequency-shift property.
First, by the time-shift property, the FS of $x_3(t) = x_1(t-3)$ has coefficients

$$\gamma_k = \alpha_k e^{-\jmath 2\pi \frac{k}{4} 3}.$$

By the frequency-shift property, the FS of $x_3(t)\, e^{\jmath 2\pi \frac{1}{4} t}$ is γ_{k-1}. Combining using linearity, the FS of $y(t)$ is

$$\beta_k = \frac{1}{2}\gamma_{k-1} + \frac{1}{2}\gamma_{k+1} = \frac{1}{2}\alpha_{k-1} e^{\jmath 2\pi \frac{k-1}{4} 3} + \frac{1}{2}\alpha_{k+1} e^{\jmath 2\pi \frac{k+1}{4} 3} = \frac{1}{2} e^{\jmath 2\pi \frac{k}{4} 3}\left[\alpha_{k-1} e^{-\jmath 3\pi/2} + \alpha_{k+1} e^{\jmath 3\pi/2}\right].$$

Now we "just" substitute in our the values for α_k computed earlier in (3a-1).

Example. The signal $x(t)$ has the following spectrum. ***(Picture)*** .

Determine the average power of $x(t)$.

From the spectrum we see that the FS coefficients are $\alpha_0 = 3$, $\alpha_k = 3/2^{|k|}\exp(\pi/k)$, $k \neq 0$.

$$\begin{aligned}\mathrm{MS}(x) &= \sum_{k=-\infty}^{\infty} |\alpha_k|^2 = \alpha_0^2 + 2\sum k = 1^{\infty} |\alpha_k|^2 = 3^2 + 2\sum k = 1^{\infty} |3/2^{|k|}\exp(\pi/k)\,|^2 = 3^2 + 2\sum k = 1^{\infty} |3/2^k|^2 \\ &= = 9 + 18\sum k = 1^{\infty} \left(\frac{1}{4}\right)^k = 9 + 18\left[\sum k = 0^{\infty}\left(\frac{1}{4}\right)^k - 1\right] = 9 + 18\left[\frac{1}{1-1/4} - 1\right] = 15\end{aligned}$$

Average Power of a Sum-of-Sinusoids

(Prelude to **parseval's theorem**.)

Fact. For a sum-of-complex-exponentials signals with different frequencies:

$$x(t) = \sum_k \alpha_k e^{\jmath 2\pi f_k t}, \qquad f_k \neq f_l, \; k \neq l,$$

the **average power** can be computed in the time domain *or* in the frequency domain:

$$\boxed{\mathrm{MS}(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \,\mathrm{d}t = \sum_k |\alpha_k|^2.}$$

The spectrum of a signal also characterizes its average power!

To prove this property, we first show the following useful limit:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{\jmath 2\pi (f_k - f_l) t} \,\mathrm{d}t = \begin{cases} 1, & f_k = f_l \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(3a-2)}$$

The case $f_k = f_l$ is obvious, so consider the case where $f_k \neq f_l$ and define $\omega = 2\pi(f_k - f_l)$:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{\jmath 2\pi (f_k - f_l) t} \,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{2T} \frac{1}{\jmath\omega} e^{\jmath \omega t} \Big|_{-T}^{T}$$

$$= \lim_{T\to\infty} \frac{e^{\jmath \omega T} - e^{-\jmath \omega T}}{\jmath 2T} = \lim_{T\to\infty} \frac{\sin(\omega T)}{T} = 0,$$

since $\sin(\cdot)$ is bounded by unity. So (3a-2) is shown.

Now we can proceed to use (3a-2) to derive the above power equation:

$$\begin{aligned}
\mathrm{MS}(x) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t)^\star \,\mathrm{d}t \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left[\sum_k \alpha_k e^{\jmath 2\pi f_k t}\right] \left[\sum_l \alpha_l e^{\jmath 2\pi f_l t}\right]^\star \mathrm{d}t \qquad \text{(why ``}l\text{'' ?)} \\
&= \sum_k \sum_l \alpha_k \alpha_l^\star \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{\jmath 2\pi (f_k - f_l) t} \,\mathrm{d}t\right] \qquad [\,]\text{ is 1 if } k = l \text{ else 0} \\
&= \sum_k \alpha_k \alpha_k^\star = \sum_k |\alpha_k|^2.
\end{aligned}$$

Interpretation. The kth frequency component in the spectrum contributes $|\alpha_k|^2$ to the overall average power of the signal. One can "see where the power is" in the spectrum.

Finite Fourier Series Approximation

Any T-periodic signal can be expressed as a (usually infinite) sum-of-complex-exponentials. Infinite sums are fine for analysis, but for practical implementation a finite sum approximation is necessary:

$$x(t) \approx \hat{x}_K(t) \triangleq \sum_{k=-K}^{K} \beta_k e^{\jmath 2\pi \frac{k}{T} t}.$$

When making such a finite-series approximation, it is natural to try to choose the coefficients $\{\beta_k\}$ to make this approximation "as good as possible."
How should we measure the "goodness of fit" ?

Choose β_k's that minimize the **mean-squared difference** between $x(t)$ and the approximation $\hat{x}_K(t)$:

$$\mathrm{MSD}(x, \hat{x}_K) = \mathrm{MS}(x - \hat{x}_K).$$

We show here that the best β_k's are those given by the analysis equation:

$$\beta_k = \frac{1}{T} \int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt. \tag{3a-3}$$

In other words, the same FS coefficients that work "perfectly" if we use *all* of them also are optimal in the MS sense for *any* finite series approximation!

Useful facts for the derivation.

- For any T-periodic signal $z(t)$, $\mathrm{MS}(z) = \frac{1}{T}\int_{\langle T \rangle} |z(t)|^2 \, dt = \frac{1}{T}\int_{\langle T \rangle} z(t)\, z^*(t)\, dt = C_T(z, z)$, where $C_T(x, y)$ denotes a time-normalized correlation.
- Bilinearity: $C(\sum_k x_k, \sum_l y_l) = \sum_k \sum_l C(x_k, y_l)$, likewise for C_T
- Harmonic complex exponentials are uncorrelated: $C_T(e^{\jmath 2\pi \frac{k}{T} t}, e^{\jmath 2\pi \frac{l}{T} t}) = \begin{cases} 1, & k = l \\ 0, & \text{otherwise} \end{cases}$
- Average power of finite series: $\mathrm{MS}(\hat{x}_K(t)) = \sum_{k=-K}^{K} |\beta_k|^2$
- Completing the square: $|\beta|^2 - 2\mathrm{Re}\{\beta^\star \gamma\} = |\beta|^2 - \beta^\star\gamma - \beta\gamma^\star + |\gamma|^2 - |\gamma|^2 = |\beta - \gamma|^2 - |\gamma|^2$

Derivation

$$\begin{aligned}
\mathrm{MS}(x - \hat{x}_K) &= C_T(x - \hat{x}_K, x - \hat{x}_K) = C_T(x, x) - C_T(x, \hat{x}_K) - C_T(\hat{x}_K, x) + C_T(\hat{x}_K, \hat{x}_K) \\
&= \mathrm{MS}(x) - [C_T(x, \hat{x}_K) + C_T^\star(x, \hat{x}_K)] + \mathrm{MS}(\hat{x}_K) \\
&= \mathrm{MS}(x) - 2\mathrm{Re}\{C_T(x, \hat{x}_K)\} + \sum_{k=-K}^{K} |\beta_k|^2 \\
&= \mathrm{MS}(x) - 2\mathrm{Re}\left\{ \frac{1}{T}\int_{\langle T \rangle} x(t) \left[\sum_{k=-K}^{K} \beta_k^\star \left(e^{\jmath 2\pi \frac{k}{T} t} \right)^\star \right] dt \right\} + \sum_{k=-K}^{K} |\beta_k|^2 \\
&= \mathrm{MS}(x) + \sum_{k=-K}^{K} \left[|\beta_k|^2 - 2\mathrm{Re}\left\{ \beta_k^\star \frac{1}{T}\int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \right\} \right] \\
&= \mathrm{MS}(x) + \sum_{k=-K}^{K} \left[\left| \beta_k - \frac{1}{T}\int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \right|^2 - \left| \frac{1}{T}\int_{\langle T \rangle} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \right|^2 \right].
\end{aligned}$$

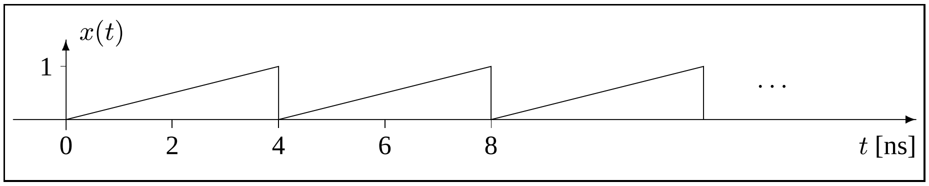
From this final expression, we see that the MS difference is minimized when the middle term (the only term that depends on β_k) is zero, *i.e.*, when when we use (3a-3).

When we use this optimal choice for β_k, the MSD simplifies as follows:

$$\begin{aligned}\mathrm{MS}(x - \hat{x}_K) &= \mathrm{MS}(x) - \sum_{k=-K}^{K} \left| \frac{1}{T} \int_{\langle T \rangle} x(t)\, \mathrm{e}^{-\jmath 2\pi \frac{k}{T} t} \, \mathrm{d}t \right|^2 \\ &= \sum_{k=-\infty}^{\infty} |\alpha_k|^2 - \sum_{k=-K}^{K} |\alpha_k|^2 = \sum_{|k|>K} |\alpha_k|^2\end{aligned}$$

Signal synthesis

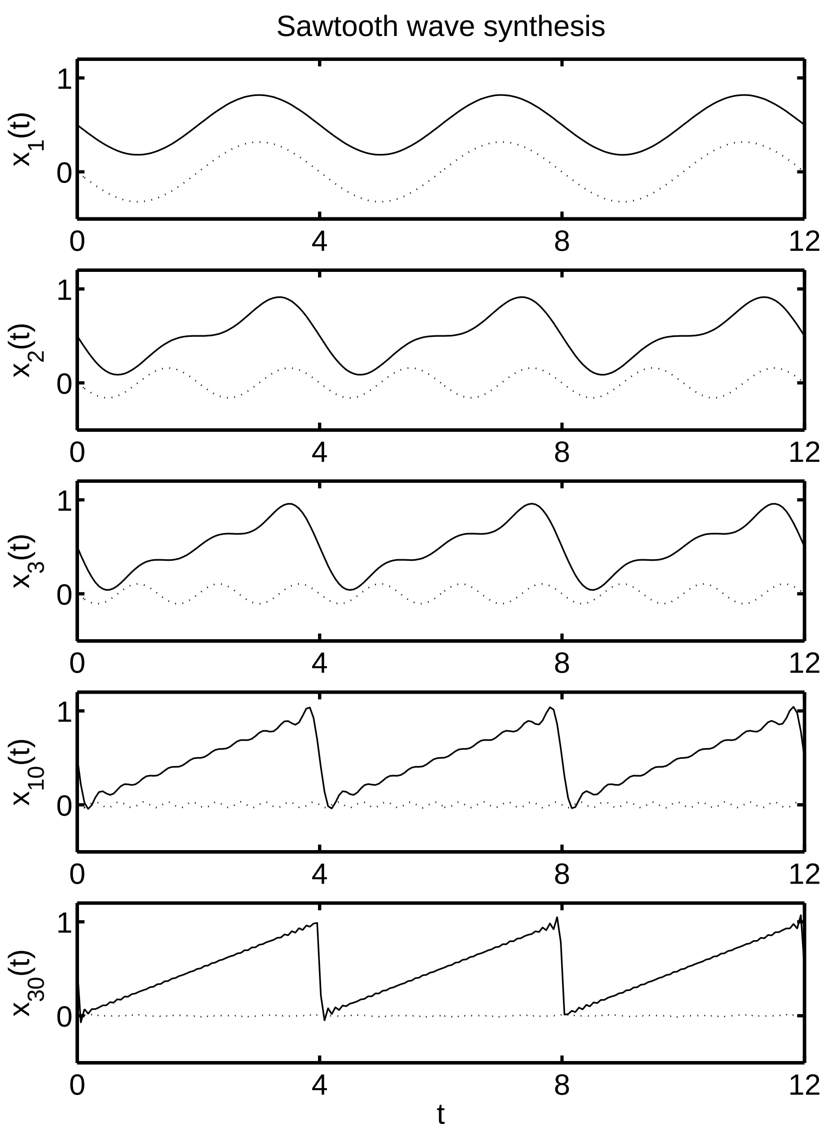
Example.



The FS coefficients of the above sawtooth signal are given by (exercise):

$$\alpha_k = \begin{cases} 1/2, & k = 0 \\ \frac{1}{-\jmath 2\pi k}, & k \neq 0. \end{cases}$$

The following figure shows the finite-series approximations $x_K(t)$ to $x(t)$ for various number of terms K.



The clock signal design problem revisited

We now return to the problem of choosing between the square wave and the pulse train for a high-speed clock signal in the presence of imperfect interconnects.

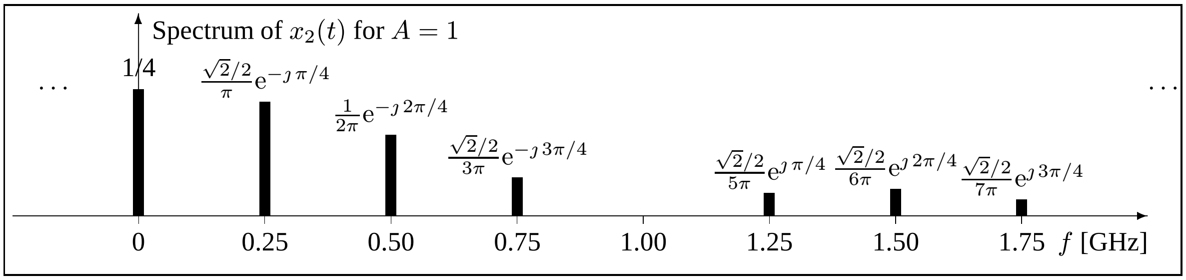
We have already computed the spectrum of $x_1(t)$, the square wave.

By similar manipulations, the spectrum of $x_2(t)$, the pulse train, has Fourier coefficients given by

$$\beta_k = \frac{A}{4}\int_0^1 1 \cdot e^{-\jmath 2\pi \frac{k}{T} t}\, dt = \cdots = \begin{cases} \frac{A}{4}, & k = 0 \\ \frac{A}{\jmath 2\pi k}\left[1 - e^{-\jmath \pi k/2}\right], & k \neq 0, \end{cases}$$

where

$$\frac{1}{\jmath 2\pi k}\left[1 - e^{-\jmath \pi k/2}\right] = \frac{1}{\jmath 2\pi k}\left[e^{\jmath \pi k/4} - e^{-\jmath \pi k/4}\right] e^{-\jmath \pi k/4} = \frac{\sin(\pi k/4)}{\pi k} e^{-\jmath \pi k/4}.$$



Now we examine the effects of the **filtering** caused by the imperfect interconnects. We can model these effects using the following block diagram

[clock signal] → [interconnects] → [other components],

$x(t) \to$ [filter] $\to y(t)$,

where the output signal $y(t)$ consists of all frequency components up to 5GHz, *i.e.*,

$$y(t) = \sum_{k=-K}^{K} x(t)\, e^{\jmath 2\pi \frac{k}{T_0} t}$$

where $f_0 = 1/T_0 = 0.25$ GHz and where $K = 5\text{GHz}/0.25\text{GHz} = 20$. So the DC term and the first 20 harmonics are passed by the interconnects, whereas the higher frequency components are removed.

A natural measure of the signal distortion introduced by the imperfect interconnects is the normalized RMS difference: $\text{NRMS} = \frac{\text{RMS}(x-y)}{\text{RMS}(x)}$. By Parseval's theorem,

$$\text{RMS}(x) = \sqrt{\text{MS}(x)} = \sqrt{\sum_{k=-\infty}^{\infty} |\alpha_k|^2} = \sqrt{\alpha_0^2 + 2\sum_{k=1}^{\infty} |\alpha_k|^2},$$

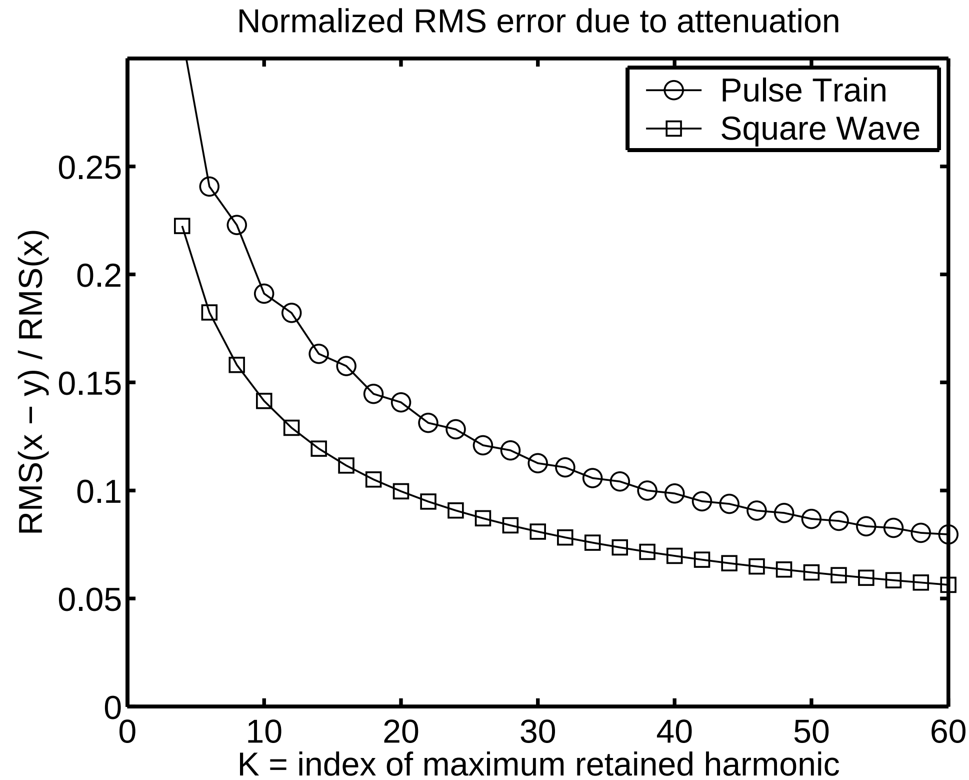
where the second expression is due to conjugate symmetry of the α_k's. Similarly, since $y(t)$ and $x(t)$ have the same low-frequency components, the spectrum of the error signal $x(t) - y(t)$ only consists of the attenuation high frequency components, so

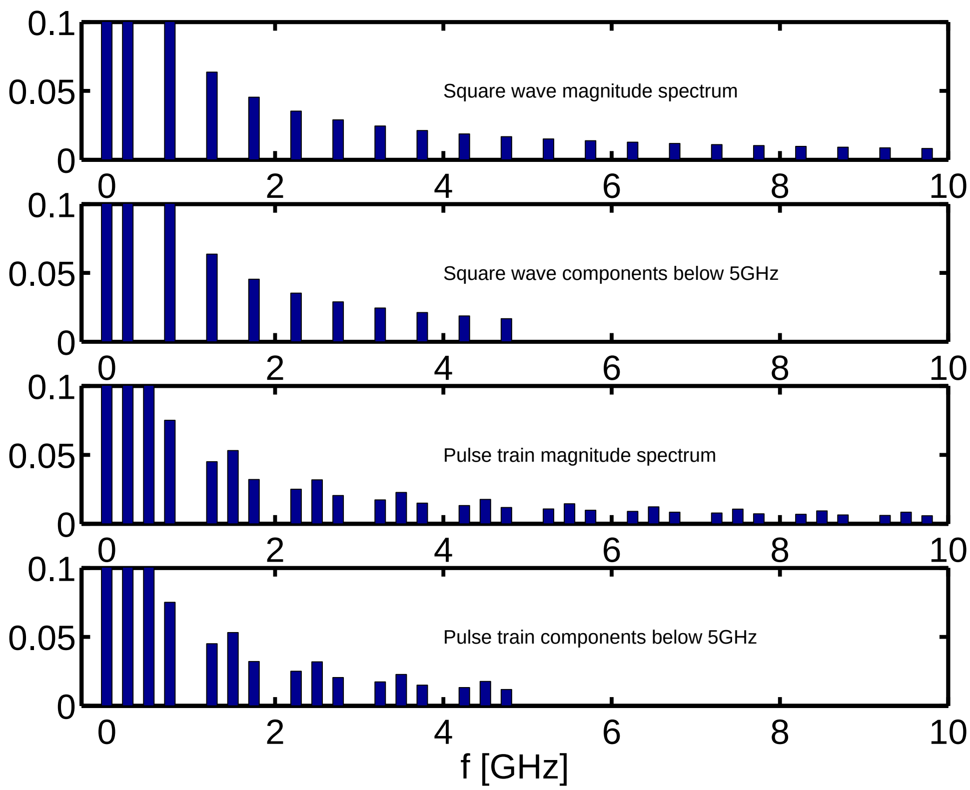
$$\text{RMS}(x-y) = \sqrt{\sum_{|k|>K} |\alpha_k|^2} = \sqrt{2\sum_{k=K+1}^{\infty} |\alpha_k|^2}.$$

Like most interesting practical problems, there is no analytical expression for this summation, so we compute it numerically using MATLAB. (We take a very large number of terms in the sum and use $A = 1$ without loss of generality.)

The numerical results are $\frac{\text{RMS}(x_1 - y_1)}{\text{RMS}(x_1)} = \frac{0.0704}{0.707} = 0.0996$ and $\frac{\text{RMS}(x_2 - y_2)}{\text{RMS}(x_2)} = \frac{0.0704}{0.5} = 0.1408$.
The NRMS error is higher for the pulse train, because a larger fraction of its power is above 5GHz. So the square wave is

preferable by this criterion. There are other criteria that must also be considered in practice. This has been a frequency-domain analysis; time-domain perspectives are also important for such problems.





D. The spectra of segments of a signal

Question: How can we assess the spectrum of a signal that is not periodic?
Example.

- What if the signal has finite support?
- What if the signal has infinite support, but is not periodic?

Observation: The Fourier series analysis formula works with a finite segment of a signal. It is just a question of how we interpret the synthesis formula!

Signals with finite support

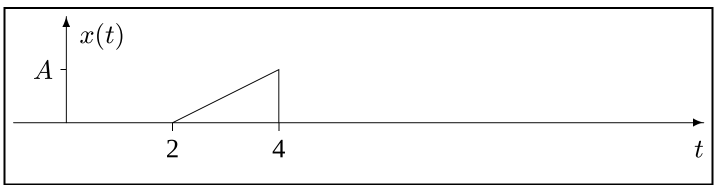
We will see that to assess the spectrum of a signal $x(t)$ with finite support $[t_1, t_2]$ we can apply the Fourier series analysis formula directly to the signal over its support interval. Let us begin by defining $\tilde{x}(t)$ to be a periodic signal that equals $x(t)$ on the interval $[t_1, t_2]$ and simply repeats this behavior on other intervals of the same length. That is, let $T = t_2 - t_1$, and let

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t - mT)\,.$$

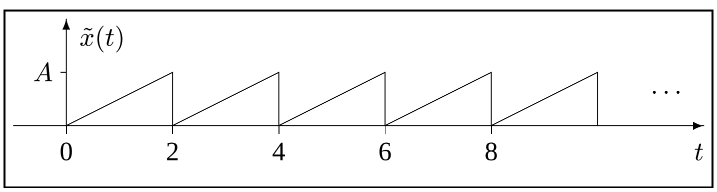
This is called the **periodic extension** of $x(t)$. Its period T is the support length of $x(t)$.

Example.

Here is a finite-support signal.



Here is its periodic extension.



Since $\tilde{x}(t)$ is T-periodic, we can represent it by a sum-of-complex-exponentials:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t} \quad \text{(synthesis formula)}$$

with FS coefficients

$$\alpha_k = \frac{1}{T} \int_{\langle T \rangle} \tilde{x}(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt = \frac{1}{T} \int_{t_1}^{t_2} \tilde{x}(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \quad \text{(analysis formula)},$$

where we have used the fact that the limits in the analysis formula integral can by any interval of length T.

Now we note that since

$$x(t) = \begin{cases} \tilde{x}(t)\,, & t_1 \le t \le t_2 \\ 0, & \text{otherwise,} \end{cases}$$

we also have the following expression for the FS coefficients:

$$\alpha_k = \frac{1}{T} \int_{t_1}^{t_2} x(t)\, e^{-\jmath 2\pi \frac{k}{T} t}\, dt \quad \text{(analysis formula)},$$

and more importantly,

$$x(t) = \begin{cases} \tilde{x}(t)\,, & t_1 \le t \le t_2 \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} \sum_{k=-\infty}^{\infty} \alpha_k e^{\jmath 2\pi \frac{k}{T} t}, & t_1 \le t \le t_2 \\ 0, & \text{otherwise,} \end{cases} \quad \text{(synthesis formula).}$$

Thus we may view the two formulas above as synthesis and analysis formulas for a spectral representation of the finite-duration signal $x(t)$. The synthesis formula shows that on its support interval, $x(t)$ can be viewed as a sum-of-complex-exponentials having frequencies that are multiples of $1/T$. The analysis formula shows how to find the spectral components (the α_k's). It is important to note that the synthesis formula yields $x(t)$ only in the support interval. Outside the support interval it yields $\tilde{x}(t)$, rather than $x(t) = 0$. So we can use the synthesis formula as long as we remember that the values of $x(t)$ are zero outside of the support interval.

In summary, just as we did for periodic signals, for a signal with finite support we can take the spectrum to be:

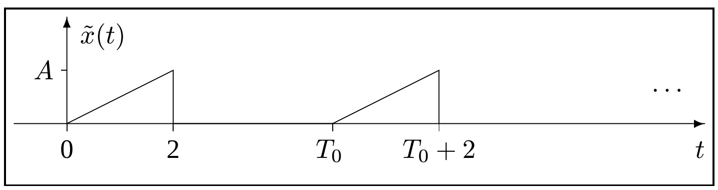
$$\left\{\ldots, (\alpha_{-2}, \frac{-2}{T}), (\alpha_{-1}, \frac{-1}{T}), (\alpha_0, 0), (\alpha_1, \frac{1}{T}), (\alpha_2, \frac{2}{T}), \ldots\right\}.$$

Note: Though we have introduced the Fourier series as fundamentally applying to periodic signals and secondarily applying to signals with finite support, some presentations take the opposite point of view, which is equally valid.

The Fourier Transform

The preceding discussion for finite-support signals is by no means the whole story. Instead of forming the periodic extension using the duration T as the period, we could instead using a larger period.

Example.



For any such $T_0 > T$, we have a periodic signal $\tilde{x}(t)$ for which we can determine its FS coefficients:

$$\alpha_k = \frac{1}{T_0} \int_{t_1}^{t_2} x(t)\, \mathrm{e}^{-\jmath 2\pi \frac{k}{T_0} t}\, \mathrm{d}t.$$

The spectra corresponding to each choice for T_0 will be different, yet somehow related. Which one to choose? The usual answer is to examine the limit as $T_0 \to \infty$, in which case the spectral line get closer and closer and, in the limit, can be though to meld into a continuous curve. The formula for that curve is

$$X(f) \triangleq \int x(t)\, \mathrm{e}^{-\jmath 2\pi f t}\, \mathrm{d}t,$$

where, for any given finite T_0,

$$\alpha_k = \frac{1}{T_0} X(f) \Big|_{f=k/T_0}.$$

The function $X(f)$ is called the **Fourier transform** of $x(t)$, and is the usual method for assessing the spectra of aperiodic signals. The Fourier transform is a primary topic in EECS 306.

Aperiodic signals with infinite support

A common approach to assessing the spectrum of an aperiodic signal with infinite support is to choose a time interval length T, divide the support interval into segments of length T, as in $[0, T]$, $[T, 2T]$, $[2T, 3T]$, $\ldots$, and apply the previous approach to each segment.

With this approach, we obtain a sequence of spectra, one for each segment. Notice that with this approach the spectra "varies with time." Indeed, there are aperiodic signals for which it makes very good sense that the spectrum should differ from segment to segment. For example, the signal produced by a musical instrument can be viewed as having a spectrum that changes with each note. This and other examples can be found in Section 3.5 of the text and in the Demos on the CD-ROM relating to Chapter 3.

There are lots of issues here, for example, what choice of T? But we will leave this discussion to future courses.

This same approach is also useful for "long" aperiodic signals with **finite support**. A grayscale picture of such spectra is called a **spectrogram**.

E. Bandwidth

One of the primary motivations for assessing the spectrum of signal is to find the range of frequencies occupied by it. This range is often called the signal's "band of spectral occupancy" or, more simply, its **frequency band**. The width of the frequency band is called the **bandwidth** of the signal. As one example of the usefulness of the concept of frequency band, signals with non-overlapping spectra do not interfere with each other. So if we know the frequency band occupied by each of a set of signals, we can determine if they interfere. As another example, certain communication media, *e.g.*, a conductor on a printed circuit board, limit propagation to signals with spectral components in a certain range. If we know the frequency band occupied by a signal, we can determine if it will propagate.

Most signals of practical interest, such as that shown in the previous section, have spectral components that extend over a broad range of frequencies. We are rarely interested in the *entire* range of frequencies over which the spectrum is not zero. Usually we are more interested in the range of frequencies over which the spectrum is "significantly large". As such, we need a definition of "significantly large" to define the concepts of "frequency band" and "bandwidth". There are several such definitions in use. The definition given below is based on one such definition.

Definition:

The "band of spectral occupancy" or **frequency band** of a signal $s(t)$ is the smallest interval of frequencies that includes all frequencies at which the magnitude spectrum is at least one half as large as the maximum value of the magnitude spectrum.

Example. For the magnitude spectrum shown below, where the horizontal axis is the frequency in Hz, the frequency band is, approximately, [800, 2100] Hz, and the bandwidth is, approximately, 1300 Hz.