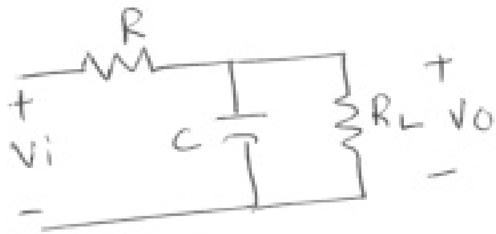
HOMEWORK #11 SOLUTIONS

[2] The lowpass series RC filter with a load resistor R_L can be represented as shown in the following diagram.

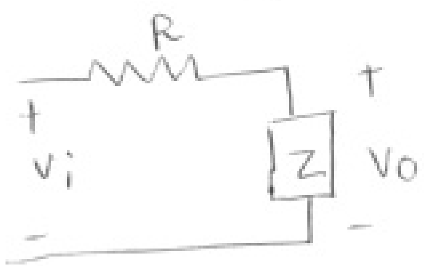


(a) Transfer function of the filter

The parallel combination of resistor R_L and capacitor can be represented by an impedance Z, where

$$Z = \frac{\frac{1}{sC} \cdot R_L}{\frac{1}{sC} + R_L} \quad \text{, where } s = j\omega$$

The equivalent ckt would be



and $H(s) = \dfrac{Z}{Z+R} = \dfrac{V_o}{V_i}$

$$H(s) = \frac{\dfrac{\frac{1}{sC} \cdot R_L}{\frac{1}{sC} + R_L}}{R + \dfrac{\frac{1}{sC} \cdot R_L}{\frac{1}{sC} + R_L}}$$

$$= \frac{R_L}{R + sRR_LC + R_L}$$

$$= \frac{R_L / RR_LC}{s + \frac{R+R_L}{RR_LC}}$$

$$= \frac{1/RC}{s + \frac{1}{KRC}} \quad , \text{where } K = \frac{R_L}{R+R_L}$$

Thus $H(s) = \dfrac{1/RC}{s + \frac{1}{KRC}}$, where $K = \dfrac{R}{R+R_L}$

Replacing s by $j\omega$ (since $s = j\omega$),

$$\boxed{H(j\omega) = \frac{1/RC}{j\omega + \frac{1}{KRC}} \quad , K = \frac{R}{R+R_L}}$$

(b) max $|H(j\omega)|$

$$|H(j\omega)| = \frac{1/RC}{\sqrt{\omega^2 + \left(\frac{1}{KRC}\right)^2}}$$

$|H(j\omega)|$ will be maximum for $\omega = 0$

$$\therefore |H(j\omega)|_{max} = \frac{1/RC}{\sqrt{0 + \left(\frac{1}{KRC}\right)^2}}$$

$$\therefore \boxed{|H(j\omega)|_{max} = K = \frac{R_L}{R + R_L}}$$

(c) cutoff frequency ω_c

at the cutoff frequency ω_c,

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}\left(|H(j\omega)|_{max}\right)$$

$$\therefore |H(j\omega_c)| = \frac{1}{\sqrt{2}}(K) = \frac{1/RC}{\sqrt{(\omega_c)^2 + \left(\frac{1}{KRC}\right)^2}}$$

squaring both sides,

$$\frac{K^2}{2} = \frac{(1/RC)^2}{\omega_c^2 + \left(\frac{1}{KRC}\right)^2}$$

$$\therefore K^2\omega_c^2 + K^2\left(\frac{1}{KRC}\right)^2 = 2\left(\frac{1}{RC}\right)^2$$

$$K^2\omega_c^2 = \left(\frac{1}{RC}\right)^2$$

$\therefore$ $\boxed{\omega_c = \frac{1}{K}\left(\frac{1}{RC}\right)}$ is the cutoff frequency

(d)

As we see in parts (a), (b) & (c), loading does affect the low pass filter. It changes the transfer function of the filter, which in turn leads to:

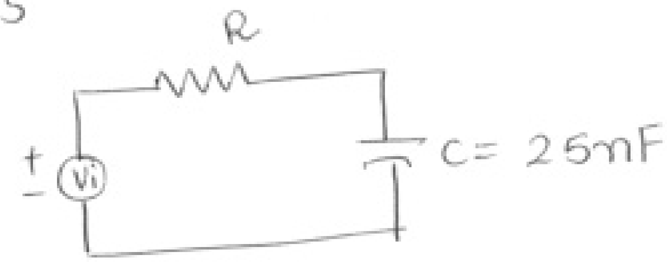
(i) Passband magnitude of the transfer function, which is also its maximum magnitude, is $|H(j\omega)|_{max} = K = \frac{R_L}{R+R_L}$, for the filter with loading. Without R_L, the value of $|H(j\omega)|_{max}$ would have been 1.
Thus loading decreases the pass band magnitude

(ii) As seen in part (c), loading increases the cutoff frequency since $\omega_c = \frac{1}{K}\left(\frac{1}{RC}\right)$.

$K = \frac{R_L}{R+R_L}$ $\therefore K < 1$

$\therefore \omega_c > \frac{1}{RC}$. The cutoff freq with no load, is $\frac{1}{RC}$.

[3] 14.5



Desired $\omega_c = 160K$ rad/s

(a) $f_c = \frac{\omega_c}{2\pi} = \frac{160K}{2\pi} = 25.46 KHz$

$$f_c = 25.46 KHz$$

(b) Now $f_c = \frac{1}{2\pi RC}$

$\therefore R = \frac{1}{2\pi f_c C}$

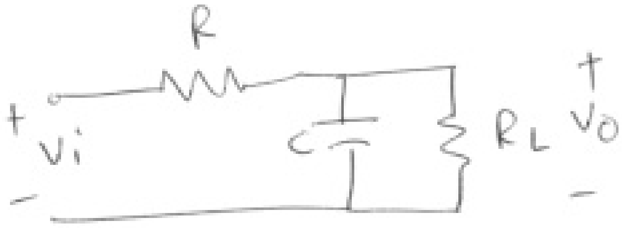
$= \frac{1}{2\pi\left(\frac{160K}{2\pi}\right) \times 25 \times 10^{-9}}$

$$\therefore R = 250 \Omega$$

(c) Now from Prob 2 - part (c), for a series RC low pass filter with load resistor R_L, as shown in ckt diagram given below, the cutoff freq is given by,

$$\omega_c = \frac{1}{K}\left(\frac{1}{RC}\right)$$

where $K = \frac{R_L}{R+R_L}$



Now given $\omega_c = 160$ krad/s

Let $\omega_{c\text{-new}}$ = 8% more than ω_c

$= 1.08 \times 160$

$= 172.8$ krad/s

$\therefore 172.8 \text{ krad/s} = \frac{1}{K}\left(\frac{1}{RC}\right)$

$= \frac{1}{K}\left(\frac{1}{250 \times 25 \times 10^{-9}}\right)$

$\therefore K = 0.9259$

[OR $\omega_{c\text{-new}} = \frac{1}{K}\omega_c$

$\therefore K = \frac{\omega_c}{\omega_{c\text{-new}}} = \frac{1}{1.08} = 0.9259$]

Since $K = \dfrac{R_L}{R + R_L}$

$$\therefore \quad 0.9259 = \frac{R_L}{250 + R_L}$$

$$\therefore \quad (0.9259)(250) = R_L(1 - 0.9259)$$

$$\boxed{\therefore \quad R_L = 3125\,\Omega}$$

is the smallest load resistor that could be connected across the o/p terminals of the filter

(d) Using expression derived in part (b) of problem 2,

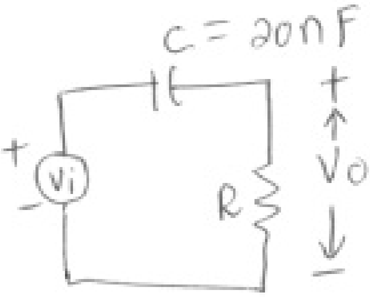
$$|H(\omega)|_{\omega=0} = |H(\omega)|_{max}$$

$$= \frac{R_L}{R + R_L}$$

$$= K$$

$$\therefore \quad \boxed{|H(\omega)|_{\omega=0} = 0.9259}$$

4   14.10



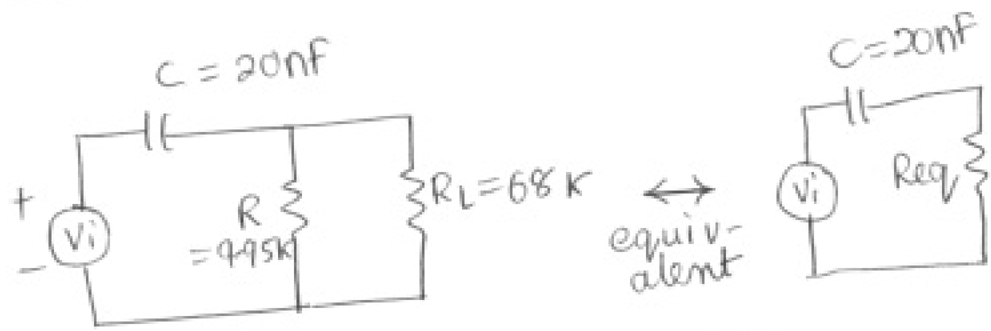
Desired $f_c = 800\,Hz$

(a) $f_c = \dfrac{1}{2\pi RC}$

$\therefore 800 = \dfrac{1}{2\pi R(20\times10^{-9})}$

$\therefore R = 9.95K$

(b)



Now $R_{eq} = R \parallel R_L = \dfrac{R \cdot R_L}{R + R_L}$

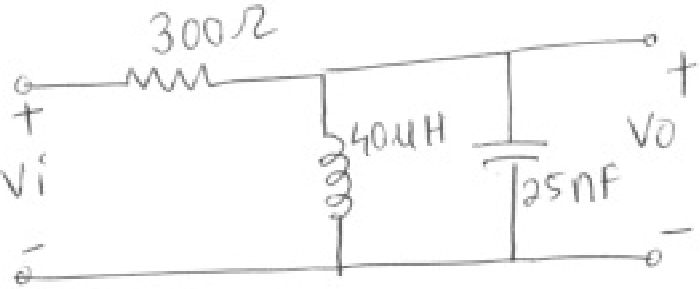
$= \dfrac{(9.95K)(68K)}{9.95K + 68K}$

$\therefore R_{eq} = 8.68K$

$f_c = \dfrac{1}{2\pi R_{eq} C} = \dfrac{1}{2\pi(8.68K)(20\times10^{-9})}$

$\therefore f_c = 917.26\,Hz$

5 14.12



Note:- For derivations of formulae used here, refer to Example 14.6, of the Text Book.

(a) $\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{40\times10^{-6}\times25\times10^{-9}}}$

$$\omega_0 = 10^6 \text{ rad/s} = 1 \text{ Mrad/s}$$

(b) $f_0 = \frac{\omega_0}{2\pi} = \frac{10^6}{2\pi} = 159.15 \text{ KHz}$

$$f_0 = 159.15 \text{ KHz}$$

(c) $\beta = \frac{1}{RC} = \frac{1}{(300)(25\times10^{-9})} = 133.33 \text{ krad/s}$

and $Q = \omega_0/\beta$

$\therefore Q = \frac{10^6}{133.33\text{K}}$

$$\therefore Q = 7.5$$

(d) $\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right) + \omega_0^2}$

substituting, $\beta = 133.33$ krad/s
and $\omega_0 = 1$ Mrad/s, we get

$$\boxed{\omega_{c1} = 935.55 \text{ krad/s}}$$

(e) $f_{c1} = \frac{\omega_{c1}}{2\pi} = \frac{935.55\text{K}}{2\pi}$

$$\therefore \boxed{f_{c1} = 148.90 \text{ KHz}}$$

(f) $\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_0^2}$

$$\therefore \boxed{\omega_{c2} = 1068.89 \text{ krad/s}}$$

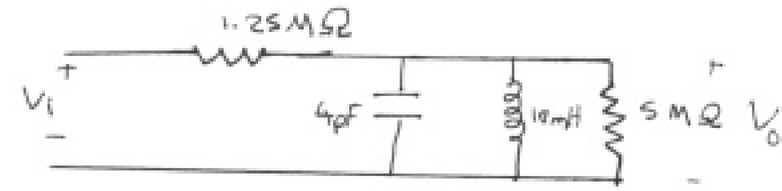
(g) $\therefore f_{c2} = \frac{\omega_{c2}}{2\pi}$

$$\therefore \boxed{f_{c2} = 170.12 \text{ KHz}}$$

(h) $\beta = \frac{\omega_0}{Q} = \frac{1}{RC}$

$$\therefore \boxed{\beta = 133.33 \text{ krad/s or } 21.22 \text{ KHz}}$$

Problem 6. (14.17)



Let $Z_p = Z_C // Z_L // R_L$

$Z_C // Z_L = \dfrac{Z_C Z_L}{Z_C + Z_L}$ $\Rightarrow Z_p = \dfrac{\dfrac{Z_C Z_L R_L}{Z_C + Z_P}}{\dfrac{Z_C Z_L}{Z_C + Z_L} + R_L} = \dfrac{Z_C Z_P R_L}{Z_C Z_L + R_L(Z_C + Z_L)}$

$Z_L = sL$

$Z_C = \dfrac{1}{sC}$ $\Rightarrow Z_p = \dfrac{\dfrac{L R_L}{C}}{\dfrac{L}{C} + R_L\left(\dfrac{1}{sC} + sL\right)}$

$$= \frac{R_L L}{L + CR_L\left(\frac{1}{sC} + sL\right)} = \frac{R_L L}{L + \frac{R_L}{s} + sCLR_L}$$

$$= \frac{R_L L s}{s^2 (R_L L C) + sL + R_L}$$

$$H(s) = \frac{V_o}{V_i} = \frac{Z_P}{Z_P + R} = \frac{\dfrac{R_L L s}{s^2(R_L L C) + sL + R_L}}{R + \dfrac{R_L L s}{s^2(R_L L C) + sL + R_L}}$$

$$= \frac{R_L L s}{s^2 R R_L L C + L(R + R_L)s + R R_L} = \frac{s/RC}{s^2 + s\frac{(R + R_L)}{C(R R_L)} + \frac{1}{LC}}$$

general expression:

$$H(s) = \frac{K \sigma s}{s^2 + \sigma s + \omega_o^2}$$

(a)

$\Rightarrow$ $\omega_o^2 = \dfrac{1}{LC} = \dfrac{1}{4\times10^{-12} \times 10\times10^{-3}} = 25\times10^{12} \Rightarrow \underline{\underline{\omega_o = 5 \text{ M rad/sec}}}$

(b) $\sigma = \dfrac{R + R_L}{C R R_L} = \dfrac{1.25\times10^6 + 5\times10^6}{4\times10^{-12} \times 1.25\times10^6 \times 5\times10^6} = \underline{\underline{250 \text{ Krad/sec}}}$

(c) $Q = \frac{\omega_0}{B} = \frac{5}{0.25} = 20$

(d) At resonance $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$

Combination of L and C has impedence $Z_P = \frac{1}{j\omega C} // j\omega L$

$$\Rightarrow Z_P = \frac{\frac{1}{j\omega C} \cdot j\omega L}{\frac{1}{j\omega C} + j\omega L} = \frac{L/C}{\frac{1}{j\omega C} + j\omega L}$$

at $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$, $Z_P = \frac{L/C}{-j\sqrt{L/C} + j\sqrt{L/C}} = \frac{L/C}{0} = \infty$

$\Rightarrow$ Can replace combination of L and C with open circuit

$$\Rightarrow H(j\omega)\Big|_{\omega=\omega_0} = \frac{V_o}{V_i} = \frac{R_L}{R + R_L} =$$

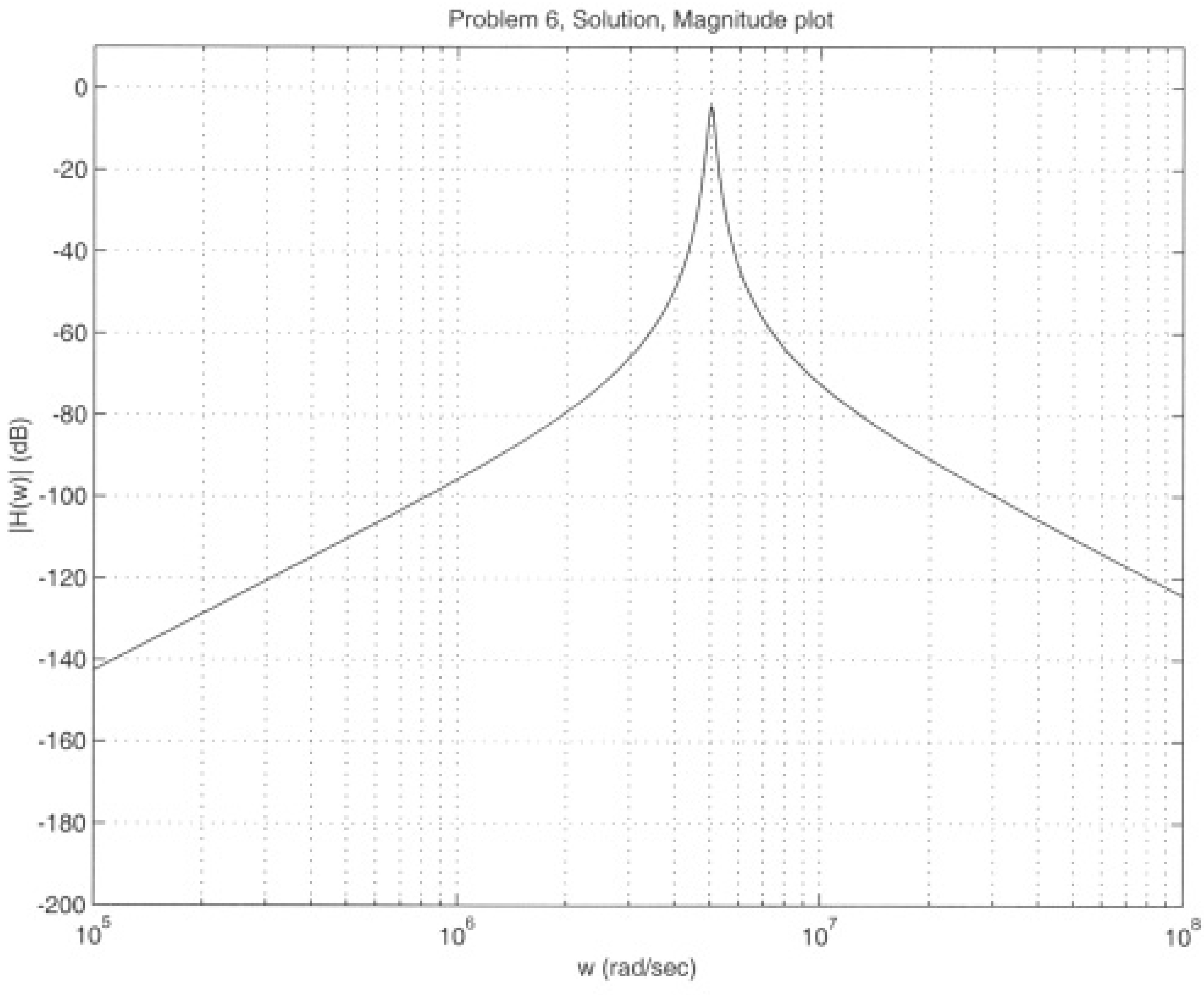
$$= \frac{5}{6.25} = 0.8 \angle 0^\circ \Rightarrow \text{Gain is } 0.8 \angle 0^\circ$$

$$\Rightarrow V_o = 0.8\, V_i = 0.8 \times 750 \cos(\omega_0 t) = 600 \cos(5\times10^6 t) \text{ mV}$$

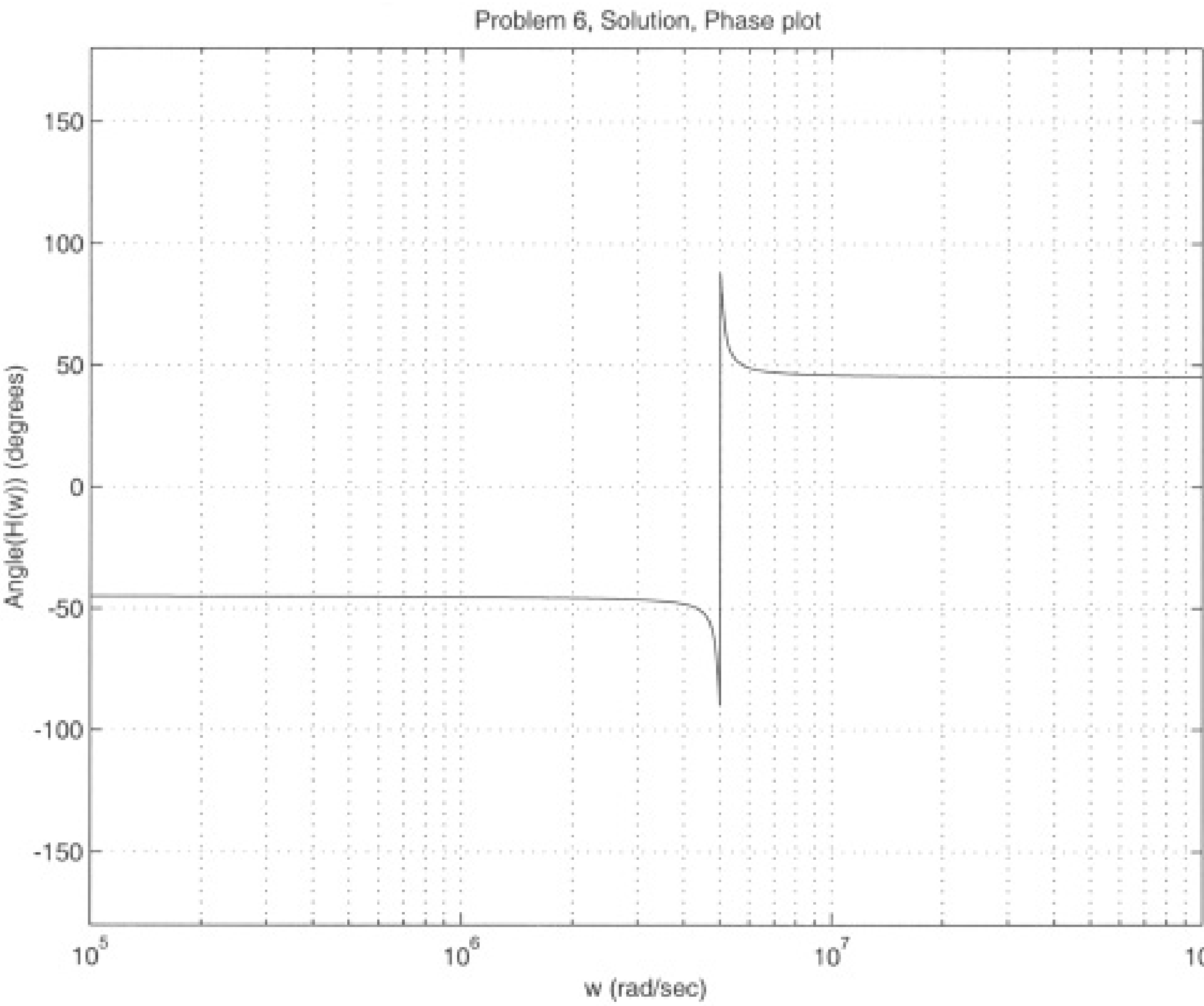
(e)

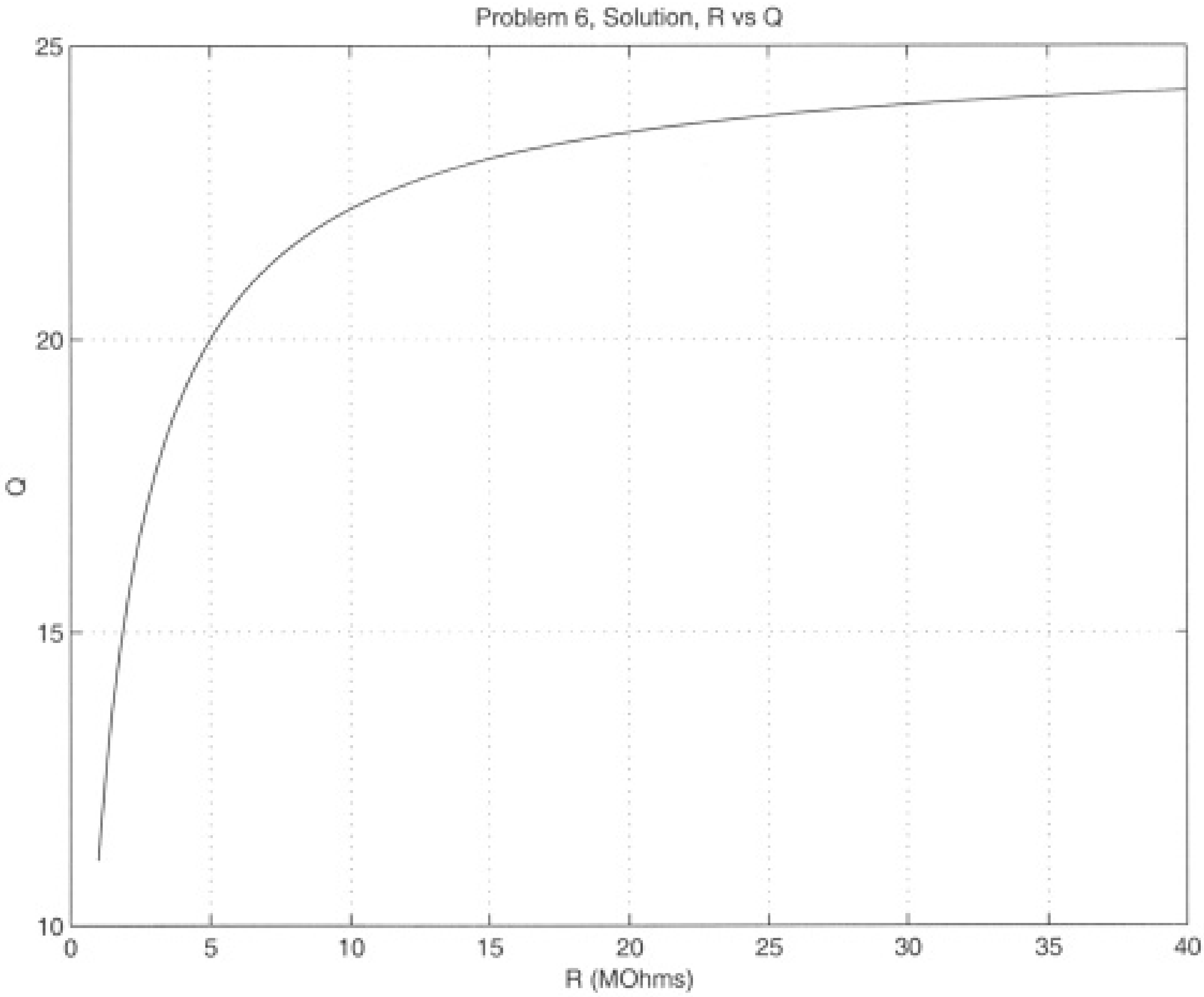
$$\left.\begin{array}{l} B = \frac{R + R_L}{C R R_L} \\ \omega_0 = \sqrt{\frac{1}{LC}} \\ Q = \frac{\omega_0}{B} \end{array}\right\} \Rightarrow Q = \frac{\omega_0}{\frac{R + R_L}{C R R_L}} = \frac{\omega_0 C R R_L}{R + R_L} = \frac{\omega_0 R C}{1 + R/R_L}$$

$$= \frac{5\times10^6 \times 5\times10^{-6}}{1 + \frac{1.25}{R_L}} = \frac{25}{1 + \frac{1.25}{R_L}} \quad \text{.. QED}$$



Problem 6, Solution, Magnitude plot
0
-20
-40
-60
-80
-100
-120
-140
-160
-180
-200
|H(w)| (dB)
10^5
10^6
10^7
10^8
w (rad/sec)

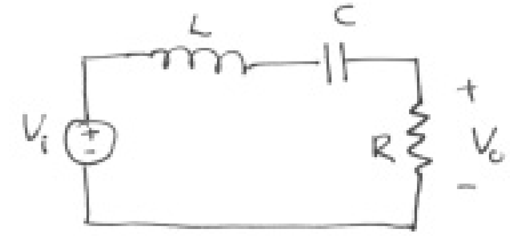

Problem 6, Solution, Phase plot
Angle(H(w)) (degrees)
150
100
50
0
-50
-100
-150
10^5
10^6
10^7
10^8
w (rad/sec)


Problem 6, Solution, R vs Q
Q
R (MOhms)
25
20
15
10
0
5
10
15
20
25
30
35
40

Problem 7. (14.19)

Note:

$\beta = \Delta\omega$



$C = 20nF$, $f_o = 20\text{ KHz} \Rightarrow$

$\omega_o = 2\pi \times 20\text{ KHz}$

$Q = 5$

(a) Following the analysis pg. 719-722,

$$\omega_o = \frac{1}{\sqrt{LC}} \quad \text{(eq. 14.31)} \Rightarrow L = \frac{1}{\omega_o^2 C} = \frac{1}{(2\pi \times 20\times10^3)^2 \times 20\times10^{-9}}$$

$$\Rightarrow L = 0.00317H = \underline{\underline{3.17mH}}$$

Recall that

$\beta = \frac{R}{L}$ and $Q = \frac{\omega_o}{\beta}$. Therefore $R = \frac{\omega_o L}{Q}$

$$\Rightarrow R = \frac{20\times10^3 \times 3.17\times10^{-3}}{5} \times 2\pi = \underline{\underline{79.7\Omega}}$$

(b)

$$\omega_{c_1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \quad \text{(eq. 14.29)}$$

$$= -\frac{79.7}{2\times 3.17\times10^{-3}} + \sqrt{\left(\frac{79.7}{2\times3.17\times10^{-3}}\right)^2 + \left(\frac{1}{3.17\times10^{-3}\times 20\times10^{-9}}\right)}$$

$= 113.65$ Krad/sec

$$f_{c_1} = \frac{\omega_{c_1}}{2\pi} = \underline{\underline{18.09\text{ KHz}}}$$

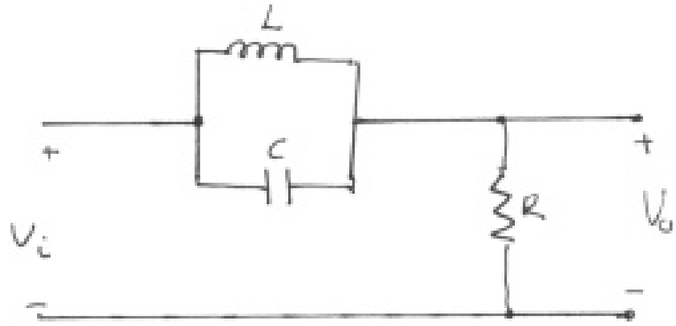
(c)

$$\omega_o = \sqrt{\omega_{c_1}\omega_{c_2}} \Rightarrow \omega_{c_2} = \frac{\omega_o^2}{\omega_{c_1}} = \frac{(2\pi\times20\times10^3)^2}{2\pi\times18.09\times10^3} = 138.95 \text{ Krad/sec}$$

$$\Rightarrow \underline{\underline{f_{c_2} = 22.11\text{ KHz}}}$$

(d). bandwidth (Hz) $= \frac{\beta}{2\pi} = \frac{\omega_o}{2\pi Q} = \frac{20\times10^3}{5} = \underline{\underline{4\text{ KHz}}}$

Problem 8 (14.27)



$L = 625\,\mu H$

$C = 25\,pF$

$R = 80\,K\Omega$

$Z_R = R$

$Z_C = \frac{1}{sC}$

$Z_L = sL$

Let $Z_p = Z_L // Z_C$

$\Rightarrow Z_p = \frac{Z_C Z_L}{Z_L + Z_C} = \frac{sL/sC}{sL + 1/sC}$

$\Rightarrow Z_p = \frac{sL}{1 + s^2 LC}$

$H(s) = \frac{V_o}{V_i} = \frac{R}{R + Z_p}$ (Voltage divider)

$= \frac{R}{R + \frac{sL}{1+s^2LC}} = \frac{R(1+s^2LC)}{sL + R(1+s^2LC)}$

$= \frac{LCR\, s^2 + R}{RLC\, s^2 + sL + R} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{CR} s + \frac{1}{LC}}$

(a) General expression of $H(s)$ is $H(s) = \frac{s^2 + \omega_o^2}{s^2 + \beta s + \omega_o^2}$ (eq. 14.55)

$\Rightarrow \omega_o = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{625\times10^{-6} \times 25\times10^{-12}}}$

$= \underline{\underline{8 \text{ Mrad/sec}}}$

(b) $f_o = \frac{\omega_o}{2\pi} = \frac{8\times10^6}{2\pi} = \underline{\underline{1.27 \text{ MHz}}}$

(c) $\beta = \frac{1}{RC} = \frac{1}{80\times10^3 \times 25\times10^{-12}} = 500 \text{ Krad/sec} \Rightarrow$ Bandwidth(Hz) $= 79.58$ KHz

$Q = \frac{\omega_o}{\beta} = \frac{8\times10^6}{500\times10^3} = \underline{\underline{16}}$

(d)

$$\omega_{c_1} = \omega_0 \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right] \quad \text{(eq. 14.53)}$$

$$\Rightarrow \omega_{c_1} = 8\times10^6 \left[-\frac{1}{32} + \sqrt{1 + \left(\frac{1}{32}\right)^2} \right]$$

$$\Rightarrow \underline{\underline{\omega_{c_1} = 7.75 \text{ Mrad/sec}}}$$

(e) $f_{c_1} = \frac{\omega_{c_1}}{2\pi} = \underline{\underline{1.23 \text{ MHz}}}$

(f)

$$\omega_{c_2} = \omega_0 \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right] \quad \text{(eq. 14.54)}$$

$$\Rightarrow \omega_{c_2} = 8\times10^6 \left[\frac{1}{32} + \sqrt{1 + \frac{1}{32^2}} \right]$$

$$\Rightarrow \omega_{c_2} = \underline{\underline{8.25 \text{ Mrad/sec}}}$$

(g) $f_{c_2} = \frac{\omega_{c_2}}{2\pi} = \frac{8.25\times10^6}{2\pi} = \underline{\underline{1.31 \text{ MHz}}}$

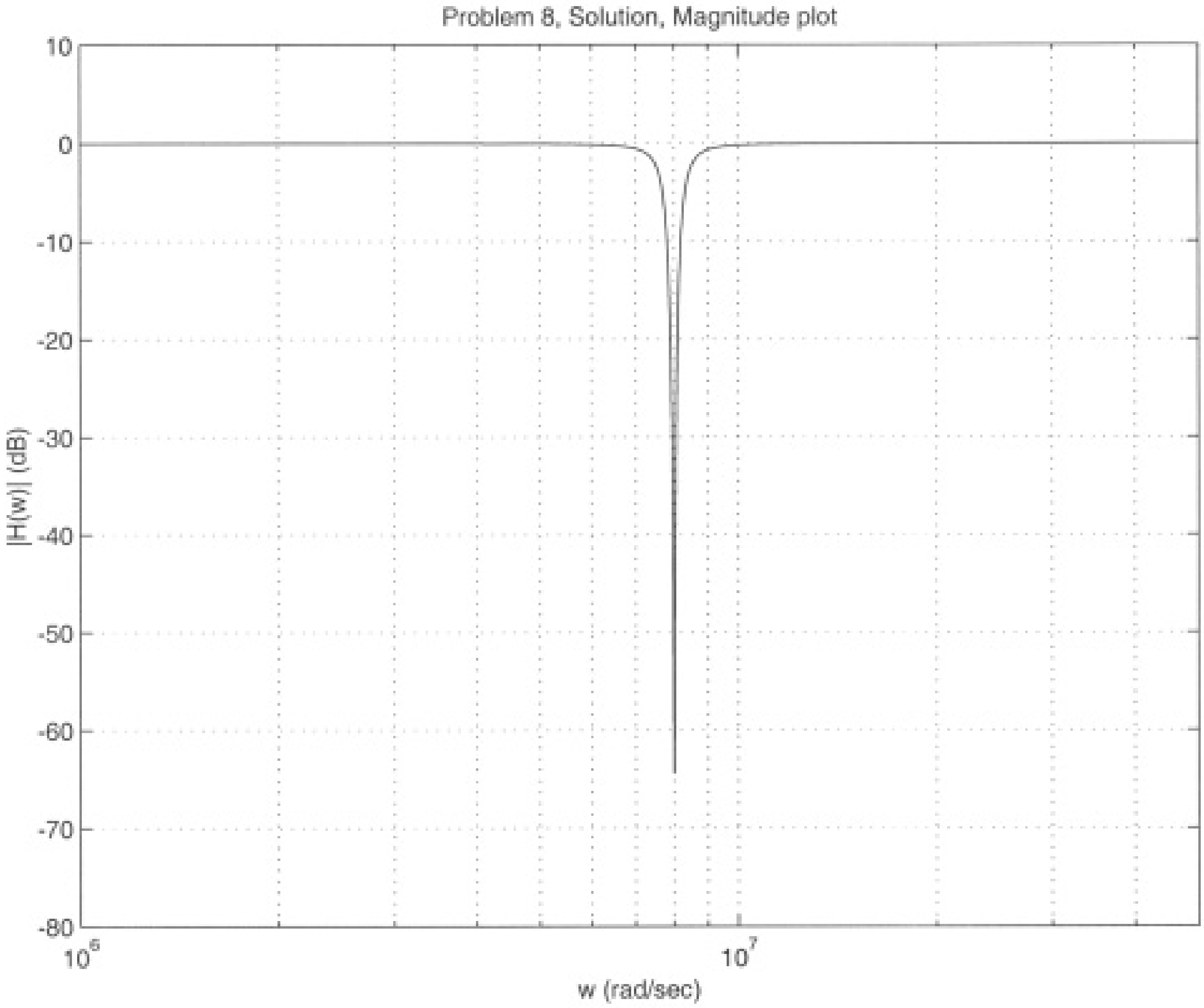
(h) $\beta = 2\pi(f_{c_2} - f_{c_1}) = 1.31 - 1.23 \simeq \underline{\underline{80 \text{ KHz}}}$

(agrees with β calculated in part (c))

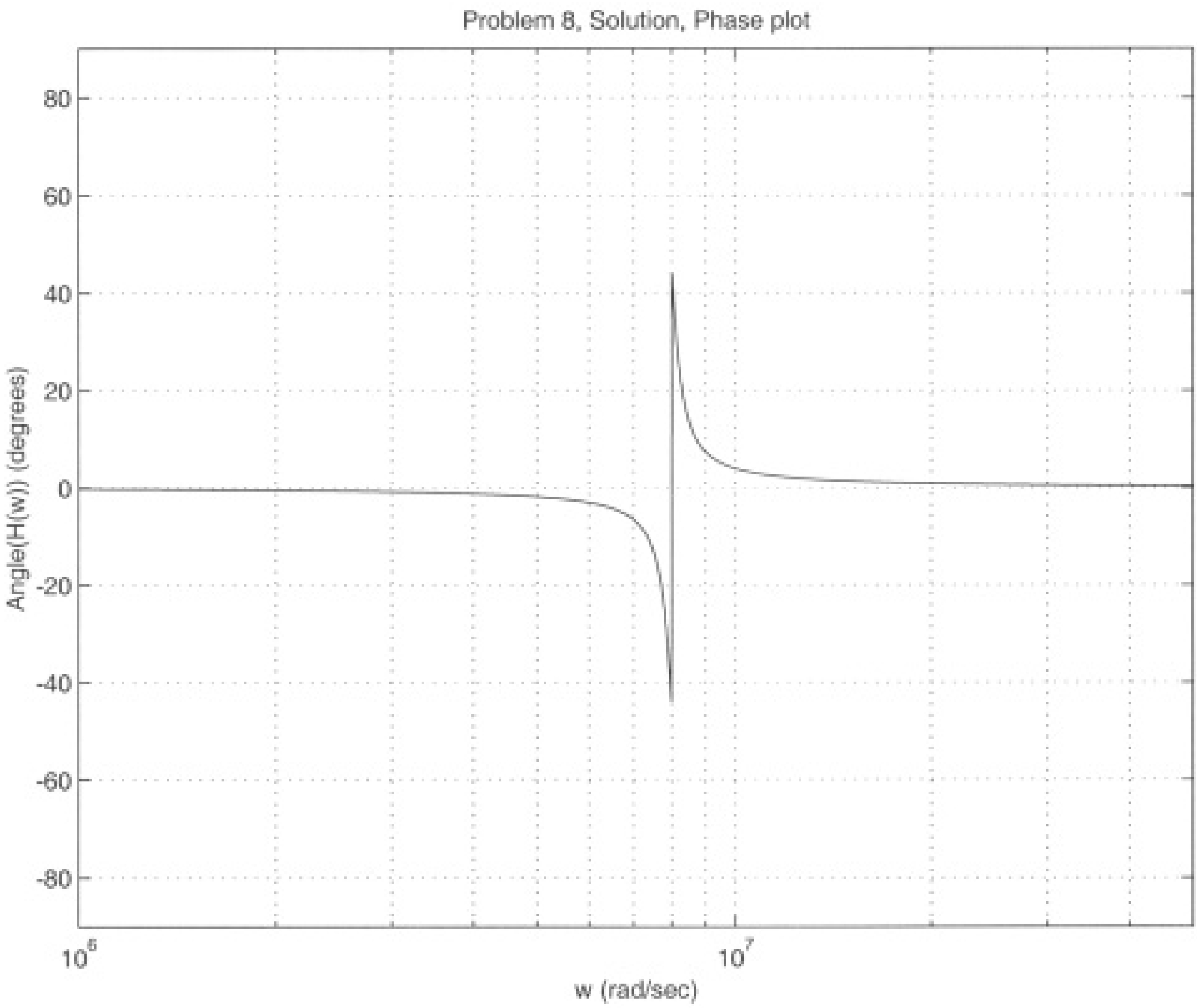
For Matlab plots:

$$\frac{1}{LC} = 64\times10^{12} \left(\frac{\text{rad}}{\text{sec}}\right)^2, \quad \frac{1}{RC} = 5\times10^5 \ \frac{\text{rad}}{\text{sec}}$$

$$\Rightarrow H(s) = \frac{s^2 + 64\times10^{12}}{s^2 + 5\times10^5 s + 64\times10^{12}}$$



Problem 8, Solution, Magnitude plot
10
0
-10
-20
-30
-40
-50
-60
-70
-80
|H(w)| (dB)
10^6
10^7
w (rad/sec)



Problem 8, Solution, Phase plot
80
60
40
20
0
-20
-40
-60
-80
Angle(H(w)) (degrees)
10^6
10^7
w (rad/sec)