2. The lowpass series RC filter with a load resistor $R_L$ can be represented as shown in the following diagram.

(a) Transfer function of the filter
The parallel combination of resistor $R_L$ and capacitor $C$ can be represented by an impedance $Z$, where

$$Z = \frac{1}{sC} \cdot R_L,$$

where $s = j\omega$

The equivalent circuit would be

and

$$H(s) = \frac{Z}{Z + R} = \frac{V_o}{V_i},$$

$$H(s) = \frac{\frac{1}{sC} \cdot R_L}{\frac{1}{sC} + R_L} \frac{R}{R + \frac{1}{sC} \cdot R_L \frac{1}{sC} + R_L}.$$
\[ H(s) = \frac{1}{RC} \cdot \frac{1}{s + \frac{1}{KRC}} \text{, where } K = \frac{R}{R+R_L} \]

Thus, \[ H(j\omega) = \frac{1}{RC} \cdot \frac{1}{j\omega + \frac{1}{KRC}} \text{, where } K = \frac{R}{R+R_L} \]

Replacing \( s \) by \( j\omega \) (since \( s = j\omega \))

\[(b) \quad \text{max } |H(j\omega)| \]
\[ |H(j\omega)| = \frac{1}{RC} \sqrt{\omega^2 + \left(\frac{1}{KRC}\right)^2} \]
\[ |H(j\omega)| \text{ will be maximum for } \omega = 0 \]

\[ |H(j\omega)|_{\text{max}} = \frac{1}{\sqrt{1 + \left(\frac{1}{kRC}\right)^2}} \frac{1}{R + R_L} \]

\[ |H(j\omega)|_{\text{max}} = k = \frac{R_L}{R + R_L} \]

(c) Cutoff frequency \( \omega_c \)

at the cutoff frequency \( \omega_c \),

\[ |H(j\omega_c)| = \frac{1}{\sqrt{2}} \left( \frac{1}{k} \right) = \frac{kRC}{\sqrt{(\omega_c)^2 + \left(\frac{1}{kRC}\right)^2}} \]

Squaring both sides,

\[ \frac{k^2}{2} = \frac{(kRC)^2}{\omega_c^2 + \left(\frac{1}{kRC}\right)^2} \]

\[ k^2\omega_c^2 + k^2\left(\frac{1}{kRC}\right)^2 = 2\left(\frac{1}{RC}\right)^2 \]

\[ k^2\omega_c^2 = \left(\frac{1}{RC}\right)^2 \]

\[ \omega_c = \frac{1}{k\left(\frac{1}{RC}\right)} \] is the cutoff frequency
As we see in parts (a), (b) & (c), loading does affect the low pass filter. It changes the transfer function of the filter, which in turn leads to:

(i) Passband magnitude of the transfer function, which is also its maximum magnitude, is
\[ |H(j\omega)|_{\text{max}} = k = \frac{R_L}{R+R_L} \]
for the filter with loading. Without \( R_L \), the value of \( |H(j\omega)|_{\text{max}} \) would have been 1. Thus loading decreases the pass band magnitude.

(ii) As seen in part (c), loading increases the cutoff frequency since
\[ \omega_c = \frac{1}{k \left( \frac{1}{RC} \right)} \]
\[ k = \frac{R_L}{R+R_L} \quad \Rightarrow \quad k < 1 \]
\[ \omega_c \gg \frac{1}{RC} \quad \text{The cutoff freq with no load is } \frac{1}{RC} \]
Resonant $\omega_c = 160K \text{ rad/}s$

(a) $f_c = \frac{\omega_c}{2\pi} = \frac{160K}{2\pi} \\
\boxed{f_c = 25.46 \text{ kHz}}$

(b) Now $f_c = \frac{1}{2\pi RC}$

\[ R = \frac{1}{2\pi f_c C} = \frac{1}{2\pi \left(\frac{160K}{2\pi}\right) \times 25 \times 10^{-9}} \]
\[ \therefore R = 250 \Omega \]

(c) Now from Prob 2-part (c), for a series RLC low pass filter with load resistor $R_L$, as shown in ckt diagram given below, the cutoff freq is given by,
\[ \omega_c = \frac{1}{K} \left( \frac{1}{RC} \right) \]

where \( K = \frac{R_L}{R + R_L} \)

Now given \( \omega_c = 160 \text{krad/s} \)

Let \( \omega_{c\text{-new}} = 8.1 \text{ more than } \omega_c \)

\[ = 1.08 \times 160 \]
\[ = 172.8 \text{ krad/s} \]

\[ 172.8 \text{ krad/s} = \frac{1}{K} \left( \frac{1}{RC} \right) \]
\[ = \frac{1}{K} \left( \frac{1}{250 \times 25 \times 10^{-9}} \right) \]

\[ K = 0.9259 \]

\[ \text{or } \omega_{c\text{-new}} = \frac{1}{K} \omega_c \]
\[ \therefore k = \frac{\omega_c}{\omega_{c\text{-new}}} = \frac{1}{1.08} = 0.9259 \]
Since \( K = \frac{R_L}{R+R_L} \)

\[ 0.9259 = \frac{R_L}{250 + R_L} \]

\[ (0.9259)(250) = R_L(1-0.9259) \]

\[ R_L = 3125.52 \]

is the smallest load resistor that could be connected across the op terminals of the filter.

(d) Using expression derived in part (b) of problem 2,

\[ |H(\omega)|_{\omega=0} = |H(\omega)|_{\omega=\infty} \]

\[ = \frac{R_L}{R+R_L} \]

\[ = K \]

\[ |H(\omega)|_{\omega=0} = 0.9259 \]
Resonant frequency $f_c = 800 \text{ Hz}$

(a) $f_c = \frac{1}{2\pi RC}$

$800 = \frac{1}{2\pi R (20 \times 10^{-9})}$

$\therefore R = 9.95 \text{k}$

(b) Now, $R_{eq} = R_{11} R_L = \frac{R \cdot R_L}{R + R_L}$

$= \frac{(9.95 \text{k})(68 \text{k})}{9.95 \text{k} + 68 \text{k}}$

$= 8.68 \text{k}$

$\therefore f_c = \frac{1}{2\pi R_{eq}} = \frac{1}{2\pi (8.68 \text{k}) (20 \times 10^{-9})}$

$\therefore f_c = 917.26 \text{ Hz}$
Note: For derivations of formulae used here, refer to Example 14.6, of the Text Book.

(a) \( \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{40 \times 10^{-6} \times 25 \times 10^{-9}}} \)

\[ \omega_0 = 10^6 \text{ rad/s} = 1 \text{ MHz} \]

(b) \( f_0 = \frac{\omega_0}{2\pi} = \frac{10^6}{2\pi} = 159.15 \text{ kHz} \)

\[ f_0 = 159.15 \text{ kHz} \]

(c) \( \beta = \frac{1}{RG} = \frac{1}{(300)(25 \times 10^{-9})} \)

\[ \beta = 133.33 \text{ kHz} \]

and \( Q = \frac{\omega_0}{\beta} \)

\[ Q = \frac{10^6}{133.33 \text{ kHz}} \]

\[ Q = 7.5 \]
(d) \( \omega_c 1 = \frac{-B}{a} + \sqrt{\frac{B^2}{a^2} + \omega_0^2} \)

Substituting \( \beta = 133.33 \text{ kHz} \) and \( \omega_0 = 1 \text{ MHz} \), we get

\[ \omega_c 1 = 935.55 \text{ kHz} \]

(e) \( f_c 1 = \frac{\omega_c 1}{2\pi} = \frac{935.55}{2\pi} \text{ kHz} \)

\[ f_c 1 = 148.90 \text{ kHz} \]

(f) \( \omega_c 2 = \frac{B}{a} + \sqrt{\frac{B^2}{a^2} + \omega_0^2} \)

\[ \omega_c 2 = 1068.89 \text{ kHz} \]

(g) \( f_c 2 = \frac{\omega_c 2}{2\pi} \text{ kHz} \)

\[ f_c 2 = 170.12 \text{ kHz} \]

(h) \( \beta = \frac{\omega_0}{\Phi} = \frac{1}{RC} \)

\( \beta = 133.33 \text{ kHz} \) on 21.22 kHz
Problem 6. (14.17)

\[ Z_p = \frac{Z_c}{Z_L} \parallel R_L \]

\[ Z_c / Z_L = \frac{Z_c Z_L}{Z_c + Z_L} \]

\[ Z_p = \frac{Z_c Z_L R_L}{Z_c Z_L + R_L} = \frac{Z_c Z_p R_L}{Z_c Z_L + R_L (Z_c + Z_L)} \]

\[ Z_L = \frac{sL}{C} \]

\[ Z_c = \frac{1}{sC} \]

\[ Z_p = \frac{\frac{L R_p}{C}}{\frac{1}{sC} + sL} \]

\[ = \frac{R_L L}{L + C R_L \left( \frac{1}{sC} + sL \right)} = \frac{R_L L}{L + \frac{R_L}{s} + sC R_L} \]

\[ H(s) = \frac{V_o}{V_i} = \frac{Z_p}{Z_p + R} = \frac{\frac{R_L L s}{s^2 (R_L C) + s L + R_L}}{R + \frac{R_L L s}{s^2 (R_L C) + s L + R_L}} \]

\[ = \frac{\frac{R_L L s}{s^2 R_L C + L (R + R_L) s + R R_L}}{s^2} \]

**general expression:**

\[ H(s) = \frac{K C s}{s^2 + \frac{R}{L} s + \omega_0^2} \]

\[ \omega_0 = \frac{1}{\sqrt{L C}} = \frac{1}{4 \times 10^{12} \times 10^{-3}} = 25 \times 10^6 \Rightarrow \omega_0 = 5 \text{ M rad/sec} \]

\[ b = \frac{R + R_L}{C R R_L} = \frac{1.25 \times 10^6 + 5 \times 10^6}{4 \times 10^{-12} \times 1.25 \times 10^6 \times 5 \times 10^6} = 250 \text{ krad/sec} \]
(c) \[ Q = \frac{\omega_0}{\epsilon} = \frac{5}{0.25} = 20 \]

(d) At resonance \( w = \omega_0 = \frac{1}{\sqrt{LC}} \)

Combination of \( L \) and \( C \) has impedance \( Z_p = \frac{1}{j\omega C} \parallel j\omega L \)

\[ Z_p = \frac{\frac{1}{j\omega C} \parallel j\omega L}{j\omega C + j\omega L} = \frac{1}{j\omega C + j\omega L} \]

at \( w = \omega_0 = \frac{1}{\sqrt{LC}} \), \( Z_p = \frac{1}{j\frac{1}{\sqrt{LC}} + j\frac{1}{\sqrt{LC}}} = \frac{1/\epsilon}{0} = \infty \)

\( \Rightarrow \) can replace combination of \( L \) and \( C \) with open circuit.

\[ H(j\omega) = \frac{V_o}{V_i} = \frac{R_L}{R + R_L} \]

\[ = \frac{5}{6.25} = 0.8 \quad 10^0 \Rightarrow \text{Gain is 0.8 }10^0 \]

\[ \Rightarrow V_o = 0.8 V_i = 0.8 \times 750 \cos(\omega_0 t) = 600 \cos(5 \times 10^6 t) \text{ mV} \]

(e)

\[ b = \frac{R + R_L}{C R R_L} \]

\[ w_0 = \sqrt{\frac{1}{LC}} \]

\[ Q = \frac{\omega_0}{\epsilon} = \omega_0 \frac{c R R_L}{R + R_L} = \omega_0 \frac{R C}{1 + R/R_L} \]

\[ Q = \frac{5 \times 10^6 \times 5 \times 10^6}{1 + 1.25 \frac{R_L}{R_L}} = \frac{25}{1 + 1.25} \quad \text{QED} \]
Problem 6, Solution, Magnitude plot

-200 \lt \|H(w)\| (dB) \lt 0

10^5 \lt w (\text{rad/sec}) \lt 10^8
Problem 7. (14.19)

Note:
\[ \theta = \Delta \omega \]

\[ C = 20 \text{nF}, \ f_0 = 20 \text{ KHz} \Rightarrow \omega_0 = 2\pi \times 20 \text{ KHz} \]
\[ Q = 5 \]

(a) Following the analysis pg. 719-722,

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad (\text{eq. 14.31}) \Rightarrow \nu = \frac{1}{\omega_0^2 C} = \frac{1}{(2\pi \times 20 \times 10^3)^2 \times 2 \times 10^{-9}} \]

\[ \Rightarrow \nu = 0.00317 \text{H} = 3.17 \text{mH} \]
Recall that \[ \theta = \frac{R}{L} \] and \[ Q = \frac{\omega_0 \nu}{\theta} \]. Therefore \[ R = \frac{\omega_0 \nu}{Q} \]

\[ \Rightarrow R = \frac{2 \times 10^3 \times 3.17 \times 10^{-3} \times 2\pi}{5} = 79.7 \Omega \]

(b) \[ \omega_{c1} = -\frac{R}{2\nu} + \sqrt{\left(\frac{R}{2\nu}\right)^2 + \left(\frac{1}{\nu C}\right)^2} \quad (\text{eq. 14.29}) \]

\[ = -\frac{79.7}{2 \times 3.17 \times 10^{-3}} + \sqrt{\left(\frac{79.7}{2 \times 3.17 \times 10^{-3}}\right)^2 + \left(\frac{1}{3.17 \times 10^{-3} \times 2 \times 10^{-9}}\right)^2} \]

\[ = 113.65 \text{ Krad/sec} \]

\[ f_{c1} = \frac{\omega_{c1}}{2\pi} = 18.09 \text{ KHz} \]

(c) \[ \omega_0 = \sqrt{\omega_e \omega_k} \Rightarrow \omega_{c2} = \frac{\omega_0^2}{\omega_{c1}} = \frac{(2\pi \times 20 \times 10^3)^2}{2\pi \times 18.09 \times 10^3} = 138.95 \text{ Krad/sec} \]

\[ \Rightarrow f_{c2} = 22.11 \text{ KHz} \]

(d) Bandwidth (Hz) = \[ \frac{\theta}{2\pi} = \frac{\omega_0}{2\pi Q} = \frac{2 \times 10^3}{5} = 4 \text{ KHz} \]
Problem 8 (14.27)

\[ L = 62.5 \mu H \]
\[ C = 25 \mu F \]
\[ R = 80 \text{ k}\Omega \]

\[ Z_R = R \]
\[ Z_C = \frac{1}{sC} \]
\[ Z_L = sL \]

Let \[ Z_p = Z_L / Z_C \]
\[ \Rightarrow Z_p = \frac{Z_C Z_L}{Z_L + Z_C} = \frac{sL / sC}{sL + 1/sC} \]
\[ \Rightarrow Z_p = \frac{sL}{1 + s^2LC} \]

\[ H(s) = \frac{V_o}{V_i} = \frac{R}{R + Z_p} \quad \text{(Voltage divider)} \]
\[ = \frac{R}{R + \frac{sL}{1 + s^2LC}} = \frac{R}{R + \frac{sL}{1 + s^2LC}} \]
\[ = \frac{LCR s^2 + R}{RLC s^2 + sL + R} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{CR} s + \frac{1}{LC}} \]

(a) General expression of \[ H(s) \] is \[ H(s) = \frac{s^2 + \omega_0^2}{s^2 + 6 \beta s + \omega_0^2} \]
\[ (eq. 14.55) \]
\[ \Rightarrow \omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{62.5 \times 10^6 \times 25 \times 10^{-12}}} \]
\[ = 8 \text{ Mrad/sec} \]

(b) \[ f_0 = \frac{\omega_0}{2\pi} = \frac{8 \times 10^6}{2\pi} = 1.27 \text{ MHz} \]

(c) \[ B = \frac{1}{LRC} = \frac{1}{80 \times 10^3 \times 25 \times 10^{-12}} = 500 \text{ krad/sec} \Rightarrow \text{Bandwidth}(B) = 79.58 \text{ KHz} \]
\[ Q = \frac{\omega_0}{B} = \frac{8 \times 10^6}{500 \times 10^3} = 16 \]
(d) \[ \omega_c = \omega \left[ -\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right] \]  
(\text{eq. 14.53})
\[ \Rightarrow \omega_c = 8 \times 10^6 \left[ -\frac{1}{82} + \sqrt{1 + \left(\frac{1}{82}\right)^2} \right] \]
\[ \Rightarrow \omega_c = 7.75 \text{ Mrad/sec} \]

(e) \( \bar{f}_c = \frac{\omega_c}{2\pi} = 1.23 \text{ MHz} \)

(f) \[ \omega_{c2} = \omega \left[ \frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2} \right] \]  
(\text{eq. 14.54})
\[ \Rightarrow \omega_{c2} = 8 \times 10^6 \left[ \frac{1}{32} + \sqrt{1 + \frac{1}{32}^2} \right] \]
\[ \Rightarrow \omega_{c2} = 8.25 \text{ Mrad/sec} \]

(g) \( \bar{f}_c = \frac{\omega_{c2}}{2\pi} = \frac{8.25 \times 10^6}{2\pi} = 1.31 \text{ MHz} \)

(h) \( \delta = 2(\bar{f}_c - \bar{f}_c) = 1.31 - 1.23 = 80 \text{ kHz} \)

\( \text{(agrees with } \delta \text{ calculated in part (c))} \)

For Matlab plots:
\[ \frac{1}{\omega C} = 64 \times 10^{12} \text{ rad/sec}^2, \quad \frac{1}{\omega C} = 5 \times 10^5 \text{ rad/sec} \]
\[ \Rightarrow H(s) = \frac{s^2 + 64 \times 10^{12}}{s^2 + 5 \times 10^5 s + 64 \times 10^{12}} \]
Problem 8, Solution, Magnitude plot

$|H(w)|$ (dB)

$w$ (rad/sec)

$10^6$ $10^7$
Problem 8, Solution, Phase plot

![Phase plot diagram](image)

- **Graph Description**: The phase plot shows the angle of the frequency response as a function of frequency. The y-axis represents the phase (degrees) and the x-axis represents frequency (rad/sec).

- **Observations**:
  - There is a sharp change in the phase angle at a certain frequency, indicating a discontinuity in the system's behavior.
  - The phase retains a constant value at the lower frequencies, suggesting a steady-state response.

- **Significance**: Understanding phase plots is crucial in analyzing the stability and behavior of dynamic systems. The plot helps in identifying resonant frequencies and the phase margin, which are critical in system design and control engineering.