EECS 210: Lecture 1

Goals

- Understand periodic signals
- Understand sinusoidal decomposition of periodic signals
- Understand how sinusoids go through linear time invariant systems.

Frequency Representation of Periodic Signals

A signal $y(t)$ is periodic if for some T

$$y(t) = y(t + nT)$$

for all t and integer values of n. T is called the period of $y(t)$.

Example

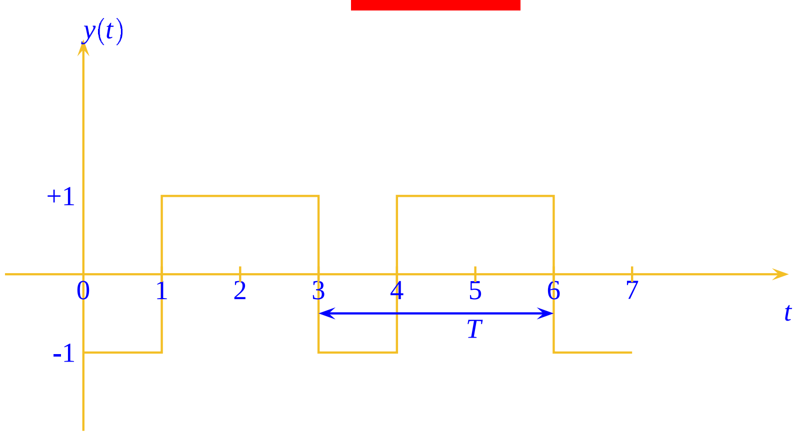


Figure 1: Periodic Function

Parameters of Periodic Functions

- y_{ppk} = peak-to-peak amplitude= max[$y(t)$]-min[$y(t)$].
- $y_{avg} = \frac{1}{T}\int_{t_0}^{t_0+T} y(t)dt$ for any t_0.
- $y_{rms} = \sqrt{\left[\frac{1}{T}\int_{t_0}^{t_0+T} y^2(t)dt\right]}$ for any t_0.

Sinusoidal Functions

$$y(t) = A\cos(2\pi f_0 t + \theta) = A\cos(\omega_0 t + \theta)$$

- $A =$ amplitude.
- $\theta =$ phase (in radians).
- $f_0 =$ fundamental frequency (cycles/second or Hz).
- $\omega_0 = 2\pi f_0$ is the radian frequency (radians/sec).

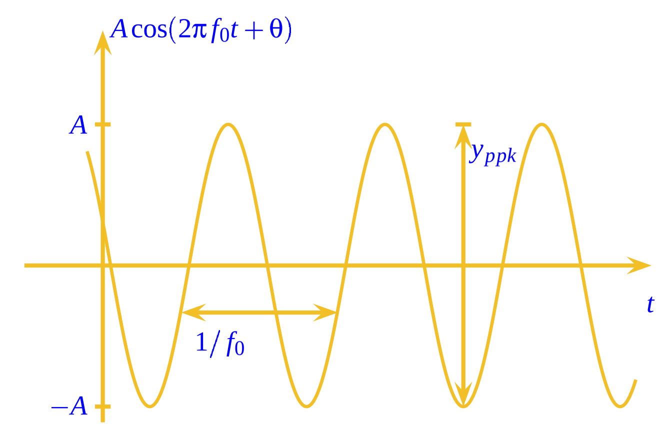


Figure 2: Sinusoidal function

Parameters for Sinusoids

For $y(t) = A\cos(2\pi f_0 t + \theta)$

- Period $T = 1/f_0$.
- $y_{ppk} = 2A$
- $y_{avg} = \frac{1}{T}\int_0^T \cos(2\pi f_0 t + \theta)dt = 0$
- $y_{rms} = \frac{A}{\sqrt{2}} = 0.707A$

Properties of Sinusoids

- Any periodic signal $y(t)$ can be decomposed as a sum or linear combination of sinusoids.
- A linear time invariant system transforms sinusoids into sinusoids with the SAME frequency but with different amplitude and different phase

Decomposition of Signals into Sinusoids

Any periodic signal $y(t)$ with period $T = 1/f_0$ can be decomposed into a sum (linear combination) of sinusoids of different frequencies (called harmonics)

$$y(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \theta_n)$$

where

- $n f_0$ is the n-th harmonic of the fundamental frequency,
- A_n is the Fourier amplitude of the n-th harmonic ($A_n > 0$),
- θ_n is the Fourier phase of the n-th harmonic ($-\pi < \theta < \pi$),
- A_0 is the average value of $y(t)$.

The decomposition is called the Fourier series.

Example

$$\begin{aligned} y(t) &= e^{t}, \quad 0 \le t < 3 \\ y(t) &= e^{t-3}, \quad 3 \le t < 6 \\ y(t) &= e^{t-6}, \quad 6 \le t < 9 \\ . & \quad . \quad . \end{aligned}$$

$$y(t) \approx \hat{y}(t) = A_0 + \sum_{n=1}^{m} A_n \cos(2\pi n f_0 t + \theta_n)$$

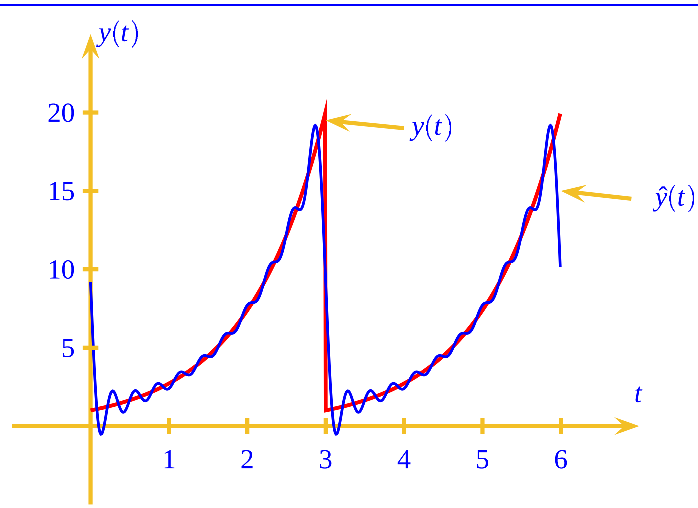
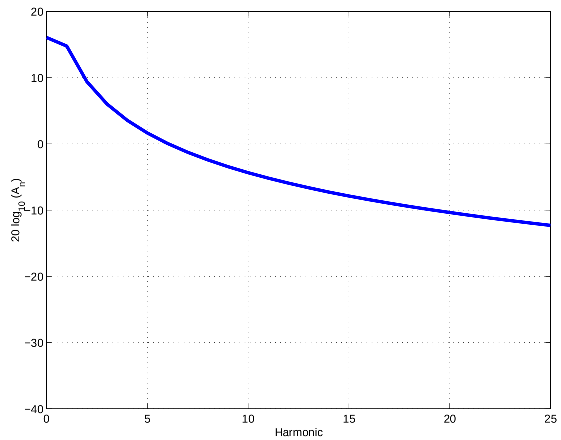


Figure 3: Fourier approximation of a periodic function

Fourier Coefficients

n	A_n	θ_n
0	6.3098	
1	5.4373	-1.1425
2	2.9302	-1.3709
3	1.9832	-1.4646
4	1.4954	-1.5208
5	1.1992	-1.5616
6	1.0005	-1.5946
7	0.8581	-1.6231
8	0.7511	-1.6488
9	0.6677	-1.6726
10	0.6010	-1.6951

Fourier Amplitudes of Signal



Linear Time Invariant (LTI) Systems

A system is linear and time invariant if the following two conditions hold.

1. If $(x_1(t), y_1(t))$ is an input-output pair and $(x_1(t), y_1(t))$ is an input-output pair then the output corresponding to the input $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. If $(x(t), y(t))$ is an input-output pair then $(x(t-\tau), y(t-\tau))$ is also an input-output pair for all τ.

LTI or Not LTI

- $y(t) = x(t) + x(t-3)$?
- $y(t) = x(t)x(t-2)$?
- $y(t) = e^{-3t}x(t)$?
- $y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau$?

Properties of LTI Systems

Fact: If

$$x(t) = A_{in}\cos(2\pi f_0 t + \theta_{in})$$

then

$$y(t) = A_{out}\cos(2\pi f_0 t + \theta_{out})$$

For LTI output frequency=input frequency

Parameters of LTI systems

- Gain of system: $G = \frac{A_{out}}{A_{in}} = |H(f_0)|$. $H(f)$ is called the transfer function of the system. It tells us the output amplitude of the signal at frequency f.

$$A_{out} = |H(f)|A_{in}$$

- Phase response of system = $\theta_{out} - \theta_{in} = \angle H(f)$

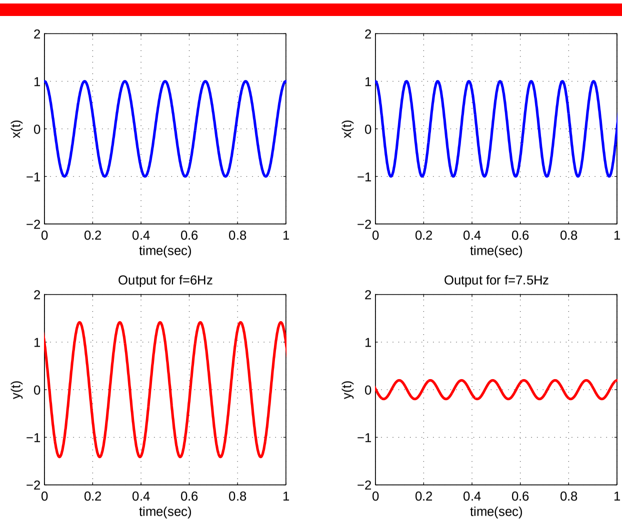
$$\theta_{out} = \theta_{in} + \angle H(f)$$

Example

$$y(t) = x(t) - x(t - 0.125)$$

This might represent what happens in a wireless communication system whereby the transmitted signal has a direct "line of sight" path to the receiver but another path which might be due to a reflection off of a building also exists.

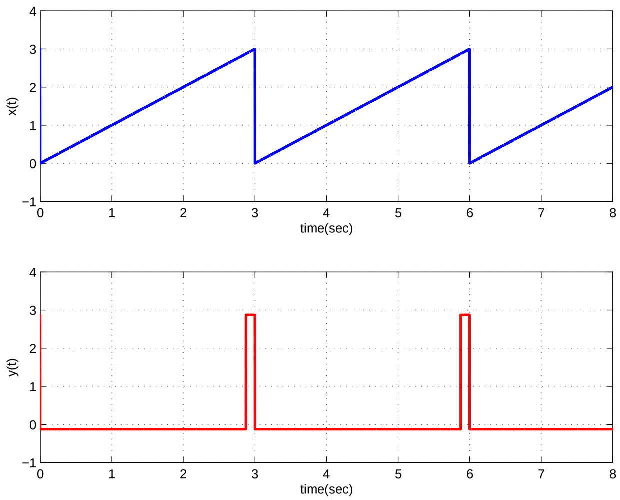
Sinusoidal In, Sinusoidal Out
Same Frequency, Different Amplitude



Transfer Function

At frequency 1 Hz the output amplitude is $A_{out} = 0.7654$ and the input amplitude is $A_{in} = 1$. Thus $|H(1)| = 0.7654$. At frequency 2 Hz the output amplitude is $A_{out} = 1.4142$ and the input amplitude is $A_{in} = 1$. Thus $|H(2)| = 1.4142$.

Triangle In, Something Else Out



DeciBel Units

Sometimes the coefficients A_n in the Fourier series are very small (e.g. 10^{-80}) and sometimes the coefficients can be very large (e.g 10^5). For ease of working with large and small numbers simultaneously a logarithmic measure has been introduced.

$$G\ (\mathrm{dB}) = 20\log_{10} G = 20\log_{10}\frac{A_{out}}{A_{in}}.$$

If the gain is positive then the system is amplifying and $A_{out} > A_{in}$. If the gain is negative then the system is attenuating and $A_{out} < A_{in}$. Operational amplifiers (opamps) have gains on the order of 60dB (i.e. $A_{out} = 1000A_{in}$).

Dynamic Range, Power, dBV

The dynamic range of a system is the ratio of the largest amplitude to the smallest amplitude. A CD player has a dynamic range on the order of 90dB.

The power gain of a system in dB is

$$\begin{aligned}\text{Power Gain (dB)} &= 10\log_{10}\frac{A_{out}^2/2}{A_{in}^2/2}\\ &= 10\log_{10}\frac{A_{out}^2}{A_{in}^2}\\ &= 20\log_{10}\frac{A_{out}}{A_{in}}\end{aligned}$$

A voltage is expressed in dBV units meaning that it is relative to a 1 volt reference so if v is voltage then $v(\mathrm{dBV}) = 20\log_{10}(v)$. For example 10 Volts =20dBV, 100 Volts=40dBV.