SOLUTIONS

Problem 1	______/20
Problem 2	______/20
Problem 3	______/20
Problem 4	______/20

TOTAL: ______/80

ELECTROMAGNETICS I
(EECS 230 - WINTER 1999)
Final Exam

Date: December 20, 1999
Time: 10:30am-12:30pm
Instructor: Professor S. Rand
Duration: 2 hours
LOCATION: EECS 1200 and 3437

The course textbook may be used for this examination. Programmable electronic calculators and note pads are NOT PERMITTED. Simple, scientific calculators are allowed. Show your work and reasoning, to be eligible for part marks. As much as possible, complete problems symbolically before substituting for parameter values.

ANSWER ALL 4 QUESTIONS

Honor Code: I have neither given nor received aid on this exam.

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Name (Print)

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Signature

1A. Short Problem Section A (**10 marks**)

A 50Ω lossless transmission line is terminated by an unknown load impedance Z_L. A standing wave ratio of 5 is measured, and the first and second voltage maxima are located 15cm and 135cm from the load. Use the Smith chart provided to find Z_L.

$\frac{\lambda}{2} = (135-15)\,cm = 120\ cm \longrightarrow \boxed{\lambda = 240\,cm}$

Hence the load is located on the chart at a position corresponding to $\frac{15cm\,\lambda}{240cm} = \frac{\lambda}{16}$

The corresponding rotation is $\frac{1}{8}$ of a full rotation WTL from the position of the first maximum (r_L^+ at $(5,0)$).

From the chart, we find $z_L = (1.1, j1.9)$ or

$\boxed{Z_L = 50\Omega\, z_L = (55 + j95)\Omega}$.

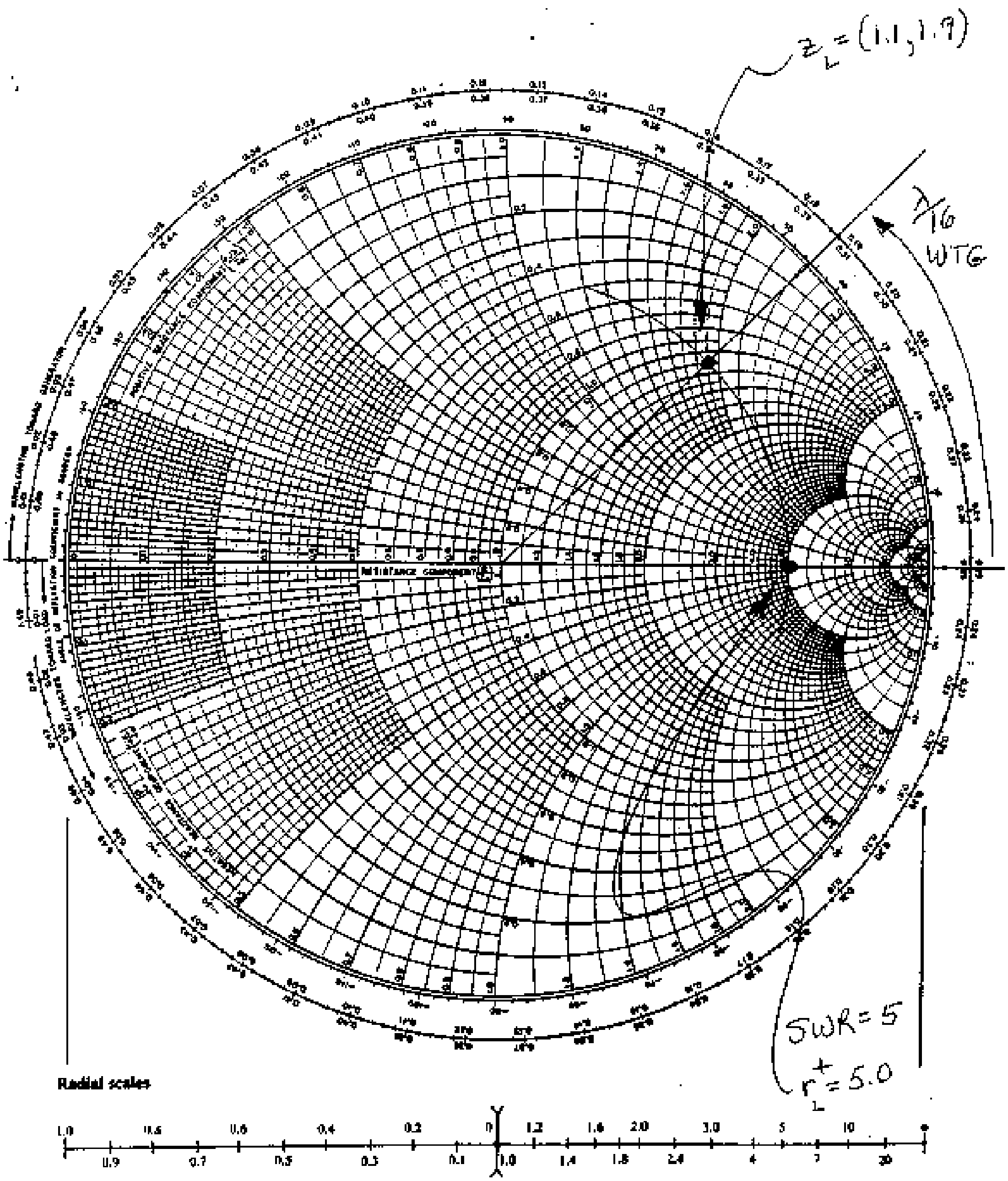
1B. Short Problem Section B (**10 marks**)

A distribution of charges consists of a point charge +q at the origin and a thin ring of radius b that has a linear charge density of $\rho_\ell = -\frac{q}{\pi b}$. The ring is centered on the origin and lies in the x-y plane.

(i) Find the axial potential V(z) of this distribution of charge. See attached

(ii) Calculate the magnitude and direction of $\overline{E}$ on the z axis from V(z). [**Important note** – you may not calculate $\overline{E}$ by a direct integration method that does not depend on V(z). No credit will be given for other approaches or for answers that are not derived.]

See attached.



$z_L = (1.1, 1.9)$
7/16 WTG
SWR = 5
$r_L^+ = 5.0$
Radial scales

SOLUTIONS

1 b. $V_b(z) = \frac{1}{4\pi\varepsilon}\int_{\phi=0}^{2\pi} \frac{\left(-\frac{q}{\pi b}\right) b\, d\phi}{\sqrt{b^2+z^2}} = \frac{-2q}{4\pi\varepsilon}\frac{1}{\sqrt{b^2+z^2}}$

The central charge produces a potential

$V_a(z) = \frac{1}{4\pi\varepsilon}\frac{q}{z}$

The total potential is

$$V(z) = V_a(z) + V_b(z)$$

$$= \frac{-2q}{4\pi\varepsilon\sqrt{b^2+z^2}} + \frac{1}{4\pi\varepsilon}\frac{q}{z}$$

(b) $\bar{E} = -\bar{\nabla} V$

$$= -\hat{z}\frac{\partial V(z)}{\partial z}$$

$$= -\hat{z}\frac{\partial}{\partial z}\left[\frac{-2q}{4\pi\varepsilon\sqrt{b^2+z^2}} + \frac{1}{4\pi\varepsilon}\frac{q}{z}\right]$$

$$= -\hat{z}\left[\frac{q}{4\pi\varepsilon}\right]\left[-2\left(\frac{-\frac{1}{2}(b^2+z^2)^{-1/2}2z}{b^2+z^2}\right) + \left(\frac{-1}{z^2}\right)\right]$$

$$= +\hat{z}\left(\frac{q}{4\pi\varepsilon}\right)\left[\frac{-z}{(b^2+z^2)^{3/2}} + \frac{1}{z^2}\right]$$

In the far field $z \gg b$ and we find

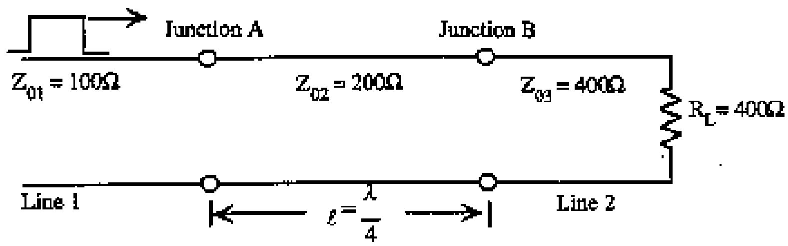
$$\bar{E} = \hat{z}\frac{q}{4\pi\varepsilon}\left[\frac{-2z}{z^3} + \frac{1}{z^2}\right]$$

$$= \hat{z}\frac{(-q)}{4\pi\varepsilon z^2}$$

2. **(20 marks)** When a pulse on a lossless transmission line encounters a junction between sections with different characteristic impedances, it is not only reflected (with voltage amplitude ratio Γ) but also transmitted (with voltage amplitude ratio τ), such that

$$\tau = 1 + \Gamma.$$

Referring to the figure, consider a square pulse of duration t_p and magnitude 1 volt incident from the left at junction A. The section of line between junctions A and B is a $\lambda/4$ quarter-wave transition section of length $\ell = 6vt_p$ where v = velocity of the wave (i.e., the transition section is 6 times longer than the pulse).



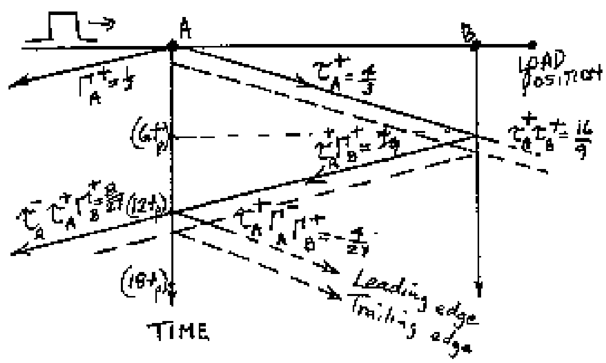
Transition section

To find out whether the quarter-wave transition section works for pulses the same way it does under steady-state conditions,

(i) State the intended function of a $\lambda/4$ section inserted between a feedline and a load. **(1 mark)** $\lambda/4$ transformer sections provide impedance matching between a feedline and an arbitrary load, provided the characteristic impedance is chosen to be $Z_{trans} = \sqrt{Z_{in} Z_L}$ as in the case at hand.

(ii) Draw a bounce diagram showing reflected and transmitted waves at <u>both</u> junctions. **(5 marks)**

NOTE: You may draw your diagram with the leading edge of the pulse in mind, or the trailing edge for full marks. However, you may find it helpful to draw lines for both the leading and trailing edges to answer part (vi).



Problem 2 (cont'd):

(iii) Calculate the reflected voltages and transmitted voltages when the pulse first encounters Junctions A and B. **(6 marks)**

Reflected voltage at A:

$$\Gamma_A^+ = \frac{Z_{02}-Z_{01}}{Z_{02}+Z_{01}} = \frac{200-100}{300} = \frac{1}{3}$$

$$V_{refl.}(A) = \Gamma_A^+ V_{in} = \tfrac{1}{3}(1) = \boxed{\tfrac{1}{3}\ V}$$

Reflected voltage at B:

$$\tau_A^+ = 1+\Gamma_A^+ = \frac{4}{3}$$

$$\Gamma_B^+ = \frac{Z_{03}-Z_{02}}{Z_{03}+Z_{02}} = \frac{400-200}{600} = \frac{1}{3}$$

$$V_{refl.}(B) = \tau_A^+ \Gamma_B^+ V_{in} = \Gamma_B^+\left(\tfrac{4}{3}\,V\right) = \boxed{\tfrac{4}{9}\ V}$$

Transmitted voltage at A:

As above,

$$V_{transm.}(A) = \tau_A^+ V_{in} = \tfrac{4}{3}(1) = \boxed{\tfrac{4}{3}\,V}$$

Transmitted voltage at B:

$$V_{transm.}(B) = \tau_B^+ \tau_A^+ V_{in} = \tau_B^+ \cdot \tfrac{4}{3}V$$

$$\tau_B^+ = 1+\Gamma_B^+ = 1+\tfrac{1}{3} = \tfrac{4}{3}$$

$$\therefore V_{transm.}(B) = \left(\tfrac{4}{3}\right)\left(\tfrac{4}{3}\right) = \boxed{\tfrac{16}{9}\ V}$$

(iv) Is there any reflection of voltage pulses on the 400Ω line from the load? **(2 marks)**

There are no reflections from the load because $Z_{03} = Z_L$, resulting in a reflection coefficient of $\Gamma = \frac{400-400}{800} = 0$.

(v) When the first reflection from junction B arrives back at A, it splits further into a transmitted and reflected part. Calculate the voltage directed back toward the generator and the voltage directed toward the load upon this second encounter with Junction A. **(4 marks)**

Toward generator: $V = \tau_A^- \tau_A^+ \Gamma_B^+ = \tau_A^-\left(\tfrac{4}{9}\right)V$

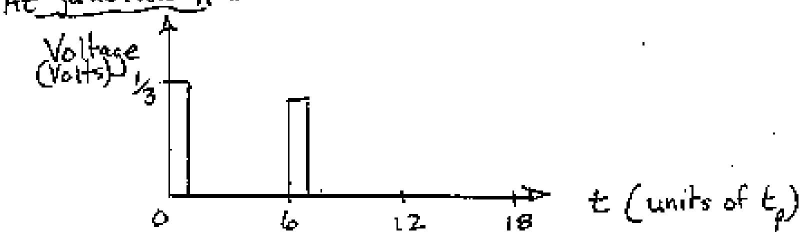
with $\tau_A^- = 1+\Gamma_A^- = 1+\frac{Z_{01}-Z_{02}}{Z_{01}+Z_{02}} = 1+\left(-\tfrac{1}{3}\right) = \tfrac{2}{3} \longrightarrow \boxed{V = 8/27}$ Volts.

Toward load: $V = \Gamma_A^- \tau_A^+ \Gamma_B^+ = \Gamma_A^-\left(\tfrac{4}{9}V\right) = \left(\frac{100-200}{300}\right)\left(\tfrac{4}{9}V\right)$

$\boxed{V = -\tfrac{4}{27}}$ Volts

(vi) Make a quick plot of reflected voltage at A versus time (in units of t_p). Does the transition section prevent reflections of pulses? **(2 marks)**

At junction A:



Reflections are not prevented from junction A or B.

3. **(20 marks)** A large flat (thin) coil of radius a with N_1 turns carries a current I in the direction shown in the accompanying figure. A small, flat coil of N_2 turns is positioned on axis a distance $2\sqrt{2}a$ from the first coil. Its radius is $\frac{a}{10}$.

(a) What is the on-axis, vector magnetic field $\bar{H}$ at the position of the second coil? **(4 marks)**

For a single loop, the axial field is

$$\bar{H} = \hat{z}\,\frac{I a^2}{2(a^2+z^2)^{3/2}}$$

For N_1 turns we find at $z = -2\sqrt{2}\,a$ that

$$\bar{H} = \hat{z}\,\frac{N_1 I a^2}{2(a^2+8a^2)^{3/2}} = \hat{z}\,\frac{N_1 I}{54a}$$

(b) What is the flux through coil 2? (Leave your answer in terms of μ). **(2 marks)**

$$\Phi = \int \bar{B}\cdot d\bar{S} = \mu_0 \int_{\text{coil 2}} \hat{z}\,\frac{N_1 I}{54a}\cdot \hat{z}\,dA = \frac{\mu_0 N_1 I}{54a}\cdot \pi\left(\frac{a}{10}\right)^2$$

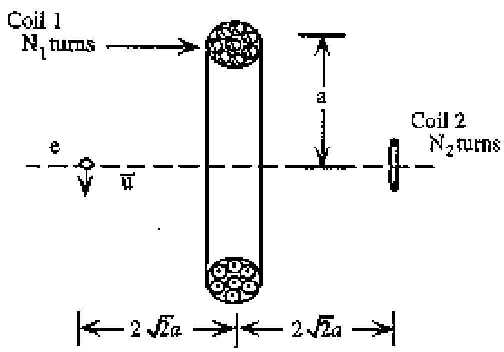
$$= \frac{\mu_0 N_1 I \pi a}{5400}$$

(c) What is the flux linkage between coils 1 and 2? **(2 marks)**

$$\Lambda = N_2 \Phi = \frac{\mu_0 N_1 N_2 I \pi a}{5400}$$

(d) What is the mutual inductance of this arrangement? **(2 marks)**

$$L_{12} = \frac{\Lambda}{I} = \frac{\mu_0 N_1 N_2 \pi a}{5400}$$

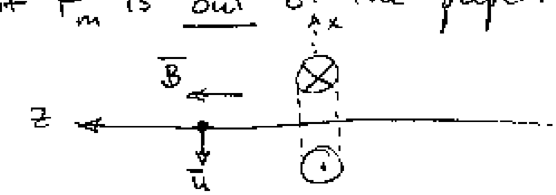


(e) What is the vectorial magnetic force (magnitude and direction) on an electron as it crosses the axis of the large coil at a distance of $2\sqrt{2a}$, travelling downward with velocity $\bar{u}$? (Ignore the small coil.) **(6 marks)**

$$\bar{F}_m = q(\bar{u}\times\bar{B}) = -e\left(\frac{u N_1 I}{54a}\right)\hat{y}$$ See diagram for directions.

$$\boxed{|\bar{F}_m| = \frac{e u N_1 I}{54a}}$$

Direction of $\bar{F}_m$ is <u>out</u> of the paper.



(f) What magnitude B_c of the magnetic field would cause the electron above, travelling at speed u perpendicular to the field, to follow a circular orbit of radius a/20? **(4 marks)** Centripetal and centrifugal forces balance.

$$\frac{mu^2}{(a/20)} = euB_c$$

$$\boxed{B_c = \frac{20mu}{ea}}$$

4 (**20 marks**)

(i) Write down the Maxwell's equation for magnetostatics that relates the field to a moving charge density. (1 **mark**)

$\bar{\nabla} \times \bar{H} = \bar{J}$

(ii) Given that a current density $\bar{J} = \frac{-2r^2}{a^4}\hat{z}$ is flowing in a straight wire of circular cross-section and radius a, find the total current by performing an appropriate integration. (**5 marks**)

$I = \int_S \bar{J} \cdot d\bar{S} = -\int_S \frac{2r^2}{a^4}\hat{z} \cdot d\bar{S}$ where S is the cross section of the wire

$I = + \int_{r=0}^{a}\int_{\phi=0}^{2\pi} \frac{2r^2}{a^4}\hat{z} \cdot \hat{z}\, r\,d\phi\,dr = -\frac{2 \cdot 2\pi}{a^4}\left[\frac{r^4}{4}\right]_0^a$

$I = +\pi \ (A).$

(iii) Apply the integration variables of part (ii) to both sides of the Maxwell equation from part (i) and use Stokes' theorem to substitute for one of the integrals. In this way, derive the circulation of the field in terms of current and find its value. (**5 marks**)

$\int_S (\bar{\nabla} \times \bar{H})\, d\bar{S} = \int \bar{J} \cdot d\bar{S} \rightarrow \oint \bar{H} \cdot d\bar{l} = I_{enclosed}$

Using the result from part (ii), the circulation is found to be

$\boxed{\oint \bar{H} \cdot d\bar{l} = +\pi \ (A)}$

(iv) Find the magnetic field (magnitude and direction) at an arbitrary radius $r < a$ inside the wire. (**6 marks**)

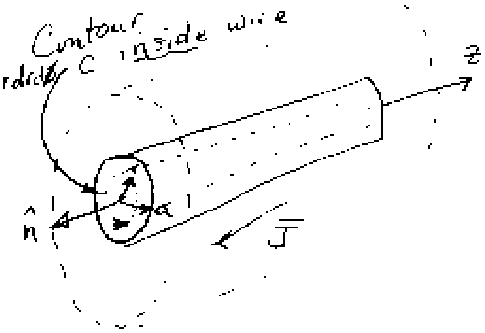
$$\oint \bar{H}\cdot d\bar{l} = I_{encl.}$$

$$I_{encl.} = -\int_{r=0}^{r}\int_{\phi=0}^{2\pi}\left(\frac{2r^2}{a^4}\right)\hat{z}\cdot(-\hat{z})\,r\,dr\,d\phi$$

$$= \pi\left(\frac{r}{a}\right)^4$$

$$2\pi r\cdot H_\phi = \pi\left(\frac{r}{a}\right)^4$$

$$\bar{H} = -\hat{\phi}\left(\frac{r^3}{2a^4}\right)$$



(v) How much power is dissipated by the magnetic field inside the wire in opposing the motion of charges? (**3 marks**)

The power, or work done, is

$$dW_m = \bar{F}\cdot d\bar{l} = q(\bar{u}\times\bar{B})\cdot d\bar{l}$$

But $d\bar{l} = \bar{u}\,dt$ and $\bar{u}\perp(\bar{u}\times\bar{B})$.

Therefore $dW_m = 0$

(no dissipation).