MARKING SCHEME & SOLUTIONS

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TOTAL: ________ / 75

ELECTROMAGNETICS I
(EECS 230 - WINTER 1998)
Mid-Term Exam #2

Date: Tuesday, March 31, 1998
Instructor: Professor S. Rand
Duration: 1.5 hours (12:00-1:30pm)
LOCATION: 1001 EECS AND 3150 DOW

No books, notes or other study materials may be used for this examination. In addition, electronic calculators and note pads are NOT PERMITTED. Show your work and reasoning, to be eligible for part marks.

ANSWER ALL QUESTIONS

Useful Formulae:

$$\overline{\nabla}\cdot\overline{D} = \rho_V \qquad \overline{\nabla}\cdot\overline{B} = 0$$

$$\overline{\nabla}\times\overline{E} = -\frac{\partial\overline{B}}{\partial t} \qquad \overline{\nabla}\times\overline{H} = \overline{J} + \frac{\partial\overline{D}}{\partial t}$$

$$V_{ba} = V_b - V_a = -\int_a^b \overline{E}\cdot d\ell$$

Honor Code: I have neither given nor received aid on this exam.

____
Name (Print)

____
Signature

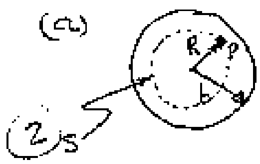
1. A <u>non-conducting</u> sphere of radius b is charged uniformly throughout its volume with a charge density ρ_V. Use the integral form of Gauss's Law to calculate

(10 marks) (a) the electric field vector at point P at a distance R from the center of the sphere, when R < b,

(10 marks) (b) the electric field vector at a point P a distance R from the center (R > b),

(5 marks) (c) and graph the magnitude of E versus R over the entire range.

(a)



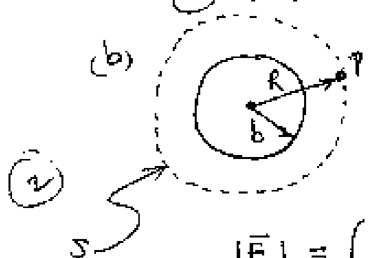
Consider a Gaussian spherical surface centered on the origin and passing through P.

$$\oint \bar{D} \cdot d\bar{S} = Q \quad 2$$

$$2 \quad \oint \varepsilon_0 |\bar{E}| \hat{R} \cdot \hat{R} R^2 \sin\theta \, d\theta \, d\phi = Q = \tfrac{4}{3}\pi R^3 \rho_v \quad 3$$

$$1 \quad |\bar{E}| = \tfrac{4\pi}{3} R^3 \rho_v / 4\pi R^2 \varepsilon_0 = \tfrac{1}{3}\left(\frac{\rho_v}{\varepsilon_0}\right) R, \text{ for } R < b$$

(b)

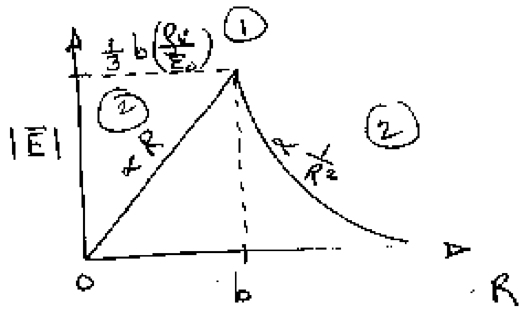


For R > b consider an external, spherical Gaussian surface passing through point P.

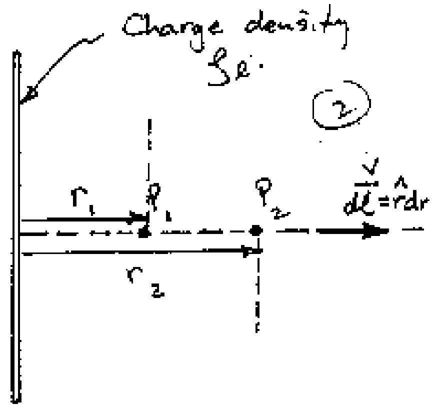
$$\oint \bar{D} \cdot d\bar{S} = Q \quad 2 \longrightarrow \varepsilon_0 |\bar{E}| 4\pi R^2 = \tfrac{4}{3}\pi b^3 \rho_v \quad 2 \quad 3$$

$$|\bar{E}| = \left(\frac{b^3 \rho_v}{3\varepsilon_0}\right) \frac{1}{R^2} \quad 1, \text{ for } R > b$$

(c)



2. (25 marks) Find the electric potential difference V_{12} between two points in air at radial distances r_1 and r_2 from an infinite line of charge with density ρ_ℓ along the z-axis.



(2)

$$V_{12} = V_1 - V_2$$
$$= -\int_2^1 \bar{E}\cdot \bar{d\ell} \quad (2)$$

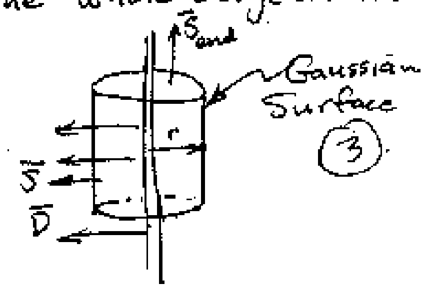
To find $\bar{E}$ explicitly we can either use Gauss's Law or integrate Coulombic field contributions over the whole charge distribution.

Gauss's Law Approach: (3)

$$\oint \bar{D}\cdot d\bar{S} = Q$$

$$L.S. = \int_{side} \bar{D}\cdot d\bar{S} + \cancel{\int_{ends} \bar{D}\cdot d\bar{S}}^{\,zero} \quad (4)$$

$$= \varepsilon E_r 2\pi r \ell \quad (3)$$



(3)

$$R.S. = \rho_\ell \ell \quad (2)$$

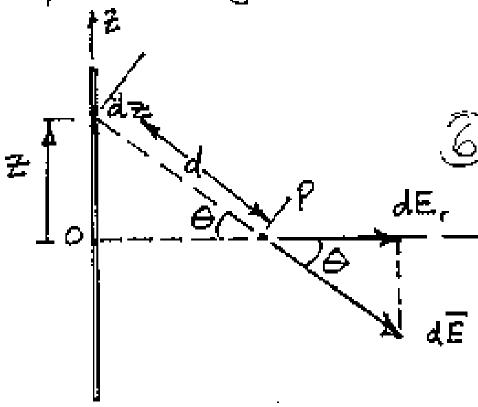
$$\therefore \varepsilon E_r 2\pi r \ell = \rho_\ell \ell \longrightarrow \boxed{E_r = \frac{\rho_\ell}{2\pi\varepsilon r}} \quad (2)$$

$$V_{12} = -\int_2^1 \frac{\rho_\ell}{2\pi\varepsilon r}\hat{r}\cdot\hat{r}\,dr = +\frac{\rho_\ell}{2\pi\varepsilon}\ln\left(\frac{r_2}{r_1}\right)$$

(3)

$$\boxed{V_{12} = \frac{\rho_\ell}{2\pi\varepsilon}\ln\left(\frac{r_2}{r_1}\right)} \quad (1)$$

Explicit Integration Approach: Alternatively we can appeal to Coulomb's Law to write



6 $|d\bar{E}| = \frac{1}{4\pi\varepsilon} \rho_\ell \frac{dz}{d^2}$

Fields along $\hat{z}$ cancel because of the existence of pairs of points with oppositely directed E_z.

Hence only the radial fields contribute to the total.

$$dE_r = |d\bar{E}| \cos\theta = \frac{1}{4\pi\varepsilon} \rho_\ell \frac{\cos\theta \, dz}{d^2} \quad 4$$

Now we replace d and dz with equivalent expressions in terms of variables r and θ.

$$\frac{z}{r} = \tan\theta \longrightarrow dz = r\sec^2\theta \, d\theta$$

$$\frac{r}{d} = \cos\theta \longrightarrow d = \frac{r}{\cos\theta} = r\sec\theta$$

$$\bar{E} = 2\hat{r} \int_{\theta=0}^{\pi/2} dE_r = \frac{\hat{r}\rho_\ell}{2\pi\varepsilon} \int_{\theta=0}^{\pi/2} \frac{\cos\theta \, r\sec^2\theta \, d\theta}{r^2 \sec^2\theta} = \frac{\rho_\ell}{2\pi\varepsilon r} \hat{r}$$

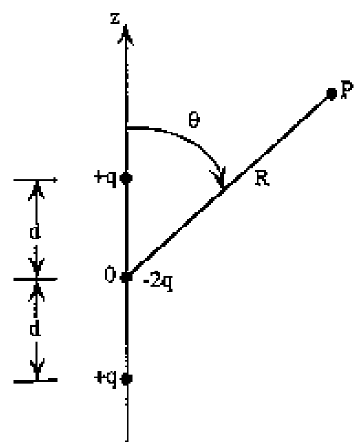
1 2 1 1

Then, in the same manner as before, we find the potential difference between two points at r_1 and r_2 to be 2

$$V_{12} = -\int_2^1 \bar{E} \cdot d\bar{\ell} = \frac{\rho_\ell}{2\pi\varepsilon} \ln\left(\frac{r_2}{r_1}\right)$$

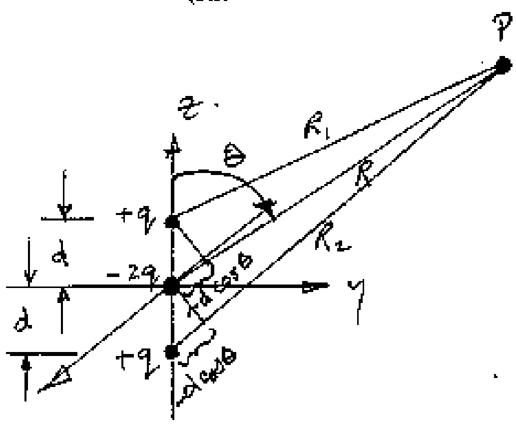
2 $\overline{d\ell}$ 2

3. A pair of electric dipoles is placed end to end as indicated in the figure below.



(15 marks) (a) Find the electric potential V at a point P specified by the distance R from the origin (R >> d) and the angle θ with respect to the z-axis.

(10 marks) (b) Find the electric field $\overline{E}$ vector at point P (again assuming R >> d). (Show how to find $\overline{E}$ from V even if you had difficulty with part (a).)



Because the charge distribution has axial symmetry along z, there is no dependence on φ. There is no loss of generality in specifying P in terms of R, θ alone.

(a) $V = \frac{1}{4\pi\varepsilon} \sum_i \frac{q_i}{R_i}$ (3)

$= \frac{1}{4\pi\varepsilon}\left[\frac{q}{R_1} - \frac{2q}{R} + \frac{q}{R_2} \right]$ 6

$$= \frac{q}{4\pi\varepsilon}\left[\frac{R_2R - 2R_1R_2 + R_1R}{R_1RR_2}\right]$$

Now $R_1 \approx R - d\cos\theta$ (2) (in the far field)

$R_2 \approx R + d\cos\theta$ (2) (in the far field)

$R_1RR_2 \approx R^3$ (in the far field)

Hence

$$V \doteq \frac{q}{4\pi\varepsilon R^3}\left[\frac{\cancel{R^2}+\cancel{dR\cos\theta} - 2(\cancel{R^2} - d^2\cos^2\theta) + \cancel{R^2} - \cancel{dR\cos\theta}}{1}\right]$$

$$= \frac{q}{4\pi\varepsilon R^3}\left[2d^2\cos^2\theta\right]$$

$$\boxed{V = \frac{qd^2}{2\pi\varepsilon}\cdot\frac{\cos^2\theta}{R^3}} \quad (1)$$

(b) $\bar{E} = -\bar{\nabla}V$ (3)

$$= -\left\{\hat{R}\frac{\partial V}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial V}{\partial\theta} + \hat{\phi}\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\right\} \quad (2)$$

$$= -\left[\hat{R}\,\frac{qd^2}{2\pi\varepsilon}\cos^2\theta\left(-\frac{3}{R^4}\right) + \hat{\theta}\,\frac{1}{R}\,\frac{qd^2}{2\pi\varepsilon R^3}\,2\cos\theta(-\sin\theta)\right]$$

(2) (2)

$$= \frac{qd^2}{2\pi\varepsilon R^4}\left[3\cos^2\theta\,\hat{R} + \sin 2\theta\,\hat{\theta}\right] \quad (1)$$