Third Problem Assignment

EECS 401

Due on January 26, 2007

PROBLEM 1 (35 points) Oscar has lost his dog in either forest A (with a priori probability 0.4) or in forest B (with a priori probability 0.6). If the dog is alive and not found by the N^{th} day of the search, it will die that evening with probability $N/(N+2)$.

If the dog is in A (either dead or alive) and Oscar spends a day searching for it in A, the conditional probability that he will find the dog that day is 0.25. Similarly, if the dos is in B and Oscar spends a day looking for it there, he will find the dog that day with probability 0.15.

The dog cannot go from one forest to the other. Oscar can search only in the daytime, and he can travel from one forest to the other only at night.

All parts of this problem are to be worked separately.

Solution Let A be the event that the dog was lost in forest A and A^c be the event that the dog was lost in forest B. Let D_n be the event that the dog dies on the nth day. Let F_n be the event that the dog is found on the nth day. Let S_n be the event that Oscar searches forest A on nth day and S_n^c be the event that he searches forest B on day n. We know that

$$\Pr(A) = 0.4 \quad \Pr(A^c) = 0.6$$

$$\Pr(D_{n+1} \mid D_n^c, F_n^c) = \frac{N}{N+2}$$

$$\Pr(F_n \mid A, S_n, F_{n-1}^c) = 0.25$$

$$\Pr(F_n \mid B, S_n^c, F_{n-1}^c) = 0.15$$

(a) In which forest should Oscar look to maximize the probability he finds his dog on the first day of search?

Solution We want to find

$$\Pr(F_1 \mid S_1) \gtrless \Pr(F_1 \mid S_1^c)$$

Now

$$\Pr(F_1 \mid S_1) = \Pr(F_1 \mid S_1, A)\Pr(A) + \underbrace{\Pr(F_1 \mid S_1, A^c)}_{=0}\Pr(A^c)$$

$$= 0.25 \times 0.4 = \boxed{0.1}$$

Similarly

$$\Pr(F_1 \mid S_1^c) = \Pr(F_1 \mid S_1^c, A)\Pr(A) + \underbrace{\Pr(F_1 \mid S_1^c, A^c)}_{=0}\Pr(A^c)$$
$$= 0.15 \times 0.6 = \boxed{0.09}$$

Thus $\Pr(F_1 \mid S_1) > \Pr(F_1 \mid S_1^c)$, so Oscar should search in forest A.

(b) Given that Oscar looked in A on the first day but didn't find his dog, what is the probability that the dog is in A?

Solution Using Baye's Rule

$$\Pr(A \mid S_1, F_1^c) = \frac{\Pr(A)\Pr(F_1^c \mid A, S_1)}{\Pr(A)\Pr(F_1^c \mid A, S_1) + \Pr(A^c)\underbrace{\Pr(F_1^c \mid A^c, S_1)}_{=1}} = \frac{0.3}{0.3 + 0.6} = \boxed{0.3333}$$

(c) If Oscar flips a fair coin to determine where to look on the first day and finds the dog on the first day, what is the probability that he looked in A?

Solution

$$\Pr(S_1 \mid F_1) = \frac{\Pr(S_1 \cap F_1)}{\Pr(F_1)}$$

Now

$$\Pr(S_1 \cap F_1) = \Pr(S_1 \cap F_1 \cap A) = \Pr(A)\Pr(S_1)\Pr(F_1 \mid S_1, A) = 0.4 \times 0.5 \times 0.25 = 0.05$$

and

$$\Pr(S_1^c \cap F_1) = \Pr(S_1^c \cap F_1 \cap A^c) = \Pr(A^c)\Pr(S_1^c)\Pr(F_1 \mid S_1^c, A^c) = 0.6 \times 0.5 \times 0.15 = 0.045$$

Using, $\Pr(F_1) = \Pr(F_1 \cap S_1) + \Pr(F_1 \cap S_1^c)$ we have

$$\Pr(S_1 \cap F_1) = \frac{0.05}{0.05 + 0.045} = \boxed{0.5263}$$

(d) Oscar has decided to look in A for the first two days. What is the a priori probability that he will find a live dog for the first time on the second day?

Solution Probability that the Oscar will find a live dog for the first time on the second day is

$$\Pr(F_2 \cap D_2^c \cap F_1^c \mid S_1, S_2) = \Pr(D_2^c|F_2, F_1^c, S_1, S_2)\Pr(F_2|F_1^c, S_1, S_2)\Pr(F_1^c|S_1, S_2)$$
$$= \Pr(D_2^c|F_1^c)\Pr(F_2|f_1^c, S_2)\Pr(F_1^c|S_1)$$
$$= \Pr(D_2^c|F_1^c) \times \left[\Pr(F_2^c|F_1^c, S_2, A)\Pr(A|F_1^c, S_1) + \underbrace{\Pr(F_2|F_1^c, S_2, A^c)}_{=0}\Pr(A^c|F_1^c, S_1)\right] \times$$
$$\left[\Pr(F_1^c|S_1, A)\Pr(A) + \underbrace{\Pr(F_1^c|S_1, A^c)}_{=1}\Pr(A^c)\right]$$
$$= 0.666 \times [0.25 \times 0.3333] \times [0.75 \times 0.4 + 0.6] = \boxed{0.05}$$

(e) Oscar has decided to look in A for the first two days. Given the fact that he was unsuccessful on the first day, determine the probability that he does not find a dead dog on the second day.

Solution Probability that Oscar does not find a dead dog at the end of the second day is

$$\begin{aligned}\Pr\left((F_2 \cap D_2)^c|F_1^c, S_1, S_2\right) &= 1 - \Pr(F_2 \cap D_2|F_1^c, S_1, S_2)\\ &= 1 - \Pr(D_2|F_2, F_1^c, S_1, S_2)\Pr(F_2|F_1^c, S_1, S_2)\\ &= 1 - \Pr(D_2|F_1^c)\Pr(F_2|F_1^c, S_1)\\ &= 1 - \frac{1}{3} \times [0.25 \times 0.333] = \boxed{0.97222}\end{aligned}$$

(f) Oscar finally found his dog on the fourth day of the search. He looked in A for the first 3 days and in B on the fourth day. What is the probability he found his dog alive?

Solution Probability that he found a live dog is

$$\begin{aligned}\Pr(D_4^c \mid F_4, F_3^c, F_2^c, F_1^c, S_1, S_2, S_3, S_4^c) &= \Pr(D_4^c \cap D_3^c \cap D_2^c|F_4, F_3^c, F_2^c, F_1^c, S_1, S_2, S_3, S_4^c)\\ &= \Pr(D_4^c|D_3^c, D_2^c, F_4, F_3^c, F_2^c, F_1^c, S_1, S_2, S_3, S_4^c)\\ &\quad \times \Pr(D_3^c|D_2^c, F_4, F_3^c, F_2^c, F_1^c, S_1, S_2, S_3, S_4^c)\\ &\quad \times \Pr(D_2^c|F_4, F_3^c, F_2^c, F_1^c, S_1, S_2, S_3, S_4^c)\\ &= \Pr(D_4^c|D_3^c, F_3^c)\Pr(D_3^c|D_2^c, F_2^c)\Pr(D_2^c|F_1^c)\\ &= \frac{2}{5} \times \frac{2}{4} \times \frac{2}{3} = \boxed{\tfrac{2}{15} = 0.13333}\end{aligned}$$

(g) Oscar finally found his dog late on the fourth day of search. If only other thing we know is that he looked in A for 2 days and in B for 2 days. What is the probability that he found his dog alive?

Solution Let L be the event that he searches twice in forest A and twice in forest B. Then

$$\begin{aligned}
\Pr(D_4^c|F_4, F_3^c, F_2^c, F_1^c, L) &= \Pr(D_4^c \cap D_3^c \cap D_2^c|F_4, F_3^c, F_2^c, F_1^c, L)\\
&= \Pr(D_4^c|D_3^c, D_2^c, F_4, F_3^c, F_2^c, F_1^c, L) \times \Pr(D_3^c|D_2^c, F_4, F_3^c, F_2^c, F_1^c, L)\\
&\quad \times \Pr(D_2^c|F_4, F_3^c, F_2^c, F_1^c, L)\\
&= \Pr(D_4^c|D_3^c, F_3^c)\Pr(D_3^c|D_2^c, F_2^c)\Pr(D_2^c|F_1^c)\\
&= \frac{2}{5} \times \frac{2}{4} \times \frac{2}{3} = \boxed{\tfrac{2}{15} = 0.13333}
\end{aligned}$$

PROBLEM 2 (12 points) The Jones family household includes Mr. and Mrs. Jones, four children, two cats, and three dogs. Every six hours there is a Jones family stroll. The rules for a Jones family stroll are:

Exactly five things (people + dogs + cats) go on each stroll.

Each scroll must include at least one parent and at least one pet.

There can never be a dog and a cat on the same stroll unless both the parents go.

All acceptable stroll groupings are equally likely.

Given that exactly one parent went on the 6 P.M. stroll, what is the probability that Rover, the oldest dog, also went?

Solution Let P the event that *exactly* one parent went and R be the event that Rover, the oldest dog, went. We want to find

$$\Pr(R|P) = \frac{\text{Number of ways exactly one parent and rover can go}}{\text{Number of ways exactly one parent goes}}$$

Now if exactly one parent and rover are going, we can choose the parent in two ways and the remaining three positons can be filled by any of the remaining two dogs or the four kids. The number of combinations of this is

$$2 \times \binom{6}{3} = 40$$

If exactly one parent is going, we can choose the parent in two ways. Both dogs and cats cannot go together, so we have two cases: either the dogs go or the cats go. Suppose the pet is a dog. So we have to choose four positions with three dogs or four kids such that at least one dog is chosen. This can be done in

$$\underbrace{\binom{4+3}{4}}_{\text{total number of ways}} - \underbrace{1}_{\text{number of ways of not choosing any dog}}$$

Suppose the pet is cat, the number of ways we can fill the remaining four positions is

$$\underbrace{\binom{4+2}{4}}_{\text{total number of ways}} - \underbrace{1}_{\text{number of ways of not choosing any cat}}$$

Thus, the total number of ways exactly one parent can go is

$$2 \times \left(\binom{7}{4} - 1 + \binom{6}{4} - 1\right) = 96$$

Hence,

$$\Pr(R|P) = \frac{40}{96} = \boxed{0.4167}$$

PROBLEM 3 (16 points) In the game of bridge the entire deck of 52 cards is dealt out to four players, each getting thirteen. Assume that cards are dealt randomly. What is the probability

Solution Number of ways of distributing 52 cards to 4 players, each getting 13 cards is

$$\binom{52}{13;13;13;13} = \frac{52!}{13!\,13!\,13!\,13!}$$

(a) one of the players receives all thirteen spades?

Solution Number of ways for one player to receive 13 spades is

$$\underbrace{\binom{4}{1}}_{\text{Choose the player}} \times \underbrace{\binom{39}{13;13;13}}_{\substack{\text{Distribute the}\\ \text{remaining 39 cards}\\ \text{between 3 players}}}$$

Thus,

$$\Pr\left(\begin{array}{c}\text{One of the players}\\ \text{receive all 13 spades}\end{array}\right) = \frac{\binom{4}{1}\binom{39}{13;13;13;13}}{\binom{52}{13;13;13;13}}$$

(b) two of the players receive all thirteen spades and each one has at least one spade?

Solution Number of ways for two of players to receive 13 spades such that each one has atleast one spade is

$$\underbrace{\binom{4}{2}}_{\substack{\text{Choose}\\ \text{the 2}\\ \text{players}}} \times \underbrace{\binom{39}{13}}_{\substack{\text{Choose}\\ \text{13 more}\\ \text{cards}}} \times \underbrace{\left[\binom{26}{13}-2\right]}_{\substack{\text{Distribute the 26}\\ \text{cards (13 spades}\\ \text{and 13 others just}\\ \text{chosen) between the}\\ \text{two players such}\\ \text{that each player}\\ \text{gets atleast 1 spade}}} \times \underbrace{\binom{26}{13}}_{\substack{\text{Distribute}\\ \text{the}\\ \text{remaining}\\ \text{26 cards}\\ \text{between the}\\ \text{remaining}\\ \text{2 players}}}$$

Thus,

$$\Pr\left(\begin{array}{c}\text{Two players receive}\\ \text{all 13 spades such}\\ \text{that each one has}\\ \text{atleast one spade}\end{array}\right) = \boxed{\frac{\binom{4}{2}\binom{39}{13}\left[\binom{26}{13}-2\right]\binom{26}{13}}{\binom{52}{13;13;13;13}}}$$

(c) each player receives one ace?

Solution Number of ways in which each player gets one ace is

$$\underbrace{\binom{48}{12;12;12;12}}_{\substack{\text{Number of ways}\\ \text{to distribute the}\\ \text{remaining 48 cards}\\ \text{between 4 players}}} \times \underbrace{4!}_{\substack{\text{Number}\\ \text{of ways of}\\ \text{distributing}\\ \text{the 4 aces}\\ \text{between}\\ \text{4 players}}}$$

Thus

$$\Pr\left(\begin{array}{c}\text{Each player}\\ \text{gets one spade}\end{array}\right) = \boxed{\dfrac{\binom{48}{12;12;12;12}4!}{\binom{52}{13;13;13;13}}}$$

(d) Suppose now we take a reduced deck of 39 cards that consists of thirteen clubs, thirteen hearts, and thirteen diamonds, and we randomly draw a hand of thirteen cards. What is the probability that this hand is void in at least one suit?

Solution

$$\Pr\left(\begin{array}{c}\text{The hand}\\ \text{drawn is}\\ \text{void in}\\ \textit{at least}\\ \text{one suit}\end{array}\right) = \Pr\left(\begin{array}{c}\text{The hand}\\ \text{drawn is}\\ \text{void in}\\ \textit{exactly}\\ \text{one suit}\end{array}\right) + \Pr\left(\begin{array}{c}\text{The hand}\\ \text{drawn is}\\ \text{void in}\\ \textit{exactly}\\ \text{two suits}\end{array}\right) \underbrace{+ \Pr\left(\begin{array}{c}\text{The hand}\\ \text{drawn is}\\ \text{void in}\\ \textit{exactly}\\ \text{three suits}\end{array}\right)}_{=0}$$

Further, number of ways to draw hands that are void in exactly one suit

$$\underbrace{\binom{3}{1}}_{\substack{\text{Number}\\ \text{of ways}\\ \text{to choose}\\ \text{the suit}}} \times \underbrace{\left[\binom{26}{13} - 2\right]}_{\substack{\text{Number of ways to draw}\\ \text{13 cards from 26 cards}\\ \text{that is not void in any suit}}}$$

and number of ways to draw hands that are void in exactly two suits

$$\underbrace{\binom{3}{2}}_{\substack{\text{Choose the}\\ \text{two suits}\\ \text{in which}\\ \text{we are void}}} \times 1$$

Thus,

$$\Pr\left(\begin{array}{c}\text{The hand}\\ \text{drawn is}\\ \text{void in}\\ \textit{at least}\\ \text{one suit}\end{array}\right) = \boxed{\frac{\binom{3}{1}\left[\binom{26}{13}-2\right]+\binom{3}{2}}{\binom{39}{13}}}$$

PROBLEM 4 (16 points)

(a) We transmit a bit of information which is 0 with probability p and 1 with probability $1-p$. It passes through a binary symmetric channel (BSC) with crossover probability ϵ. Suppose that we observe a 1 at the output. Find the conditional probability p_1 that the transmitted bit is a 1.

Solution Let A be the event that 1 is transmitted and A^c be the event that 0 is transmitted. Let B_n be the event that the n_{th} bit we receive is 1. Then using the law of Total Probability and Baye's rule we have

$$p_1 = \Pr(A|B_1) = \frac{\Pr(B_1|A)\Pr(A)}{\Pr(B_1|A)\Pr(A) + \Pr(B_1|A^c)\Pr(A^c)} = \frac{(1-\varepsilon)(1-p)}{(1-\varepsilon)(1-p)+\varepsilon p}.$$

(b) The same bit is transmitted again through the BSC and you observe another 1. Find a formula to update p_1 to obtain p_2, the conditional probability that the transmitted bit is a 1. (You may find equation (1) from the last homework assignment useful.)

Solution Note that B_1 and B_2 are conditionally independent given A. Now,

$$
\begin{aligned}
p_2 = \Pr(A \mid B_1, B_2) &= \frac{\Pr(B_2 \mid A, B_1) \Pr(A \mid B_1)}{\Pr(B_2 \mid B_1)} \\
&= \frac{\Pr(B_2 \mid A, B_1) \Pr(A \mid B_1)}{\Pr(B_2 \mid A, B_1) \Pr(A \mid B_1) + \Pr(B_2 \mid A^c, B_1) \Pr(A^c \mid B_1)} \\
&= \frac{\Pr(B_2 \mid A, B_1) \Pr(A \mid B_1)}{\Pr(B_2 \mid A) \Pr(A \mid B_1) + \Pr(B_2 \mid A^c) \Pr(A^c \mid B_1)} \\
&= \frac{(1-\varepsilon)p_1}{(1-\varepsilon)p_1 + \varepsilon(1-p_1)}
\end{aligned}
$$

(c) Using the preceding part or otherwise, calculate p_n, the probability that the transmitted bit is a 1 given than you have observed n 1's at the BSC output. What happens as $n \to \infty$?

Solution Using the same argument as in part 2, we get

$$
p_n = \frac{(1-\varepsilon)p_{n-1}}{(1-\varepsilon)p_{n-1} + \varepsilon(1-p_{n-1})} = \frac{(1-\varepsilon)^n p}{(1-\varepsilon)^n p_0 + \varepsilon^n(1-p_0)} = \boxed{\frac{p_0}{p_0 + \left(\frac{\varepsilon}{1-\varepsilon}\right)^n (1-p_0)}}
$$

where $p_0 = 1 - p$.

As $n \to \infty$, we have

$$
p_n = \begin{cases} 1, & \varepsilon < 1/2\,; \\ p_0, & \varepsilon = 1/2\,; \\ 0 & \varepsilon > 1/2\,; \end{cases}
$$

These results meet the intuitions. When $\varepsilon < 1/2$, it means the observations are positively correlated with the bit transmitted, thus knowing the observations $B_n, n = 1, 2\ldots$ will increase the conditional probability of A given the observations, until it hits 1. When $\varepsilon = 1/2$, it means the observations are independent to the bit transmitted, thus given a bunch of observations $B_n, n = 1, 2\ldots$ won't change the conditional probability of A given the observations, which is the same as the prior probability p. When $\varepsilon > 1/2$, it means the observations are negatively correlated with the bit transmitted, thus knowing the observations $B_n, n = 1, 2\ldots$ will decrease the conditional probability of A given the observations, until it hits 0.

(d) You declare that the transmitted bit is a 1 whenever p_n exceeds 0.99. How long do you have to wait? How does your answer qualitatively depend on p and ϵ? Does it make intuitive sense? Explain.

Solution By part (c), to get $p_n \geqslant 0.99$, the following condition must be true:

$$p_n = \frac{p_0}{p_0 + \left(\frac{\varepsilon}{1-\varepsilon}\right)^n (1-p_0)} \geqslant 0.99$$
$$\Longrightarrow \quad \log \frac{p_0}{99(1-p_0)} \geqslant n \log \frac{\varepsilon}{1-\varepsilon} \tag{1}$$

There are three cases

- when $\varepsilon < 1/2$, we need

$$n \geqslant \frac{\log p_0 - \log[99(1-p_0)]}{\log \varepsilon - \log[1-\varepsilon]},$$

  The intuition is the following. In this case, knowing the observations $B_n, n = 1, 2 \ldots$ will monotonously increase p_n, until it hits 1. So we expect after the number of observations n exceeds a threshold, p_n can be no less than 0.99.

- when $\varepsilon = 1/2$, the right hand side of (1) is 0. Thus for (1) to be true, the left hand side must be no less than 0, which implies $p_0 \geqslant 0.99$. If $p_0 \geqslant 0.99$, then for any n, p_n exceeds 0.99; otherwise no p_n exceeds 0.99.

  The intuition is that when $\varepsilon = 1/2$, the observations are independent to the bit transmitted. Thus the conditional probability of A given the observations won't change as we get more observations. Consequently, the only possible way to have $p_n > 0.99$ is the prior probability $p_0 \geqslant 0.99$.

- When $\varepsilon > 1/2$, we need the following to be true

$$n \leqslant \frac{\log p_0 - \log[99(1-p_0)]}{\log \varepsilon - \log[1-\varepsilon]}$$

  Since n is no less than 1, the above inequality further requires

$$\log p_0 - \log[99(1-p_0)] \geqslant \log \varepsilon - \log[1-\varepsilon] \quad \Longrightarrow \quad p_0 \geqslant \frac{99\varepsilon}{1+98\varepsilon}.$$

  The intuition is that when $\varepsilon > 1/2$, knowing the observations $B_n, n = 1, 2, \ldots$ will monotonously decrease p_n, until it hits 0. Thus the only possible way for $p_n \geqslant 0.99$ is that p_0 is large enough so as to make $p_n \geqslant 0.99$ for n less than a certain threshold.

PROBLEM 5 (21 points) In the commuication network given below, link failures are independent, and each link has a probability of failure p Consider the physical situation before you write anything. A can ommunicate with B as longs they are connected by at least one path which contains only in-service links.

(a) Given that exactly five links have failed, determine the probability that A can still communicate with B.

> **Solution** Let A be the event that "A can communicate with B" and B be the event that "exactly five links have failed". We are asked to find $\Pr(A \mid B)$.
>
> $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$
>
> Now
>
> $$\Pr(A \cap B) = \Pr(\{a\bar{b}\bar{c}d\bar{e}\bar{f}g\bar{h}, \bar{a}bc\bar{d}\bar{e}\bar{f}\bar{g}h\}) = p^5(1-p)^3 + p^5(1-p)^3 = 2p^5(1-p)^3$$
>
> and
>
> $$\Pr(B) = \binom{8}{5} p^5(1-p)^3$$
>
> Thus
>
> $$\Pr(A \cap B) = \frac{2p^5(1-p)^3}{\binom{8}{3}p^5(1-p)^3} = \boxed{\frac{1}{28}}$$

(b) Given that exactly five links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

> **Solution** Let C be the event that either g or h (*but not both*) are working.
>
> $$\Pr(C \mid B) = \frac{\Pr(C \cap B)}{\Pr(B)}$$
>
> To find $\Pr(C \cap B)$ we need to choose one gate out of g and h that is working (and the other is not) and choose four gates out of the remaining six such that these four are not working (and the remaining two are working). Thus,
>
> $$\Pr(C \mid B) = \frac{\binom{2}{1}p(1-p)\binom{6}{4}p^4(1-p)^2}{\binom{8}{5}p^5(1-p)^3} = \boxed{\frac{15}{28}}$$

(c) Given that a, d and h have failed (but no information about the condition of the other lnks), determine the probability that A can communicate with B.

Solution Let D be the event that a, d and h have failed. Given this the only way A can communicate with B is if gates b, b, f and g are working. Let this event be E. As the gates fail independently, events E and D are independent. Thus

$$\Pr(E \mid D) = \Pr(E) = (1-p)^4$$