

Fifth Problem Assignment

EECS 401

Due on Feb 9, 2007

PROBLEM 1 (24 points) Discrete random variable X is described by the PMF

$$p_X(x) = \begin{cases} K - \frac{x}{12}, & \text{if } x = 0, 1, 2 \\ 0, & \text{for all other values of } x \end{cases}$$

Let $D_1, D_2, \dots, D_N$ represent N successive independent experimental values of random variable X .

(1) Determine the numerical value of K .

Solution For $p_X(x)$ to be a valid probability measure, it must sum to 1. Thus,

$$\begin{aligned} K + K - \frac{1}{12} + K - \frac{2}{12} &= 1 \\ \Rightarrow K &= \frac{5}{12} \end{aligned}$$

(2) Determine the probability that $D_1 > D_2$.

Solution

$$\begin{aligned} \Pr(D_1 > D_2) &= \Pr(\{(1, 0), (2, 0), (2, 1)\}) \\ &= \Pr((1, 0)) + \Pr((2, 0)) + \Pr((2, 1)) \\ &= \frac{1}{3} \frac{5}{12} + \frac{1}{4} \frac{5}{12} + \frac{1}{4} \frac{1}{3} = \frac{47}{144} \end{aligned}$$

(3) Determine the probability that $D_1 + D_2 + \dots + D_N \leq 1.0$

Solution

$$\begin{aligned} \Pr(D_1 + D_2 + \dots + D_N \leq 1) &= \Pr(\text{All } D_i \text{'s are zero}) + \Pr(\text{One } D_i \text{ is one and rest are zeros}) \\ &= \left(\frac{5}{12}\right)^N + N \frac{1}{3} \left(\frac{5}{12}\right)^{N-1} \end{aligned}$$

(4) Define $R = \max(D_1, D_2)$ and $S = \min(D_1, D_2)$. Determine the following PMF's for all values of their arguments:

Solution $r = \max(D_1, D_2)$ and $s = \min(D_1, D_2)$. We have

D_1	D_2	$\Pr(D_1, D_2)$	r	s	t
0	0	$\frac{25}{144}$	0	0	1
0	1	$\frac{5}{36}$	1	0	1
0	2	$\frac{5}{48}$	2	0	1
1	0	$\frac{5}{36}$	1	0	2
1	1	$\frac{1}{9}$	1	1	1
1	2	$\frac{1}{12}$	2	1	1
2	0	$\frac{5}{48}$	2	0	3
2	1	$\frac{1}{12}$	2	1	$\frac{3}{2}$
2	2	$\frac{1}{16}$	2	2	1

Hence

(a) $p_S(s)$

Solution

$$p_S(s) = \begin{cases} \frac{25}{144}, & s = 0 \\ \frac{5}{18}, & s = 1 \\ \frac{1}{16}, & s = 2 \end{cases}$$

(b) $P_{R|S}(r|0)$

Solution

$$p_{R|S}(r|0) = \begin{cases} \frac{5}{19}, & r = 0 \\ \frac{8}{19}, & r = 1 \\ \frac{6}{19}, & r = 2 \end{cases}$$

(c) $P_{R,S}(r, s)$

Solution

$$p_{R,S}(r, s) = \begin{cases} \frac{25}{144}, & (r, s) = (0, 0) \\ \frac{5}{18}, & (r, s) = (1, 0) \\ \frac{5}{24}, & (r, s) = (2, 0) \\ \frac{1}{9}, & (r, s) = (1, 1) \\ \frac{1}{6}, & (r, s) = (2, 1) \\ \frac{1}{16}, & (r, s) = (2, 2) \end{cases}$$

(d) $P_T(t)$ with $T = (1 + D_1)/(1 + S)$

Solution

$$p_T(t) = \begin{cases} \frac{97}{144}, & t = 1 \\ \frac{1}{12}, & t = \frac{3}{2} \\ \frac{5}{36}, & t = 2 \\ \frac{5}{48}, & t = 3 \\ 0 & \text{otherwise} \end{cases}$$

- (5) Determine the expected value and variance of random variable S defined above.

Solution Using the PMF of S given by part (i) above, we have

$$E[S] = \frac{29}{72}$$

- (6) Given $D_1 + D_2 \leq 2.0$, determine the conditional expected value and conditional variance of random variable S defined above.

Solution If $D_1 + D_2 \leq 2$, then S can only take values 0 and 1. Let A be the event that $D_1 + D_2 \leq 2$. We have

$$\Pr(A) = \frac{37}{48}$$

and

$$p_{S|A}(s) = \frac{\Pr(\min(D_1, D_2) = s, A)}{\Pr(A)} = \begin{cases} \frac{95}{111}, & s = 0 \\ \frac{16}{111}, & s = 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$E[S|A] = \frac{16}{111}$$

$$\text{and } \text{Var}(S) = \frac{1520}{12321}$$

PROBLEM 2 (12 points) A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability 0.3, and his second appointment will lead independently to a sale with probability 0.6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the PMF of X , the total dollar value of the sales this salesman makes.

Solution Let (S_1, S_2) denote the amounts of sale of first and the second appointment. Thus we have,

Sale	Total	Probability
(0, 0)	0	$0.7 \times 0.4 = 0.28$
(500, 0)	500	$0.3 \times 0.5 \times 0.4 = 0.06$
(0, 500)	500	$0.7 \times 0.6 \times 0.5 = 0.21$
(500, 500)	1000	$0.3 \times 0.5 \times 0.6 \times 0.5 = 0.045$
(1000, 0)	1000	$0.3 \times 0.5 \times 0.4 = 0.06$
(0, 1000)	1000	$0.7 \times 0.6 \times 0.5 = 0.21$
(500, 1000)	1500	$0.3 \times 0.5 \times 0.6 \times 0.5 = 0.045$
(1000, 500)	1500	$0.3 \times 0.5 \times 0.6 \times 0.5 = 0.045$
(1000, 1000)	2000	$0.3 \times 0.5 \times 0.6 \times 0.5 = 0.045$

Thus the PMF is given by

$$p_X(x) = \begin{cases} 0.28, & x = 0 \\ 0.27, & x = 500 \\ 0.315 & x = 1000 \\ 0.09 & x = 1500 \\ 0.045 & x = 2000 \\ 0 & \text{otherwise} \end{cases}$$

PROBLEM 3 (12 points) Let X , Y , and Z be independent geometric random variables with the same PMF:

$$p_X(k) = p_Y(k) = p_Z(k) = p(1-p)^{k-1},$$

where p is a scalar with $0 < p < 1$. Find $P(X = k \mid X + Y + Z = n)$.

Hint: Try thinking in terms of coin tosses.

Solution Note that

$$\Pr(X = k \mid X + Y + Z = n) = \frac{\Pr(X = k, X + Y + Z = n)}{\Pr(X + Y + Z = n)}$$

We have

$$\Pr(X = k, X + Y + Z = n) = \Pr(X = k, Y + Z = n - k) = \Pr(X = k) \Pr(Y + Z = n - k)$$

where the last equality follows from the fact that X , Y and Z are independent.

Consider the following experiment. We repeatedly toss a coin with probability of head equal to p . Let X be the number of tosses up to and including the first head, Y be the number of tosses following the first head up to and including the second head, and Z be the number of tosses following the second head up to and including the third head. The random variables X , Y and Z are independent having the distribution as given above.

Then $\{Y + Z = n - k\}$ is the event that the second head occurs on the $(n - k)$ th toss. For this event to occur, we need to get two heads and $n - k - 2$ tails, and the second head must occur on toss $n - k$, but the first head could occur at any of the previous $n - k - 1$ tosses. Therefore

$$\Pr(Y + Z = n - k) = (n - k - 1)p^2(1 - p)^{n-k-2}$$

Similarly, $\{X + Y + Z = n\}$ is the event that the third head occurs on the n th toss. For this to occur, we need to get 3 heads and $n - 3$ tails. The third head must occur on the n th toss, but the first two heads could occur at any of the previous $n - 1$ tosses. The total number of ways two heads can occur in $n - 1$ tosses is $\binom{n-1}{2}$. Therefore

$$\Pr(X + Y + Z = n) = \binom{n-1}{2} p^3 (1 - p)^{n-3}, \quad n \geq 3.$$

Combining the preceding equations we have

$$\begin{aligned} \Pr(X = k | X + Y + Z = n) &= \begin{cases} \frac{(n - k - 1)p^3(1 - p)^{n-3}}{\binom{n-1}{2} p^3(1 - p)^{n-3}} & \text{if } n \geq 3, k = 1, 2, \dots, n - 2, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{2(n-k-1)}{(n-1)(n-2)} & \text{if } n \geq 3, k = 1, 2, \dots, n - 2, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

PROBLEM 4 (30 points) Nick shops for probability books for K hours, where K is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n|k) = \frac{1}{k}, \quad \text{for } n = 1, \dots, k.$$

Solution We are given that

$$p_K(k) = \begin{cases} 1/4 & \text{if } k = 1, 2, 3, 4, \\ 0 & \text{otherwise} \end{cases}$$

and

$$p_{N|K}(n|k) = \begin{cases} 1/k & \text{if } n = 1, \dots, k, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint PMF of K and N .

Solution Applying the chain rule, we have

$$p_{N,K}(n, k) = p_K(k)p_{N|K}(n|K),$$

Substituting $p_K(k)$ and $p_{N|K}(n|k)$ we obtain

$$p_{N,K}(n, k) = \begin{cases} 1/(4k) & \text{if } k = 1, 2, 3, 4 \text{ and } n = 1, \dots, k, \\ 0 & \text{otherwise} \end{cases}$$

(b) Find the marginal PMF of N .

Solution The marginal PMF $p_N(n)$ is given by

$$p_N(n) = \sum_k p_{N,K}(n, k) = \sum_{k=n}^4 \frac{1}{4k}$$

that is,

$$p_N(n) = \begin{cases} \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} = \frac{25}{48}, & \text{if } n = 1, \\ \frac{1}{8} + \frac{1}{12} + \frac{1}{16} = \frac{13}{48}, & \text{if } n = 2 \\ \frac{1}{12} + \frac{1}{16} = \frac{7}{48}, & \text{if } n = 3 \\ \frac{1}{16} = \frac{3}{48}, & \text{if } n = 4 \end{cases}$$

(c) Find the conditional PMF of K given that $N = 2$.

Solution We have

$$p_{K|N}(k|2) = \frac{p_{N,K}(2, k)}{p_N(2)} = \begin{cases} \frac{6}{13} & \text{if } k = 2, \\ \frac{4}{13} & \text{if } k = 3, \\ \frac{3}{13} & \text{if } k = 4, \\ 0 & \text{otherwise} \end{cases}$$

(d) Find the conditional mean and variance of K , given that he bought at least two but no more than three books.

Solution Let A be the event $2 \leq N \leq 3$. We know that

$$p_{K|A}(k) = \frac{\Pr(K = k, A)}{\Pr(A)},$$

$$\Pr(A) = p_N(2) + p_N(3) = \frac{5}{12},$$

$$\Pr(K = k, A) = \begin{cases} \frac{1}{8} & \text{if } k = 2, \\ \frac{1}{12} + \frac{1}{12} & \text{if } k = 3, \\ \frac{1}{16} + \frac{1}{16} & \text{if } k = 4, \\ 0 & \text{otherwise} \end{cases}$$

and finally

$$p_{K|A}(k) = \begin{cases} \frac{3}{10} & \text{if } k = 2, \\ \frac{2}{5} & \text{if } k = 3, \\ \frac{3}{10} & \text{if } k = 4, \\ 0 & \text{otherwise} \end{cases}$$

The conditional PMF of K given A is symmetric around $k = 3$, so

$$E[K|A] = \boxed{3}$$

The conditional variance of K given A is given by

$$\text{var}(K|A) = E[(K - E[K|A])^2|A] = \frac{3}{10} \cdot (2 - 3)^2 + \frac{2}{5} \cdot 0 + \frac{3}{10} \cdot (4 - 3)^2 = \boxed{\frac{3}{5}}$$

- (e) The cost of each book is a random variable with mean \$30. What is the expected value of his total expenditure?

Hint: Condition on the events $\{N = 1\}, \dots, \{N = 4\}$, and use the total expectation theorem.

Solution We are given that $E[C_i] = 30$, where C_i is the cost of book i . Let T be the total cost, so that $T = C_1 + \dots + C_N$. (Notice that N here is a random variable.) Using the total expectation theorem we have

$$\begin{aligned} E[T] &= E[E[T|N]] \\ &= E\left[E\left[\sum_{i=1}^N C_i | N\right]\right] \\ &= E[N \cdot 30] = 30E[N] \\ &= 30 \cdot \left(1 \cdot \frac{25}{48} + 2 \cdot \frac{13}{48} + 3 \cdot \frac{7}{48} + 4 \cdot \frac{3}{48}\right) = 30 \cdot \frac{84}{48} \\ &= \boxed{52.5} \end{aligned}$$

PROBLEM 5 (12 points)

- (a) Let X be a random variable that takes on nonnegative integer values. Show that

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

Hint: Start with the above formula and interchange the order of the summation.

Solution Consider the right hand side

$$\begin{aligned}\sum_{k=1}^{\infty} \Pr(X \geq k) &= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p_X(X=i) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i p_X(X=k) = \sum_{i=1}^{\infty} i p_X(X=i) \\ &= E[X]\end{aligned}$$

- (b) Use the formula in the previous part to compute the expectation of a random variable Y with the geometric PMF $p_Y(l) = p(1-p)^{l-1}$, $l = 1, 2, \dots$ (Here p is a constant between 0 and 1.)

Solution For a geometric distribution

$$\Pr(X \geq k) = \sum_{l=k}^{\infty} p(1-p)^{l-1} = p(1-p)^{k-1} \sum_{i=0}^{\infty} (1-p)^i = (1-p)^{k-1}$$

From the previous part we know that

$$E[X] = \sum_{k=1}^{\infty} \Pr(X \geq k) = \sum_{k=1}^{\infty} (1-p)^{k-1} = \boxed{\frac{1}{p}}$$

PROBLEM 6 (10 points) Consider the following two cellphone billing plans.

- (a) You are charged a monthly fee of \$20, which includes thirty free minutes. Beyond this you pay \$0.50 for each extra minute.

Solution Let $c(X)$ denote the amount of money that one has to pay for using X minutes. Then

$$c(x) = \begin{cases} 20 & x \leq 30 \\ 20 + (x - 30) \times 0.5 & x > 30 \end{cases}$$

Thus,

$$\begin{aligned}E[X] &= \sum_{x=1}^{\infty} c(x) p_X(x) = 20 + \sum_{x=31}^{\infty} 0.5(x-30)p(1-p)^{x-1} \\ &= 20 + \sum_{y=1}^{\infty} 0.5yp(1-p)^{y+29} = 20 + 0.5(1-p)^{30} \sum_{y=1}^{\infty} yp(1-p)^{y-1} \\ &= \boxed{20 + 0.5 \frac{(1-p)^{30}}{p}}\end{aligned}$$

- (b) You are charged a monthly fee of \$15, which has no free minutes. You pay \$1 for each minute you use.

Solution In this case we have $c(x) = 15 + x$. Thus,

$$E[c(X)] = E[15 + X] = 15 + E[X] = 15 + \frac{1}{p}$$

Suppose that the number of minutes you use every month is a geometric random variable X with mean $1/p$. What is the expected monthly cost of each of the two plans?
