EECS 401 **Midterm 1 Solutions** **Feb. 25, 2000**

Exam Statistics: Average = 104.5, Median = 107, Std. deviation = 13.9, High score = 124

Grade ranges: A = 125-110 (25 A's); B = 110-90 (29 B's); C = 90-65 (7 C's); D = 65- (1 D)

1. *Short answer questions (no explanations or justifications required)*

[8] (a) ***Complete*** *the following: A probability model for a random experiment consists of the items in the following list (just name the items):*

A variable name, a sample spac]e, a probability law

[7] (b) ***State*** *the "law of total probability". (There are two versions, either will suffice.)*

If $B_1,\ldots,B_n$ are a partition of the sample space (i.e. they are disjoint events and their union is the entire sample space) and A is some other event then

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i) \quad \text{or} \quad P(A) = \sum_{i=1}^{n} P(B_i)\, P(A|B_i)$$

(Note on terminology: the entire collection $\{B_1,...,B_n\}$ is called a partition. The individual sets B_i are called elements of the partitions. They are not themselves called "partitions".)

[5] (c) ***Give*** *a brief precise definition of a "continuous" random variable. (You don't have to define "random variable", just "continuity" of a random variable.)*

A random variable X is continuous if $Pr(X=x) = 0$ for all $x \in (-\infty,\infty)$.

The book gives a less direct definition: A r.v. X is continuous if its cdf F_X is a continuous function. It also says that the F_X must be smooth enough so that it can be written as the integral of some function. Actually, this last condition defines a special kind of continuous random variable

[5] (d) ***Give*** *a brief precise formula for the "conditional probability of A given B" in terms of unconditional probabilities.*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{assuming } P(B) > 0; \text{ it is undefined if } P(B) = 0$$

[5] (e) ***Give*** *a brief precise definition of the "probability density function" of a continuous random variable X.*

The probability density function of a cont. r.v. X is a function $f: (-\infty,\infty) \to [0,\infty)$ such that

$$Pr(X \in A) = \int_A f(x)\, dx$$

Alternatively, an indirect definition is: the probability density fucntion is the derivative of the cdf.

2. *Suppose events E and F are independent with probabilities $P(E) = .7$ and $P(F) = .4$, respectively.*

[10] (a) ***Find*** *the probability that E or F occurs.*

$\mathbf{P(E \cup F)}$ = $P(E) + P(F) - Pr(E \cap F) = P(E) + P(F) - P(E)P(F) = .7 + .4 - .28 =$ **.82**.

[10] (b) ***Find*** *the probability that exactly one of the events E or F occurs. (Hint: A Venn diagram might help.)*

The event "exactly one of the events E or F occurs" corresponds to the set $(E-F) \cup (F-E)$, whose probability is

$P(E-F)+P(F-E) = P(E)-P(E \cap F) + P(F)-P(E \cap F) = .7 - .28 + .4 - .28 =$ **.54**

3. *A rental car company has 10 compact cars and 5 full-size cars. Four of the 15 cars are randomly selected for a safety inspection.* ***What is*** *the probability of getting two of each kind?*

Each outcome of this experiment is a set of 4 cars (unordered). There are $\binom{15}{4}$ ways of choosing 4 cars from a set of 15 (ordering doesn't matter)); i.e. there are $\binom{15}{4}$ outcomes in the sample space. Each outcome has probability $1/\binom{15}{4}$; i.e. all outcomes have the same probability. The event/set of interest is the set of all sets of 4 cars such that 2 are compact and 2 are full size. The probability of this event equals the number of outcomes in the set times $1/\binom{15}{4}$. So we must count how many outcomes are in the set. There are $\binom{10}{2}$ ways of choosing 2 compacts out of the 10, and there are $\binom{5}{2}$ ways of choosing two full size cars out of the 5. So the total number of ways of choosing 2 compacts and 2 full-size cars is $\binom{10}{2} \times \binom{5}{2}$. Therefore,

[20] $$\textbf{Pr(2 cars of each kind)} = \binom{10}{2} \times \binom{5}{2} \times \frac{1}{\binom{15}{4}} = \frac{45 \times 10}{1365} = \mathbf{0.3297}$$

4. *At an electronics plant, it is known from past experience that the probability is 0.86 that a new worker who has attended the company's training program will meet the production quota and that the corresponding probability is 0.35 for a new worker who has not attended the training program. It is also known that 80% of all new workers attend the training program.* ***Find*** *the probability that a randomly chosen new worker who meets the production quota will have attended the training program.* ***Show*** *your work.*

[20] Let M = event that work meets production quota.
Let T = event that worker attended the training program.
We need to find P(T|M) = prob worker worker has attended training program given he/she met quota
From the problem statement we deduce:

$$P(M|T) = 0.86, \quad P(M|T^c) = 0.35, \quad P(T) = 0.8, \quad P(T^c) = 0.2.$$

We now use Bayes rule:

$$P(T|M) = \frac{P(M|T)P(T)}{P(M)}$$

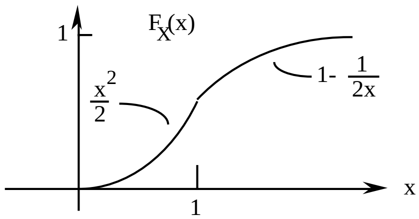
To find P(M) we use the law of total probability

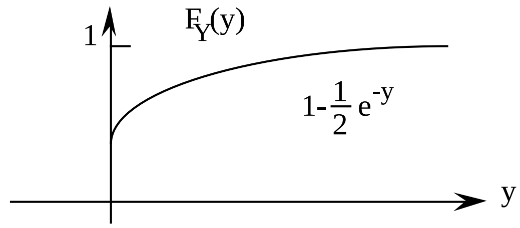
$$P(M) = P(M|T)\,P(T) + P(M|T^c)\,P(T^c) = 0.86 \times 0.8 + 0.35 \times 0.2 = .758$$

Therefore

$$\mathbf{P(T|M)} = \frac{0.86 \times 0.8}{.758} = \mathbf{0.908}$$

5. *Random variables X and Y have the probability distribution functions shown below.*





Note: On this problem, points were reassigned to total 35 rather than 40.

[5] (a) *What kind of random variable is X?*

continuous

[7] (b) *Find the probability density function of X.*

$$\mathbf{f_X(x)} = \frac{d}{dx} F_X(x) = \begin{cases} \mathbf{0}, & \mathbf{x < 0} \\ \mathbf{x}, & \mathbf{0 \le x \le 1} \\ \mathbf{\frac{1}{2x^2}}, & \mathbf{1 < x} \end{cases}$$

[5] (c) *Find $Pr(X < 2)$.*

$$\mathbf{Pr(X<2)} = F_X(2) - Pr(X=2) = 1 - \frac{1}{2\times 2} - 0 = \mathbf{\frac{3}{4}}$$

[5] (d) *What kind of random variable is Y?*

mixed (notice that $Pr(Y=0) = \frac{1}{2}$

[8] (e) *Find the probability density function of Y.*

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \mathbf{\frac{1}{2}\,\delta(y) + \frac{1}{2}\,u(y)} = \mathbf{\frac{1}{2}\,\delta(y)} + \begin{cases} \mathbf{\frac{1}{2} e^{-y}}, & \mathbf{y \ge 0} \\ \mathbf{0}, & \mathbf{y < 0} \end{cases}$$

[5] (f) *Find $Pr(Y < \frac{1}{2})$.*

$$\mathbf{Pr(Y<\tfrac{1}{2})} = F_Y(\tfrac{1}{2}) - Pr(Y=\tfrac{1}{2}) = 1 - \frac{1}{2} e^{-1/2} - 0 = 1 - \frac{1}{2} e^{-1/2} = \mathbf{.697}$$

[125 points total]