Average Grade: 83.5 out of 100. High grade = 100 (3 of them)

Grade Ranges: A = 100 to 85, B = 85 to 67, C = 67 to 45. 44 A's, 27 B's, 8 C's.

1. *Short answer questions*

[5] (a) ***State*** *the formula for the conditional probability of event B given event A.*

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

[5] (b) ***State*** *Bayes rule. (Two versions were given in class; either will do)*

One version:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Another version:

If $B_1, \quad , B_n$ are disjoint events such that $B_1 \quad B_2 \qquad B_n$ contains A, then

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{j=1}^{n} P(A|B_j)P(B_j)}$$

[10] (c) ***State*** *the axioms of probability.*

(1) $P(A) \geq 0$, for every event A

(2) $P(S) = 1$, where S is the set of all possible otucomes

(3) $P(A_1 \quad A_2 \quad \ldots) = \sum_i P(A_i)$ when $A_1, A_2, \qquad$ are disjoint.

[5] (d) ***State****, mathematically, what it means for three events A, B, C to be independent.*

All of the following must be true:

$P(A \cap B) = P(A)\,P(B)$, $P(A \cap C) = P(A)\,P(C)$, $P(B \cap C) = P(B)\,P(C)$,

$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

One call state this in terms of conditional probabilities.

2. *E, F and G are events such that*

$P(E \cap F) = 0.2$, $P(E \cap G) = 0.6$, $P(F \cap G) = 0$, $P(F \quad G) = 1$, $P(F) = 0.25$

[7] (a) ***Find*** *the probability of E.*

Notice that F and G are essentially disjoint because $P(F \cap G) = 0$. They could have a small overlap but the probability of the overlap is zero. Similarly, F G is essentially everything because $P(F \quad G) = 1$. There may be a few points not in F G but their probability is zero. We may as well assume F and G are disjoint that F G equals the sample space S. Then

P(E) $= P(E \cap F) + P(E \cap G) = .2 + .6 =$ **.8**

[7] (b) ***Find*** *the probability of* $E \quad F$

$$P(E \quad F) = P(E) + P(F) - P(E \cap F) = .8 + .25 - .2 = \mathbf{.85}$$

[6] (c) ***Are*** *E and F independent?* ***Justify your answer.***

No, they are independent because

$$P(E \cap F) = .2 = P(E)\,P(F)$$

3. Suppose A, B and C are events such that

A and B are independent, B and C are independent, $P(A) = P(B) = P(C) \neq 0$

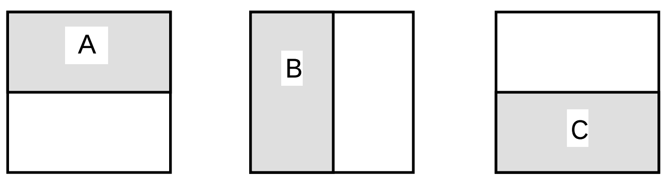
Which *of the following must necessarily be true? (In case of mistakes, you can get partial credit if you justify your answer and your justification is partially correct.)*

[5] (a) *A and B are disjoint.* **NOT TRUE**

(Since A and B are independent they must overlap, so they are not disjoint.)

[5] (b) *A and C are independent.* **NOT NECESSARILY TRUE**

For example, in the Venn diagram below, A and B are independent, B and C are independent, but A and C are disjoint, so they are not independent.



[5] (c) $P(A \quad B) = 2P(A) - (P(A))^2$ **NECESSARILY TRUE**

$P(A \quad B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$ since A&B indep.

$= 2P(A) - (P(A))^2$ since $P(B) = P(A)$

[5] (d) $P(A|B) = P(B|A)$. **NECESSARILY TRUE**

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} = P(B|A)$$

4. *In a certain community 36% of the families own a dog, and 22% of the families that own a dog also own a cat. In addition, 15% of the families who do not own a dog own a cat.*

From the problem we deduce:

$P(D) = P(\text{own a dog}) = .36$, $P(C|D) = P(\text{own a cat} \mid \text{own a dog}) = .22$, and

$P(C|D^c) = P(\text{own a cat} \mid \text{don't own a dog}) = .15$.

[10] (a) ***Find*** *the probability that a randomly selected family owns both a dog and a cat.*

P(own a dog and a cat) $= P(D \cap C) = P(D)\,P(C|D) = .36 \times .22 = \mathbf{.0792}$

[10] (b) ***Find*** *the probability that a randomly selected family owns a dog given that it owns a cat.*

$$\text{P(own a dog | own a cat)} = P(D|C) = \frac{P(C|D)\,P(D)}{P(C)} \text{ by Bayes rule}$$

and by the law of total probability

$$P(C) = P(D)\,P(C|D) + P(D^c)\,P(C|D^c) = .36 \times .22 + (1\text{-}.36) \times .15 = .175$$

So

$$\mathbf{P(\text{own a dog | own a cat})} = \frac{.36 \times .22}{.1752} = \mathbf{.452}$$

Note to some: We do not ordinarily express probabilities as "percents". Probabilities are useful for predicting frequencies of occurrence, which may be expressed as percents. And percents may be used as suggesting how to assign probabilities. But probabilities are numbers between 0 and 1. You might want to consider the following: Suppose the probability of an event is expressed as 50 percent. Percent of what?

5. *Suppose that on the average 3% of people who make airplane reservations do not show up for their flight. Knowing this, an airline accepts 103 reservations for a plane that seats 100.* ***Find*** *the probability that when the plane is ready to leave, more passengers with reservations show up than there are seats. (If you are short of time, you can get substantial partial credit by giving just a correct expression for the probability, as opposed to the final numerical value.)*

[15]

We need to convert the problem description into a probabilistic model. Specifically, we assume that each of the 103 passengers who made a reservation will Showup with probability 0.97 or will Notshowup with probability 0.03. We also assume that passengers make their decisions independently. As a result we have a sequence of 103 Bernoulli trials. The event "more passengers with reservations show up than there are seats" is the event that number of "successes" in these Bernoulli trials is 101, 102 or 103. Therefore,

P(more passengers with reservations show up than there are seats)

$$= \sum_{i=101}^{103} \binom{103}{i} p^i (1-p)^{103-i}, \quad \text{where } p = .97$$

$$= \binom{103}{101} p^{101} (1-p)^2 + \binom{103}{102} p^{102} (1-p) + \binom{103}{103} p^{103} (1-p)^0$$

$$= \frac{103\times102}{2\times1} (.97)^{101}(.03)^2 + 103\,(.97)^{102}\,(.03) + 1\,(.97)^{103}$$

$$= 5253 \times .0461 \times .0009 + 103 \times .0447 \times .03 + .0413$$

$$= .218 + .138 + .0413 = \mathbf{.398}$$