Exam Statistics: Average = 67/105, Median = 68, Std. deviation = 18.7, High score = 105

Grade ranges: A = 105-87 (11 A's); B = 87-65 (25 B's); C = 65-39 (19 C's); D = 39- (6 D's)

1. *Short answer questions (no explanations or justifications required)*

[5] (a) ***Define*** *the joint probability mass function for a pair of random variables X and Y.*

The joint probability density of X and Y is the function $p_{XY}(x,y) = Pr(X=x,Y=y)$.

[5] (b) ***State*** *the condition guaranteeing that a pair of random variables possess a joint density.*

The pair of random variables X,Y possess a joint density if

$$Pr((X,Y)\in A) = 0 \text{ for every set } A \text{ with area zero.}$$

Alternatively, if there exists a function $f(x,y)$ such that

$$Pr((X,Y)\in A) = \int_A f(x,y)\, dx\, dy \quad \text{for every set } A.$$

[5] (c) ***Define*** *what it means for two random variables to be independent. (For full credit your definition should not apply just to some special case.)*

Random variables X and Y are independent if

$$Pr(X\in A, Y\in B) = Pr(X\in A)\, Pr(Y\in B) \quad \text{for all sets } A, B$$

Alternatively, X and Y are independent if

$$F_{XY}(x,y) = F_X(x)\, F_Y(y) \quad \text{all } x,y$$

[5] (d) ***Define*** *what is meant by $Pr(X\in A|Y=3)$ for the situation that Y is a continuous random variable.*

When Y is continuous, $Pr(Y=3) = 0$ so instead of the conventional definition of conditional probability, we use

$$Pr(X\in A|Y=3) = \lim_{\Delta\to 0} Pr(X\in A|3\le Y\le y+\Delta)$$

[0] (e) ***Define*** *what it means for two random variables to be identical.*

X and Y are identical if they have the same probability distribution, e.g. the same cdf, pdf or pm.

This problem was not graded because it wasn't supposed to be part of the coverage of the exam.

2. *A rectangle is randomly drawn with height H and width W being independent and identical random variables that are each uniformly distributed between 8 and 9 inches.* ***Find*** *the probability that the area of the rectangle is less than 72 square inches.*

[20]

Since the area of the rectangle is H×W, we need to find $Pr(HW \le 72)$. The problem tells us that H and W are continuous random variables with densities

$$f_H(h) = \begin{cases} 1, & 8\le h\le 9 \\ 0, & \text{else} \end{cases}, \quad f_W(w) = \begin{cases} 1, & 8\le w\le 9 \\ 0, & \text{else} \end{cases}$$

Since it also says that H and W are independent, their joint density is

$$f_{HW}(h,w) = f_H(h)\, f_W(w) = \begin{cases} 1, & 8\le h\le 9, 8\le w\le 9 \\ 0, & \text{else} \end{cases}$$

Now

$$\mathbf{Pr(HW \le 72)} = \int_8^9 \int_8^{72/h} f_{HW}(h,w)\, dw\, dh = \int_8^9 \int_8^{72/h} 1\, dw\, dh = \int_8^9 \left(\frac{72}{h} - 8\right) dh = 72 \ln h \Big|_8^9 - 8$$

$$= 72 (\ln 9 - \ln 8) - 8 = 8.48 - 8 = \mathbf{0.4804}$$

3. *It has been found that when slowly driving a car along a country road on a warm summer evening at v miles per hour, the time T in seconds it takes for the next bug to hit the windshield can be modelled as random with an exponential distribution with average 10/v. Assume the speed of the car (in miles per hour) is also random with probability density function*

$$f_V(v) = \begin{cases} \frac{c}{v}, & 5 \le v \le 10 \\ 0, & \text{else} \end{cases} \qquad \text{where } c = \frac{1}{\ln(2)}$$

[13] (a) ***Find** the probability density function of T.*

We deduce from the problem statement that

$$f_{T|V}(t|v) = \begin{cases} \frac{v}{10} e^{-(v/10)t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

We now find that for $t \ge 0$,

$$f_T(t) = \int_{-\infty}^{\infty} f_{T|V}(t|v)\, f_V(v)\, dv = \int_5^{10} \frac{c}{10} e^{-(v/10)t}\, dv = -\frac{c}{10}\frac{10}{t} e^{-(v/10)t}\Big|_5^{10} = \frac{c}{t}\left(e^{-t/2} - e^{-t}\right)$$

And for $t < 0$, $f_T(t) = 0$, $t < 0$. In summary

$$\mathbf{f_T(t)} = \begin{cases} \frac{c}{t}\left(e^{-t/2} - e^{-t}\right), & t \ge 0 \\ 0, & \text{else} \end{cases}$$

[12] (b) *If the next bug hits the windshield at 1 second,* ***what is*** *the probability that the car is travelling less than 8 miles per hour.*

We must find $\Pr(V \le 8|T=1)$. To compute this we first find the conditional density of V given T: $f_{V|T}(v|t)$. For $t \ge 0$ and $5 \le v \le 10$

$$f_{V|T}(v|t) = \frac{f_{T|V}(t|v)\, f_V(v)}{f_T(t)} = \frac{\frac{c}{10} e^{-(v/10)t}}{\frac{c}{t}\left(e^{-t/2} - e^{-t}\right)} = \frac{t\, e^{-(v/10)t}}{10\left(e^{-t/2} - e^{-t}\right)}$$

$$= \frac{1}{10}\frac{e^{-v/10}}{e^{-1/2} - e^{-1}} \quad \text{when } t = 1$$

and $f_{V|T}(v|t) = 0$, if $v<5$ or $v>10$.
Now,

$$\mathbf{Pr(V \le 8|T=1)} = \int_{-\infty}^{8} f_{V|T}(v|t)\, dv = \int_5^8 \frac{1}{10}\frac{e^{-v/10}}{e^{-1/2} - e^{-1}}\, dv = \frac{1}{10}\frac{1}{e^{-1/2} - e^{-1}}(-10)\, e^{-v/10}\Big|_5^8$$

$$= \frac{e^{-1/2} - e^{-4/5}}{e^{-1/2} - e^{-1}} = \frac{.157}{.239} = \mathbf{0.6587}$$

4. *X is a Gaussian random variable with mean 2 and variance 3. Let $Y = 4X^2+X+7$.*

[5] (a) *What type of random variable is Y?*

Y is continuous. This is because $\Pr(Y=y) = 0$ for all y. This is because $\Pr(Y=y)$ = probability of the set of x values such that $4x^2+x+7=y$. For any value y, there are at most two values of x inthis set, and since X is continuous each of these probabilities is zero, and so the whole set has zero probability. Hence, $\Pr(Y=y)=0$. (Y is not Gaussian.)

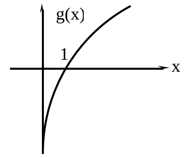
[15] (b) *Find the expected value of Y.*

$$\mathbf{E[Y]} = E[4X^2 + X + 7] = 4\,E[X^2] + E[X] + 7 \quad \text{by linearity of expectation}$$

$$= 4\left(\sigma_X^2 + (E[X])^2\right) + E[X] + 7 = 4\,(3 + 2^2) + 2 + 7 = \mathbf{37}$$

5. Suppose $Y = g(X)$, where X is a Gaussian random variable with mean zero and variance one, and $g(x)$ is the function

$$g(x) = \begin{cases} \ln(x), & x>0 \\ 0, & x \le 0 \end{cases}$$



[5] (a) What type of random variable is Y?

Y is mixed.

First, we notice that $\Pr(Y=0) = \Pr(X\le 0) = \frac{1}{2}$. Hence, Y is not continuous.

Next, for $y \neq 0$, $\Pr(Y=y) = \Pr(\ln(X) = y) = \Pr(X=e^y) = 0$. Since there does not exist a finite or countably infinite set of y's whose probabilities some to one, Y is not discrete.

[15] (b) Find the probability density of Y.

Approach 1: Because $\Pr(Y=0) = \frac{1}{2}$, the density $f_Y(y)$ contains the delta function $\frac{1}{2}\delta(y)$.

Next since for any y the only value of x such that $g(x) = y$ is $x = e^y$.

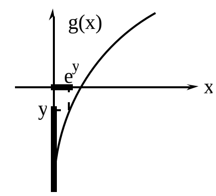
Therefore, the density $f_Y(y)$ also contains the term

$$f_X(x) \frac{1}{g'(x)} \Big|_{x=e^y} = f_X(e^y) \frac{1}{\frac{1}{e^y}}$$

Combining these two terms yields

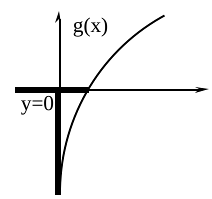
$$\mathbf{f_Y(y)} = \frac{1}{2}\delta(y) + f_X(e^y)\, e^y = \boldsymbol{\frac{1}{2}\delta(y) + \frac{1}{\sqrt{2\pi}} e^{-e^{2y}/2}\, e^y}$$

Approach 2: find the cdf $F_Y(y)$ and then differentiate



If $y<0$, $F_Y(y) = \Pr(Y \le y) = \Pr(X>0 \text{ and } \ln(X)\le y) = \Pr(0<X\le e^y)$

$$= F_X(e^y) - F_X(0) = F_X(e^y) - \frac{1}{2}$$

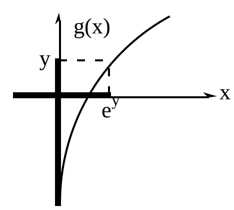


If $y=0$, $F_Y(0) = \Pr(Y\le 0) = \Pr(X<0) + \Pr(X>0 \text{ and } \ln X \le 0) = \Pr(X<0) + \Pr(0<X\le 1)$

$$= F_X(0) + F_X(1) - F_X(0) = F_X(1)$$

Note that there is a jump of height $\frac{1}{2}$ in $F_Y(y)$ at $y = 0$, becuase

$$F_Y(0-) = F_X(1) - \frac{1}{2} \quad \text{and} \quad F_Y(0) = F_X(1)$$



If $y>0$, $F_Y(y) = \Pr(Y\le y) = \Pr(\ln(X)\le y) = \Pr(X\le e^y) = F_X(e^y)$

In summary,

$$F_Y(y) = \begin{cases} F_X(e^y) - \frac{1}{2}, & y<0 \\ F_X(1), & y=0 \\ F_X(e^y), & y>0 \end{cases}$$

Taking derivatives (using chain rule of differentiation) yields

$$\mathbf{f_Y(y)} = f_X(e^y)\, e^y + \frac{1}{2}\delta(y) = \boldsymbol{\frac{1}{2}\delta(y) + \frac{1}{\sqrt{2\pi}} e^{-e^{2y}/2}\, e^y}$$