EECS 401 **Midterm 2 Solutions** **April 8, 1998**

Exam Statistics:

Average = 71/110, High Score = 100.

Grade Ranges: A = 110 to 90, B = 90 to 68, C = 68 to 44.

14 A's, 35 B's, 19 C's, 7 D's, 1 E

1. ***Give*** *a brief precise definition or formula for each of the following. (No explanation is needed.)*

[5] (a) *A continuous random variable.*

A random variable X is continuous if P(X = x) = 0 for all x. Equivalently a random variable is continuous if its cumulative distirbution function is continuous everywhere.

[5] (b) *A discrete random variable*

A random variable X is discrete if there exists a discrete set A (finite or countably infinite) such that P(X   A) = 1. Equivalently, a random variable is discrete if its cdf consists only of jumps and "flats".

[5] (c) *The joint cdf (cumulative distribution function) for random variables X and Y.*

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

[5] (d) *Independence of two discrete random variables X and Y.*

Discrete random variables X and Y are independent if any of the following equivalent conditions hold: (You *only* need to state one.)

$p_{XY}(x,y) = p_X(x)\, p_Y(y)$ for all x,y

$F_{XY}(x,y) = F_X(x)\, F_Y(y)$ for all x,y

P(X   A, Y   B) = P(X   A, Y   B) for all events A and B

$p_{X|Y}(y|x) = p_X(x)$ for all x,y

$p_{Y|X}(y|x) = p_Y(y)$ for all x,y

$F_{X|Y}(y|x) = F_X(x)$ for all x,y

$F_{Y|X}(y|x) = F_Y(y)$ for all x,y

[5] (e) *The Gaussian density with mean 3 and variance 5.*

$$f_X(x) = \frac{1}{\sqrt{10\pi}} \exp\left\{-\frac{(x-3)^2}{10}\right\}$$

2.

[18] (a) *For each of the three cdf's shown below,* ***state*** *the type of the random variable to which it corresponds, and* ***find*** *the probability that the outcome of the random variable is greater or equal to 3 and less than 5.*

Note: $P(3 \le X < 5) = P(X \le 5) - P(X=5) - P(X \le 3) + P(X=3)$

(i) **mixed, $P(3 \le X < 5) = .2$** (Note: $P(X=1) = .2 > 0$)

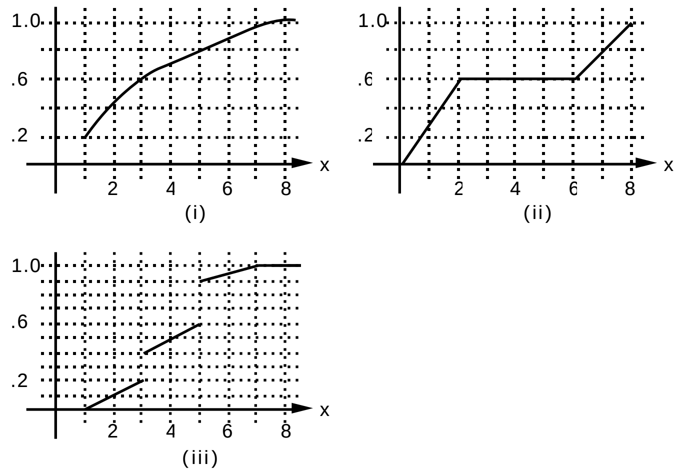
(ii) **continuous, $P(3 \le X < 5) = 0$**

(iii) **mixed, $P(3 \le X < 5) = .4$**

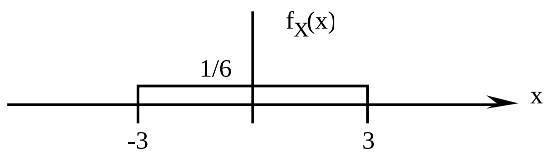
[7] (b) ***Find** the expected value of the random variable whose cdf is given in (ii).*

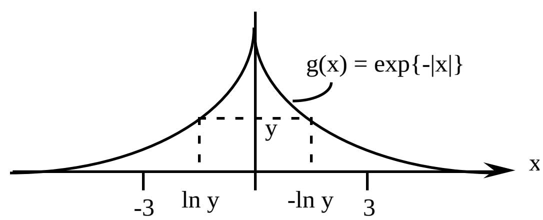
The density of this random variable is

$$f_X(x) = \begin{cases} .3 \ , & 0 \le x \le 2 \\ .2 \ , & 6 \le x \le 8 \\ 0 \ , & \text{elsewhere} \end{cases} \qquad \text{and so} \quad \mathbf{E[X]} = \int_{-\infty}^{\infty} f_X(x)\,dx = \mathbf{3.4}$$



3. *A random variable X is uniformly distributed on the interval [-3,3]. Let Y = g(X), where* $g(x) = e^{-|x|}$.





[5] (a) ***What** kind of random variable is Y?*

continuous, because each value of Y has zero probability, because it is caused by just two values of X, each of which has zero probability

[10] (b) ***Find** the pdf (probability density function) of Y.*

We find the cdf and then differentiate. We first notice that Y can range from e^{-3} to 1. For y $[e^{-3},1]$,

$$F_Y(y) = P(Y \le y) = P(-\ln y \le X \le 3) + P(-3 \le X \le \ln y)$$

$$= \int_{-\ln y}^{3} f_X(x)\,dx + \int_{-3}^{\ln y} f_X(x)\,dx \ , \quad \text{where } f_X(x) = \frac{1}{6}, \ -3 \le x \le 3$$

$$= \frac{1}{6}(3+\ln y) + \frac{1}{6}(\ln y + 3) = \frac{\ln y}{3} + 1$$

For $y < e^{-3}$, $F_Y(y) = 0$ and for $y > 1$, $F_Y(y) = 1$.

Differentiation $F_Y(y)$ gives
$$f_Y(y) = \begin{cases} \frac{1}{3y} \ , & e^{-3} \le y \le 1 \\ 0 \ , & \text{elsewhere} \end{cases}$$

[5] (c) ***Find*** *the probability that* $Y \ge 1/2$.

Approach 1: $\mathbf{P(Y\ge\frac{1}{2})} = \int_{1/2}^{1} f_Y(y)\,dy = \int_{1/2}^{1} \frac{1}{3y}\,dy = \frac{1}{3}(\ln y)\Big|_{1/2}^{1} = \mathbf{\frac{\ln 2}{3}} = \mathbf{.23}$

Approach 2: $\mathbf{P(Y\ge\frac{1}{2})} = P(\ln\frac{1}{2} \le X \le -\ln\frac{1}{2}) = \int_{\ln 1/2}^{-\ln 1/2} f_X(x)\,dx = \frac{1}{6}(-\ln 1/2 - \ln 1/2) = \mathbf{\frac{\ln 2}{3}}$

[5] (d) ***Find*** *the expected value of* Y.

Approach 1: $\mathbf{E[Y]} = \int_{-\infty}^{\infty} y\; f_Y(y)\,dy = \int_{e^{-3}}^{1} y\,\frac{1}{3y}\,dy = \mathbf{\frac{1}{3}(1-e^{-3}) = .32}$

Approach 2: $\mathbf{E[Y]} = E[g(X)] = \int_{-\infty}^{\infty} g(x)\; f_X(x)\,dx = \int_{-3}^{3} e^{-|x|}\frac{1}{6}\,dx = 2\,\frac{1}{6}\int_{0}^{3} e^{-x}\,dx$

$= \frac{1}{3}\left(-e^{-x}\right)\Big|_0^3 = \mathbf{\frac{1}{3}(1-e^{-3})}$

4. *Suppose* X *and* Y *are jointly continuous random variables with joint pdf*

$$f_{XY}(x,y) = \begin{cases} x\,e^{-(x+y)}, & x \ge 0,\ y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

Note: $\int x\,e^{-x}\,dx = -e^{-x}(x+1)$

[7] (a) ***Are*** *X and Y independent?* **YES**

To check independence, it is sufficient to find the marginal densities $f_X(x)$ and $f_Y(y)$ and see if $f_{XY}(x,y) = f_X(x)\,f_Y(y)$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\infty} x\,e^{-(x+y)}\,dy = x\,e^{-x}\int_0^{\infty} e^{-y}\,dy = x\,e^{-x},\ x \ge 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\infty} x\,e^{-(x+y)}\,dx = e^{-y}\int_0^{\infty} x e^{-x}\,dy = e^{-y}\left(-(x+1)\,e^{-x}\right)\Big|_0^{\infty}$$

$$= e^{-y},\ y \ge 0$$

$f_X(x)\,f_Y(y) = x\,e^{-x}\,e^{-y}$, $x \ge 0$, $y \ge 0$, which is the same as $f_{XY}(x,y)$

[8] (b) ***Find*** $P(X>3, Y>4)$.

Since X and Y are independent,

$$\mathbf{P(X>3,\ Y>4)} = P(X>3)\,P(Y>4) = \int_3^{\infty} f_X(x)\,dx \int_4^{\infty} f_Y(y)\,dy$$

$$= \int_3^{\infty} x\,e^{-x}\,dx \int_4^{\infty} e^{-y}\,dy = \left(-(x+1)\,e^{-x}\right)\Big|_3^{\infty} \times \left(-e^{-y}\right)\Big|_4^{\infty}$$

$$= 4\,e^{-3}\,e^{-4} = \mathbf{4\,e^{-7} = .00365}$$

Note: $P(X>3, Y>4) \ne 1 - P(X\le 3, Y\le 4)$ (draw the picture to see this)

5. *A random variable X is exponentially distributed with expected value 10. Let Y be a random variable whose conditional pdf given X = x is*

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{10}\ e^{(x-y)/10}, & y \geq x \\ 0, & \text{else} \end{cases}$$

[15] (a) ***Find*** *the conditional pdf of X given Y = y.*

By Bayes rule for conditional densities: $f_{X|Y}(x|y) = \dfrac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$

Since X is exponentially distributed with expected value 10: $f_X(x) = \frac{1}{10}\ e^{-x/10}$, $x \geq 0$.

To find $f_Y(y)$ we use $f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\, f_{Y|X}(y|x)\, dx$

$$f_X(x)\, f_{Y|X}(y|x) = \frac{1}{10}\ e^{-x/10}\ \frac{1}{10}\ e^{(x-y)/10} = \frac{1}{100}\ e^{-y/10}, \quad 0 \leq x \leq y$$

So

$$f_Y(y) = \int_0^y \frac{1}{100}\ e^{-y/10}\, dx = \frac{1}{100}\ y\ e^{-y/10}$$

Substituting into Bayes rule gives:

$$\mathbf{f_{X|Y}(x|y)} = \frac{\frac{1}{100}\ e^{-y/10}}{\frac{1}{100}\ y\ e^{-y/10}} = \mathbf{\frac{1}{y}}, \quad \mathbf{0 \leq x \leq y}$$

[5] (b) ***Find*** *the conditional expected value of X given Y = y.*

$$\mathbf{E[X|Y=y]} = \int_{-\infty}^{\infty} x\ f_{X|Y}(x|y)\, dx = \int_0^y x\ \frac{1}{y}\, dx = \frac{1}{y}\ \frac{x^2}{2}\Big|_0^y = \mathbf{\frac{y}{2}}$$

[110 points total]