EECS 401 **Final Exam Solutions** **April 28, 2000**

Exam Statistics: Average = 106/155, Median = 111, Std. deviation = 25.2, High score = 150

Grade ranges: A = 155-125 (16 A's); B = 125-93 (29 B's); C = 93-55 (13 C's); D = 55- (3 D's)

1. *Give a brief precise formula or definition for each of the following:*

[5] (a) *The autocorrelation function of a random process.*

The autocorrelation function of a random process $\{X(t)\}$ is

$$R_X(t,s) = E\, X(t)\, X(s)$$

[5] (b) *Widesense stationarity of a random process.*

A random process $\{X(t)\}$ is widesense stationary if $E\, X(t)$ has no dependence on t and if $R_X(t,s)$ depends only on t-s.

[5] (c) *The power spectral density of a widesense stationary random process.*

The power spectral density of a widesense stationary random process $\{X(t)\}$ is $S_X(f)$ is the Fourier transform of the autocorrelation function $R_X(\tau)$, i.e.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

Alternatively, the power spectral density is a function $S_X(f)$ such that power in $\{X(t)\}$ in frequency band $[f_1,f_2]$ = $\int_{-\infty}^{\infty} S_X(f)\, df$

[5] (d) *A continuous random variable.*

A random variable X is continuous if $\Pr(X=x) = 0$ all x.

[5] (e) *A discrete random variable.*

A random variable X is discrete if $\Pr(X\in A) = 1$ for some finite or countably infinite set A.

2. *A light bulb is purchased from company A or B with equal probability. The bulbs from company A have a lifetime that is exponentially distributed with mean 3 years. Those from company B have a lifetime that is exponentially distributed with mean 4 years. If the bulb is still working after 5 years of use,* ***what is*** *the probability that it came from company B?*

[20]

Let T denote the lifetime of the bulb. Let A, B denote the event that the bulb comes from company A, B, respectively. We need to find $P(B|T\geq 5)$. Note: The event "the bulb is working after 5 years of use" is the event "$T\geq 5$". From the problem statement we deduce: $P(A) = P(B) = \frac{1}{2}$,

the conditional pdf of T given A is $f_T(t|A) = \begin{cases} \frac{1}{3}\exp\{-\frac{t}{3}\}, & t\geq 0 \\ 0, & t<0 \end{cases}$,

and the conditional pdf of T given B is $f_T(t|B) = \begin{cases} \frac{1}{4}\exp\{-\frac{t}{4}\}, & t\geq 0 \\ 0, & t<0 \end{cases}$

We now have: $P(B|T\geq 5) = \dfrac{P(B \text{ and } T\geq 5)}{P(T\geq 5)} = \dfrac{P(B)\, P(T\geq 5|B)}{P(A)P(T\geq 5|A)+P(B)P(T\geq 5|B)}$

$$P(T\geq 5|A) = \int_5^{\infty} f_T(t|A)\, dt = \int_5^{\infty} \frac{1}{3} e^{-t/3}\, dt = e^{-5/3}$$

$$P(T\geq 5|B) = \int_5^{\infty} f_T(t|B)\, dt = \int_5^{\infty} \frac{1}{4} e^{-t/4}\, dt = e^{-5/4}$$

Therefore,

$$\mathbf{P(B|T\geq 5)} = \frac{1/2\, e^{-5/4}}{1/2\, e^{-5/3}+1/2\, e^{-5/4}} = \mathbf{\frac{1}{e^{-5/12}+1}} = \mathbf{0.603}$$

Note: Some people found $P(B|T=5) = \dfrac{1}{4/3\, e^{-5/12} + 1} = 0.532$.

3. *A pair of continuous random variables (X,Y) has joint pdf*

$$f_{XY}(x,y) = \begin{cases} x e^{-x(y+1)}, & x \ge 0 \text{ and } y \ge 0 \\ 0, & \text{else} \end{cases}$$

[8] (a) ***Find** the pdf of X.*

$$\mathbf{f_X(x)} = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \int_0^{\infty} x\, e^{-xy}\, e^{-x}\, dy = e^{-x} \int_0^{\infty} a\, e^{-ay} dy, \quad \text{with } a = x$$

$$= \mathbf{e^{-x}, \quad for \quad x \ge 0}$$

$$f_X(x) = 0, \text{ for } x < 0$$

[8] (b) ***Are** X and Y independent?*

No, they are not independent.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_0^{\infty} x\, e^{-x(y+1)}\, dx = \frac{1}{a} \int_0^{\infty} x\, a\, e^{-ax}\, dx, \text{ with } a = y+1$$

$$= \frac{1}{a}\frac{1}{a} = \frac{1}{(y+1)^2}, \quad y \ge 0$$

And we see that $f_X(x)\, f_Y(y) \ne f_{XY}(x,y)$

[12] (c) ***Find** the estimate of Y given $X = x$ that minimizes mean squared error.*

If $X = x$, the estimate of Y that minimizes the minimizes MSE is

$$\mathbf{E[Y|X=x]} = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_{-\infty}^{\infty} y\, \frac{f_{XY}(x,y)}{f_X(x)}\, dy = \int_0^{\infty} y\, \frac{x\, e^{-x(y+1)}}{e^{-x}}\, dy$$

$$= \int_0^{\infty} y\, x\, e^{-xy}\, dy = \mathbf{\frac{1}{x}}$$

If $x < 0$, then $E[Y|X=x]$ is undefined.

[7] (d) ***Find** an integral expression for the probability that X is larger than Y. You need not evaluate the integral.*

$$\Pr(X>Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x,y)\, dy\, dx = \int_0^{\infty} \int_0^{x} x\, e^{-x(y+1)}\, dy\, dx$$

4. *CAEN needs to decide how many workstations are needed in its labs to meet the demand of engineering students at 3 PM on weekdays, which is considered the busiest time. Assume that there are 4000 engineering students, that at 3 PM each engineering student will independently decide to use a workstation with probability 0.2 . **Find**, approximately, the minimum number of workstations CAEN must have in its labs so that the demand for workstations at 3 PM is met on 95% of weekdays.*

[20]

We use the Central Limit Theorem (CLT) to solve this problem. Let X_i be a random variable that is 1 if the ith student decides to use a workstation and 0 if not. Then $X_1, X_2, \ldots, X_{4000}$ are independent and identical random variables with

$$p_{X_i}(1) = 0.2, \quad p_{X_i}(0) = 0.8,$$

$$EX = 0.2, \quad EX^2 = 0.2 \text{ and } \sigma_X^2 = 0.2 - (0.2)^2 = 0.16.$$

The number of students needing a workstation is $\sum_{i=1}^{400} X_i$.

Let n denote the number of CAEN workstations. In this problem, we need to find n such that

$$\Pr\left(\sum_{i=1}^{400} X_i \geq n\right) \cong 0.05 \ .$$

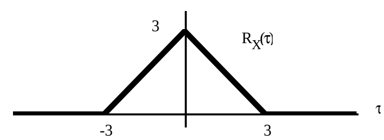
We have:

$$\Pr\left(\sum_{i=1}^{4000} X_i \geq n\right) = \Pr\left(\frac{1}{\sqrt{4000}\sigma_X} \sum_{i=1}^{4000} (X_i\text{-}EX) \ \geq \ \frac{n\text{-}4000EX}{\sqrt{4000}\sigma_X} \right)$$

$$\cong Q\left(\frac{n\text{-}4000EX}{\sqrt{4000}\sigma_X}\right)$$ because the CLT shows that $\frac{1}{\sqrt{n}\sigma_X} \sum_{i=1}^{4000} (X_i\text{-}EX)$ is approximately Gaussian with mean 0 and variance 1.

$$= Q\left(\frac{n\text{-}4000\times.2}{\sqrt{4000\times0.4}}\right) = Q\left(\frac{n\text{-}800}{25.3}\right) = 0.05 \ \Rightarrow \ \frac{n\text{-}800}{25.3} \cong 1.65 \ \Rightarrow \ \mathbf{n} \ \cong \ \mathbf{842}$$

5. *It is found that the noise at a certain node of an electronic circuit can be modelled as a continuous-time, wide-sense stationary random process $\{X(t)\}$ that is Gaussian with zero mean and with the autocorrelation function $R_X(\tau)$ shown below:*



[15] (a) ***Find*** $Pr\left(\left(X(5) + 2X(6)\right)^2 > 7\right)$

Let Y = X(5)+2X(6). Then since {X(t)} is Gaussian, X(5) and X(6) is jointly Gaussian, and therefore Y is Gaussian since it is a linear combination of Gaussian random variables.

$$\Pr\left(\left(X(5) + 2X(6)\right)^2 > 7\right) = \Pr(Y^2 > 7) = \Pr(Y > \sqrt{7}) + \Pr(Y < -\sqrt{7}) = 2 \Pr(Y>\sqrt{7}) = 2 Q\left(\frac{\sqrt{7}\text{-EY}}{\sigma_Y}\right)$$

$$EY = E\,X(5) + 2\,E\,X(6) = 0 + 2\times0 = 0$$

$$\sigma_Y^2 = E\,Y^2 - (EY)^2 = E\,Y^2 = E\,(X(5)+2X(6))^2 = E\,X^2(5) + 4\,E\,X^2(6) + 2\,E\,X(5)X(6)$$

$$= R_X(0) + 4\,R_X(0) + 4\,R_X(1) = 3 + 4\times3 + 4\times2 = 23$$

$$\mathbf{Pr}\left(\left(\mathbf{X(5) + 2X(6)}\right)^{\mathbf{2}} > \mathbf{7}\right) = 2\,Q\left(\frac{\sqrt{7}\text{-}0}{\sqrt{23}}\right) = 2\,Q(0.55) \cong 2 \times 0.29 = \mathbf{0.58}$$

If you have difficulty, you can get approximately two-thirds credit for computing Pr(X(5) + 2X(6)> 7).

$$\Pr(X(5) + 2X(6) > 7) = \Pr(Y>7) = Q\left(\frac{7\text{-EY}}{\sigma_Y}\right) = Q\left(\frac{7\text{-}0}{\sqrt{23}}\right) = Q(1.46) \cong 0.07$$

For each of the following three parts, you must describe the set of (t,s) pairs for which a given condition holds. Partial credit goes to answers that partially describe the set.

[5] (b) *For what values of t and s, if any, are X(t) and X(s) independent?*

|t-s| ≥ 3. Since X(t) and X(s) are jointly Gaussian, they are independent if and only if they are uncorrelated. Since their means are zero, they are uncorrelated if and only if their correlation is zero. The autocorrelation function shows that that their correlation is zero when aond only when |t-s| ≥ 3.

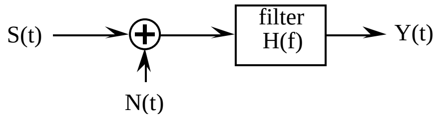
[5] (c) *For what values of t and s, if any, are X(t) and X(s) identical?*

For all t,s. Because the random process is widesense stationary and Gaussian it is also stationary. Hence, all of its random variables are identical.

[5] (d) *For what values of t and s, if any, are X(t) and X(s) equal?*

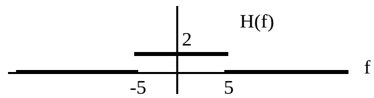
For t=s. For t≠s, the random variables are not equal.

6. *As illustrated in the diagram below, a signal S(t) is corrupted by additive noise N(t) and then filtered. The output of the filter is denoted Y(t).*



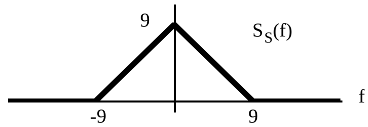
The filter has frequency response

$$H(f) = \begin{cases} 2, & -5 \le f \le 5 \\ 0, & else \end{cases}$$

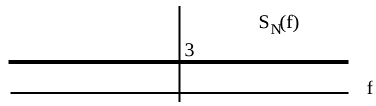


Assume that $\{S(t)\}$ and $\{N(t)\}$ are widesense stationary, continuous-time random processes with zero means and power spectral densities

$$S_S(f) = \begin{cases} 9-|f|, & -9 \le f \le 9 \\ 0, & else \end{cases}$$



$$S_N(f) = 3 \quad all\ f$$



Assume that $\{S(t)\}$ and $\{N(t)\}$ are independent of each other

[6] (a) ***Find** the power in $\{N(t)\}$*

Power in {N(t)} $= E\,N^2(t) = R_N(0) = \int_{-\infty}^{\infty} S_N(f)\,df = \int_{-\infty}^{\infty} 3\,df$ $\mathbf{=}$ ∞

[13] (b) ***Find** the power in $\{Y(t)\}$.*

Power in $\{Y(t)\} = E\,Y^2(t) = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df$

$S_Y(f) = S_X(f)\,|H(f)|^2$, where $S_X(f)$ is the power spectral of $\{X(t)\}$ with $X(t) = S(t) + N(t)$.

As derived in HW, $S_Y(f) = S_X(f) + S_N(f)$. However, in the following we derive it again:

$R_X(t,s) = E\,X(t)\,X(s) = E\,(S(t)+N(t))(S(s)+N(s))$

$= E\,S(t)S(s) + E\,S(t)N(s) + E\,S(s)N(t) + E\,N(t)N(s))$

$= R_S(s-t) + E\,S(t)\,E\,N(s) + E\,S(s)\,E\,N(t) + R_N(s-t)$, since $\{N(t)\}$ and $\{S(t)\}$ are indep.

$= R_S(s-t) + R_N(s-t)$, since both random processes have zero mean.

$\Rightarrow R_X(\tau) = R_S(\tau) + R_N(\tau)$

$\Rightarrow S_s(f) = F\{R_X(\tau)\} = F\{R_S(\tau) + R_N(\tau)\} = F\{R_S(\tau)\} + F\{R_N(\tau)\} = S_S(f) + S_N(f).$

Finally,

Power in {Y(t)} $= \int_{-\infty}^{\infty} S_X(f)\,|H(f)|^2\,df = \int_{-\infty}^{\infty} (S_S(f)+S_N(f))\,|H(f)|^2\,df$

$$= \int_{-\infty}^{\infty} S_S(f)\,|H(f)|^2\,df + \int_{-\infty}^{\infty} S_N(f)\,|H(f)|^2\,df$$

$$= \int_{-5}^{5} S_S(f)\,4\,df + \int_{-5}^{5} 3 \times 4\,df$$

$= 260 + 120$ $\mathbf{= 380}$ (from the area under the integrands)

[6] (c) ***How much** of the power in $\{Y(t)\}$ is due to the noise?*

Power in {Y(t)} due to noise $= \int_{-\infty}^{\infty} S_N(f)\,|H(f)|^2\,df = \int_{-5}^{5} 3 \times 4\,df$ $\mathbf{= 120}$

[155 points total]