More Detailed Solution to the Game Show Problem:

The previous solution shows correctly that the "always switch" strategy wins with probability 2/3 and the "never switch" strategy wins with probability 1/3. But it does this without taking into account what the host actually does. Note for example that if I choose Door 1 and the prize is behind Door 1, the host has to choose whether to open Door 1 or Door 2? Might the probability of winning depend on how how the host makes this decision? The following solution shows that it does not. Also, the previous solution did not rule out the possibility that there might be some better strategy than "always switch". The following shows there is not.

Some notation:

P_1, P_2, P_3 denote the events that the prize is behind Doors 1, 2 and 3, respectively

C_1, C_2, C_3 denote the events that my first choice is Door 1, 2 and 3, respectively

H_1, H_2, H_3 denote the events that the host chooses Doors 1, 2 and 3, respectively

q_1, q_2, q_3 are numbers in the interval [0,1]. Their purpose will become clear shortly.

W = event that I win

The hosts strategy:

It is necessary to model the possible actions of the host. In the situation where the first Door I pick is the correct door, the host has a choice of which door to open. Consider the following model for the host's actions.

When I first choose Door 1 and the prize is behind Door 1; i.e. the event $C_1 \cap P_1$ occurs, the host opens Door 2 with probability q_1 and Door 3 with probability $(1-q_1)$. That is, $P(H_2|C_1 \cap P_1) = q_1$ and $P(H_3|C_1 \cap P_1) = 1 - q_1$.

When I first choose Door 2 and the prize is behind Door 2, the host opens Door 1 with probability q_2 and Door 3 with probability $(1-q_2)$. That is, $P(H_1|C_2 \cap P_2) = q_2$ and $P(H_3|C_2 \cap P_2) = 1 - q_2$).

When I first choose Door 3 and the prize is behind Door 3, the host opens Door 1 with probability q_3 and Door 2 with probability $(1-q_3)$. That is, $P(H_1|C_3 \cap P_3) = q_3$ and $P(H_2|C_3 \cap P_3) = 1 - q_3$.

The numbers q_1, q_2 and q_3 determine the hosts strategy. By varying them one considers a broad range of host strategies. For example, $q_1 = q_2 = q_3 = 1/2$ seems a natural choice. However, we will see that if I choose the second door in the best possible way, my probability of winning the prize will not be affected by the choice of q_1, q_2 and q_3.

My strategy for deciding whether to switch or not:

When it is time for me to decide whether to switch or not, I know my choice of door and I know the hosts choice of door. In other words, I know which one of the following disjoint events has occurred: $C_1 \cap H_2$, $C_1 \cap H_3$, $C_2 \cap H_1$, $C_2 \cap H_3$, $C_3 \cap H_1$, $C_3 \cap H_2$. For each of these events, I need to calculate the conditional probability that the prize is behind the door I choose first. If it is bigger than 1/2, I'll stick with that door. If it is less than 1/2, I'll switch. This will maximize the probability that I win.

Calculating the conditional probabilities of Door 1:

Here we calculate the conditional probability that the prize is behind Door 1 given that I chose Door 1 and the host chose Door 2:

$$P(P_1|C_1 \cap H_2) = \frac{P(P_1 \cap H_2|C_1)}{P(H_2|C_1)} \quad \text{(this is the defn of cond. prob, with all prob's conditioned on } C_1)$$

$$= \frac{P(P_1|C_1)P(H_2|C_1 \cap P_1)}{P(P_1|C_1)P(H_2|C_1 \cap P_1) + P(P_2|C_1)P(H_2|C_1 \cap P_2) + P(P_3|C_1)P(H_2|C_1 \cap P_3)}$$

$$= \frac{\frac{1}{3} q_1}{\frac{1}{3} \times q_1 + \frac{1}{3} \times 0 + \frac{1}{3} \times 1} = \frac{q_1}{q_1 + 1}$$

Since $\frac{q_1}{q_1+1} \le \frac{1}{2}$, switching to Door 3 is the best thing to do. Actually, if $q_1 = 1$, either choice is equally good. But if $q_1 < 1$, then switching to Door 3 is definitely better. (To make winning difficult in this situation, the host might want to choose $q_1 = 1$.) For future reference, we note that if we always switch in this situation, then

$$P(W|C_1 \cap H_2) = \frac{1}{q_1 + 1}$$

One may similarly compute

$$P(P_1|C_1 \cap H_3) = \frac{1 - q_1}{1 - q_1 + 1} \quad \text{(this answer is obtained by replacing } q_1 \text{ in the previous answer with } 1 - q_1)$$

Since $\frac{1 - q_1}{1 - q_1 + 1} \le 1$, switching to Door 2 is the best thing to do. Actually, if $q_1 = 0$, then either choice is equally good. But if $q_1 > 0$, switching to Door 2 is definitely better. (To make winning difficult, the host might want to choose $q_1 = 0$, but this conflicts with the hosts ideal strategy in the other case.) For future reference, we note that if we always switch in this situation, then

$$P(W|C_1 \cap H_3) = \frac{1}{1 - q_1 + 1}$$

We could similarly analyze the situation when my first choice is C_2 or C_3. But we already have done enough to be able to compute the probility of winning given that my first choice is C_1 and that I use the strategy described above, which in effect says to switch to the other choice no matter what the host strategies. In this case,

$$P(W|C_1) = P(W|C_1 \cap H_1)P(H_1|C_1) + P(W|C_1 \cap H_2)P(H_2|C_1) + P(W|C_1 \cap H_3)P(H_3|C_1)$$

$$= P(W|C_1 \cap H_1) \times 0 + \frac{1}{q_1 + 1} \times \frac{1}{3}(q_1 + 1) + \frac{1}{1 - q_1 + 1} \times \frac{1}{3}(1 - q_1 + 1) = \frac{2}{3}$$

where $P(H_2|C_1)$ and $P(H_3|C_1)$ were computed previously.

We see from this that the "always switch strategy", which was shown to be the best strategy, has probability of winning equal to 2/3, no matter what q_1 is. A similar argument applies when we first choose Door 2, or when we first choose Door 3.