Solution to Game Show Problem EECS 401 **March 13, 2000**

Game Show Problem:

> *On a certain game show, you will win the $10K Grand Prize if you can guess the door behind which it is located. The situation is the following.*
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> *The prize is equally likely to be located each of three doors. You select one of the doors. However, after making your choice, the game show host opens a different door and shows you that the prize is not there. (The host knows the location of the prize and is careful not to open the corresponding door.) The host then gives you the option of choosing the "other" closed door or sticking with your original choice.*
>
> *Question: Should you stick with your present choice, switch to the other door, or does it make no difference?*
>
> *Please, justify your answer. Indeed the problem will be graded mainly on your justification, rather than the answer itself. This is because the tough thing is to find an argument that is convincing.*

Consider two different strategies for playing the game.

Strategy 1: I always stay with the door I choose first.

Strategy 2: I always switch to the other unopened door.

Analysis of Strategy 1: With this strategy, I will win when and only when my first choice is correct. So the probability of winning the prize is the probability that the first choice is correct, which is 1/3.

Analysis of Strategy 2: This strategy wins when and only when the first chosen door is wrong. Since the first chosen door is wrong with probability 2/3, the probability of winning is 2/3. Equivalently, this strategy wins if and only if the prize is behind one of the two doors I did not pick with my first choice, which happens with probability 2/3.

Answer to the question: I should switch to the other door because if I do, the probability of winning is 2/3, whereas if I don't, the probability of winning is 1/3.

Comment: In effect, switching gives me two chances to win whereas has staying with my first choice gives me just one.

Mathematical analysis:

Let's rework the problem in a more formal way. We may assume without loss that the first door I choose is always Door 1. Now let us find a probability model for the random experiment of randomly placing the prize between one of the three doors. The outcome of the experiment is the identity of the door, the sample space is $S = \{1, 2, 3\}$ and the probability assignment is such that $P(\{1\}) = P(\{2\}) = P(\{3\}) = 1/3$. Let $A = \{1\}$ and $B = \{2,3\}$. With this setup

$$P(\text{winning if I stick with Door 1}) = P(A) = 1/3$$

and

$$P(\text{winning if I switch}) = P(B) = 2/3$$

If one prefers not to make the assumption that the first door chosen is always Door 1, then one may consider the above analysis to give the conditional probabilities given the first door chosen is Door 1. One may then repeat the analysis given that the first door chosen is Door 2 and then again assuming Door 3. The conditional probabilities will turn out to be the same, so by the law of total probability, the unconditional probabilities will be the same as computed above no matter how one makes the choice about the first door.

An alternative derivation: Instead of assuming the first door I choose is alway Door 1, one may assume that the prize is always behind Door 1, and my first guess is equally likely to be any of the three doors. This analysis turns out to be the same.