Homework Set 10 Solutions *DRAFT* **EECS 401** **April 7, 2000**

1. 4.76, p. 264

   This problem was cancelled because the given density function was invalid. Its exponent was inconsistent with the constant multiplying the exponential. However, if the density were valid, the idea in this problem would have been to match the coefficients of the polynomial in the exponent of the given Gaussian density with coefficients in the general polynomial expression appearing in the exponent of a Gaussian random variables. There are 5 unknowns EX, EY, σ_X^2, σ_Y^2, and ρ_{XY}. So one one needs to find 5 equations.

2. 4.105 a, p. 268

   $Y = X + N$, where X and N are independent zero-mean Gaussian random variables with variances σ_X^2 and σ_N^2, respectively.

   a. The correlation coefficient:

$$\rho_{XY} = \frac{EXY - EX\,EY}{\sigma_X \sigma_Y}$$

$$EY = E(X+N) = EX + EN = 0 + 0 = 0$$

$$EXY = E\,X(X+N) = EX^2 + EXN = EX^2 + EX\,EN \text{ because X and N are indep.}$$

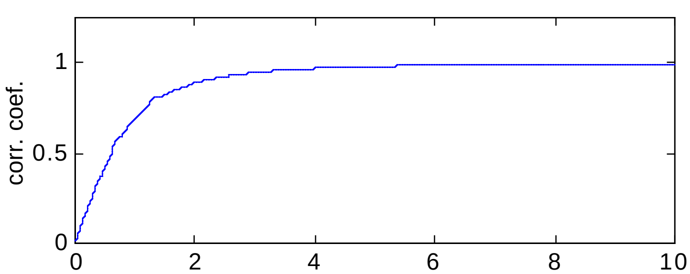
$$= \sigma_X^2 \quad \text{because } \sigma_X^2 = EX^2 - (EX)^2 = EX^2$$

$$\sigma_Y^2 = \text{var}(X+N) = \text{var}(X) + \text{var}(N) \text{ because X and N are uncorrelated}$$

$$= \sigma_X^2 + \sigma_N^2$$

Therefore,
$$\boldsymbol{\rho_{XY}} = \frac{\sigma_X^2 - 0\times 0}{\sigma_X\sqrt{\sigma_X^2+\sigma_N^2}} = \frac{1}{\sqrt{1+\sigma_N^2/\sigma_X^2}} = \frac{\mathbf{1}}{\sqrt{1+(\sigma_X/\sigma_N)^{-2}}}$$

Plot of ρ_{XY} vs. σ_X/σ_N



3. 5.2, p. 317 $S_n = \sum_{i=1}^{n} X_i$

$$\mathbf{E[S_n]} = \sum_{i=1}^{n} E[X_i] = \mathbf{n\mu}$$

$$\mathbf{var(S_n)} = \sum_{i=1}^{n}\sum_{j=1}^{n} \text{cov}(X_i,X_j) = \sum_{i=1}^{n} \text{cov}(X_i,X_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \text{cov}(X_i,X_j)$$

$$= \mathbf{n}\,\sigma^{\mathbf{2}} + \mathbf{2(n-1)}\,\rho\,\sigma^2$$

4. 5.25, p. 320

Let $X_1,...,X_{16}$ denote the lifetimes of the life bulbs. They are independent and each is exponential with mean 36. Let $S_{16} = X_1 + ... + X_{16}$. Let $W_{16} = (S_{16}\text{-}nE[X])/(\sqrt{n}\,\sigma)$, where $\sigma^2 = var(X) = 36^2$. The CLT tells us that W_{16} is approximately Gaussian with mean 0 and variance 1. Therefore,

$$\mathbf{P(S_{16} < 600)} = P\left(W_{16} < \frac{600\text{-}16\times36}{\sqrt{16}36}\right) = P(W_{16} < \frac{1}{6}) = 1 - Q\left(\frac{1}{6}\right) = 1\text{-}.43 = \mathbf{.57}$$

5. 5.26, p. 320

Let X_i be the lifetime of the ith pen. The X_i's are independent and each is exponential with mean 1. If the student buys n pens, let $S_n = X_1 + ... + X_n$ be the time at which the last pen dies. n should be chosen so that $P(S_n \geq 15) = .99$. Let $W_n = (S_n\text{-}nE[X])/(\sqrt{n}\,\sigma)$, where $\sigma^2 = var(X) = 1$. The CLT tells us that W_{16} is approximately Gaussian with mean 0 and variance 1.

$$P(S_n \geq 15) = P\left(W_n \geq \frac{15\text{-}n\times1}{\sqrt{n}1}\right) = Q\left(\frac{15\text{-}n}{\sqrt{n}}\right) = 1\text{-}\ Q\left(\frac{n\text{-}15}{\sqrt{n}}\right) = .99$$

$\Rightarrow \frac{n-15}{\sqrt{n}} = 2.35 \Rightarrow n = 27.04$ since this is not an integer we round up to $\mathbf{n = 28.}$

Rounding down to 27 is also acceptable.

6. 6.2, p. 389

(a) There are only two possible sample paths:

If the coin lands heads: then the sample function is: $\mathbf{X_1, X_2, \ldots = 1, 1, 1, 1, \ldots}$

If the coin lands tails: then the sample function is: $\mathbf{X_1, X_2, \ldots = -1, -1, -1, -1, \ldots}$

(b) The pmf of X_n is $\mathbf{p_{X_n}(x)} = \begin{cases} \mathbf{\frac{1}{2}}, & \mathbf{x = 0, 1} \\ \mathbf{0}, & \text{else} \end{cases}$

(c) The joint pmf of X_n and X_{n+k} is

$$\mathbf{p_{X_n,X_{n+k}}(x_1,x_2)} = \begin{cases} \mathbf{\frac{1}{2}}, & \mathbf{x_1=x_2=1 \text{ or } x_1=x_2=-1} \\ \mathbf{0}, & \mathbf{else} \end{cases}$$

(d) Mean function: $\mathbf{m_X(n)} = E[X_n] = \mathbf{0}$

Autovariance function: $\mathbf{C_X(m,n)} = cov(X_mX_n) = E[X_mX_n] - E[X_m]E[X_n] = 1 - 0 = \mathbf{1}$

7. 6.11, p. 391

$$\mathbf{E\left[|X(t_2)\text{-}X(t_1)|^2\right]} = E\left[\,X^2(t_1) - 2\,X(t_2)X(t_1) + X^2(t_1)\,\right]$$

$$= E\left[X^2(t_1)\right] - 2\,E\left[X(t_2)X(t_1)\right] + E\left[X^2(t_1)\right] = \mathbf{R_X(t_1,t_1) - 2\,R_X(t_1,t_2) + R_X(t_2,t_2)}$$

8. *An elementary continuous-time random process $\{X(t): -\infty<t<\infty\}$ has four equiprobable sample functions:*

$$X(t,1) = 1, \quad X(t,2) = -2, \quad X(t,3) = \sin \pi t, \quad X(t,4) = \cos \pi t.$$

(a) *Find the mean function.*

The outcome of the random process is one of the four sample functions above, each with probability 1/4. Therefore,

$$\mathbf{m_X(t)} = E[X(t)] = \frac{1}{4} X(t,1) + \frac{1}{4} X(t,1) + \frac{1}{4} X(t,1) + \frac{1}{4} X(t,1)$$

$$= \mathbf{\frac{1}{4}\left(1 - 2 + \sin \pi t + \cos \pi t\right)}$$

(b) *Find the autocorrelation function.*

The outcome of the random process is one of the four sample functions above, each with probability 1/4. Therefore,

$$\mathbf{R_X(t_1,t_2)} = E[X(t_1)X(t_2)]$$

$$= \frac{1}{4} X(t_1,1)X(t_2,1) + \frac{1}{4} X(t_1,2)X(t_2,2) + \frac{1}{4} X(t_1,3)X(t_2,3) + \frac{1}{4} X(t_1,4)X(t_2,4)$$

$$= \frac{1}{4}\left(1\times 1 + 2\times 2 + \sin \pi t_1 \sin \pi t_2 + \cos \pi t_1 \cos \pi t_2\right)$$

$$= \mathbf{\frac{1}{4}\left(5 + \cos \pi(t_2 - t_1)\right)}, \qquad (\sin a \sin b + \cos a \cos b = \cos a\text{-}b)$$

(c) *Find $Pr(-2.5 < X(-.5) < 2.5, -.5 < X(1) < .5)$. (Hint: It helps to draw pictures.)*

Note that $X(-.5,1) = 1$, $X(-.5,2) = -2$, $X(-.5,3) = \sin -\pi/2 = -1$, and $X(-.5,4) = \cos -\pi/1 = 0$

So the event $\{-2.5 < X(-.5) < 2.5\}$ is equivalent to the event $\{1,2,3,4\}$.

(If one draws pictures of the four sample functions one will see that at time -.5 their values are all between -2.5 and +2.5.)

Similarly, $X(1,1) = 1$, $X(1,2) = -2$, $X(1,3) = \sin \pi = 0$, and $X(1,4) = \cos \pi = 1$

So the event $\{-.5 < X(1) < .5\}$ is equivalent to the event $\{3\}$. (If one draws pictures of the four sample functions one will see that at time 1 only $x(.,s_3)$ takes value between -.5 and +.5.) Therefore

$$\mathbf{Pr(-2.5 < X(-.5) < 2.5, -.5 < X(1) < .5)} = P(\{1,3,4,4\}\cap\{3\}) = Pr(\{3\}) = \mathbf{\frac{1}{4}}$$

(d) *Find $Pr(-.8 < X(-.25) < .8, -.8 < X(.25) < .8)$*

Note: $X(-.25,1) = 1$, $X(-.25,2) = -2$, $X(-.25,3) = \sin -\pi/4 = -.707$, $X(-.25,4) = \cos \pi/4 = .707$

So the event $\{-.8 < X(-.25) < .8\}$ is equivalent to the event $\{3,4\}$. (If one draws pictures of the four sample functions one will see that at time -.25, only $X(.,3)$ and $X(.,4)$ have values between -.8 and +.8.)

Next, $X(.25,1) = 1$, $X(.25,2) = -2$, $X(.25,3) = \sin \pi/4 = .707$, $X(.25,4) = \cos \pi/4 = .707$

So the event $\{-.8 < X(.25) < .8\}$ is equivalent to the even $\{3,4\}$. (If one draws pictures of the four sample functions one will see that at time .25, only $X(.,3)$ and $X(.,4)$ have values between -.8 and +.8.) Therefore,

$$\mathbf{Pr(-.8<X(-.25)<.8, \ -.8<X(.25)<.8)} = P(\{3,4\}\cap\{3,4\}) = P(\{3,4\}) = \mathbf{\frac{1}{2}}$$