Homework Set 11 Solutions **EECS 401** **April 18, 2000**

1. 6.14, p. 391

   Since $\{X(t)\}$ is a zero-mean Gaussian random process, any collection of its random variables has a jointly Gaussian density. In particular, $X(t)$ and $X(t+s)$ have a jointly Gaussian density, namely, from equation (4.79), p. 238:

$$f_{X(t_1),X(t_2)}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{x_1^2}{\sigma_1^2} - 2\rho\frac{x_1}{\sigma_1}\frac{x_2}{\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right]\right\}$$

   where $\sigma_1^2 = \mathrm{var}(X(t_1)) = C_X(t_1,t_1) = \sigma^2$ and $\sigma_2^2 = \mathrm{var}(X(t_2)) = C_X(t_2,t_2) = \sigma^2$

   and $\rho = \dfrac{\mathrm{cov}(X(t_1),X(t_2))}{\sigma_1\sigma_2} = \dfrac{C_X(t_1,t_2)}{\sigma_1\sigma_2} = e^{-|t_1-t_2|}$

2. 6.15, p. 391. *$Z(t) = Xt + Y$*

   (a) mean function: $\mathbf{m_Z(t)} = E[Z(t)] = E[Xt + Y] = t\,E[X] + E[Y] = \mathbf{t\,m_X + m_Y}$

   autocovariance function: $\mathbf{C_Z(t,s)} = \mathrm{cov}(Z(t),Z(s)) = E[(Z(t)-m_Z(t))(Z(s)-m_Z(s))]$

$$= E[(tX + Y - tm_X - m_Y)(sX + Y - sm_X - m_Y)]$$

$$= E[(t(X-m_X) + (Y-m_Y))(s(X-m_X) + (Y-m_Y))]$$

$$= ts\,E[(X-m_X)^2] + (t+s)\,E[(X-m_X)(Y-m_Y)] + E[(Y-m_Y)^2]$$

$$= \mathbf{ts\,\sigma_X^2 + (t+s)\,\rho_{XY}\,\sigma_X\sigma_Y + \sigma_Y^2} \text{ jj}$$

   (b) Given that X and Y are jointly Gaussian and that $Z(t)$ is a linear combination of X and Y, we may conclude immediately that $Z(t)$ is Gaussian. Therefore, its pdf is

$$f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi\sigma_{Z(t)}^2}} \exp\left\{-\frac{(z-m_Z(t))^2}{2\sigma_{Z(t)}^2}\right\}$$

   where $\mathbf{m_Z(t) = t\,m_X + m_Y}$ and $\sigma_{Z(t)}^2 = C_Z(t,t) = \mathbf{t^2\,\sigma_X^2 + 2t\,\rho_{XY}\,\sigma_X\sigma_Y + \sigma_Y^2}$

3. 6.16, a, p. 391. *$X(t) = A\cos\omega t + B\sin\omega t$*

   (a) mean function: $\mathbf{m_X(t)} = E[X(t)] = E[A\cos\omega t + B\sin\omega t] = E[A]\cos\omega t + E[B]\sin\omega t = \mathbf{0}$

   autocovariance function: $\mathbf{C_X(t,s)} = \mathrm{cov}(X(t),X(s))$

   $= E[(A\cos\omega t + B\sin\omega t)(A\cos\omega s + B\sin\omega s)]$, since $m_X(t)=m_X(s) = 0$

   $= E[A^2]\cos\omega t\cos\omega s + E[AB](\cos\omega t\sin\omega s + \sin\omega t\cos\omega s) + E[B^2]\sin\omega t\sin\omega s$

   $= \sigma^2\cos\omega t\cos\omega s + \sigma^2\sin\omega t\sin\omega t$, since $E[AB] = E[A]\,E[B] = 0$

   $= \mathbf{\sigma^2\cos\omega(t-s)}$

4. 6.24, a,b p. 392 *Notes: X_n's are a Bernoulli random process. Skip the question: "Is the sample mean meaningful ...?"*

(a) Here's one possible sequence of X_n's:

$$X_1, X_2, ..., X_{10} = 1, 0, 1, 1, 0, 0, 0, 1, 0, 1$$

Since $W_n = 2 W_{n-1} + X_n$, the W_n realization (i.e. sample function) resulting from this is:

$$\mathbf{W_1, W_2, \ldots, W_{10} = 1, 2, 5, 11, 22, 44, 88, 177, 354, 709}$$

Since $Z_n = \frac{1}{2} Z_{n-1} + X_n$, the W_n realization (i.e. sample function) resulting from this is:

$$\mathbf{Z_1, Z_2, ..., Z_{10} = 1, \frac{1}{2}, \frac{5}{4}, \frac{13}{8}, \frac{13}{16}, \frac{13}{32}, \frac{13}{64}, \frac{141}{128}, \frac{141}{256}, \frac{653}{512}}$$

Both $\{W_n\}$ and $\{Z_n\}$ are autoregressive random processes. Because the coefficient multiplying W_{n-1} is larger than 1, the W_n's are nonstationary, which is consistent with the fact that the W_n's are increasing exponentially, and the sample mean $\frac{1}{n}\sum_{i=1}^{n} W_n \to \infty$, which the book seems to want to refer to as meaningless. My interpretation is that this is a nonstationary reandom process and since the probability distributions are changing, $E[W_n]$ does not have a great deal of significance.

On the other hand, since the coefficient multiplying Z_{n-1} is less than 1, the Z_n's approach a steady state behavior (they have a kind of asymptotic stationarity). The sample mean will approach the asymptotic mean value, namely, $\lim_{N\to\infty} E[Z_n]$ and the Z_n's have a useful (meaningful?) sample mean. We'll find $\lim_{N\to\infty} E[Z_n]$ in the next part.

(b) $\mathbf{W_n} = 2 W_{n-1} + X_n = 2 (2 W_{n-2} + X_{n-1}) + X_n = ...$

$$\mathbf{= 2^{n-1} X_1 + 2^{n-2} X_2 + \ldots + 4 X_{n-2} + 2 X_{n-1} + X_n}$$

$$\Rightarrow \mathbf{E[W_n] = 2^{n-1} E[X_1] + 2^{n-2} E[X_2] + \ldots + 4 E[X_{n-2}] + 2 E[X_{n-1}] + E[X_n]}$$

$$= (2^n-1) E[X] = \mathbf{\frac{1}{2} (2^n - 1)}$$

$$\mathbf{Z_n} = \frac{1}{2} Z_{n-1} + X_n = \frac{1}{2}(\frac{1}{2} Z_{n-2} + X_{n-1}) + X_n = ...$$

$$\mathbf{= (1/2)^{n-1} X_1 + (1/2)^{n-2} X_2 + \ldots + (1/4) X_{n-2} + (1/2) X_{n-1} + X_n}$$

$$\Rightarrow \mathbf{E[Z_n]} = (1 + (1 - (1/2)^{n-1})) E[X] = (2 - (1/2)^{n-1}) \frac{1}{2} = \mathbf{1 - 2^{-n}}$$

Notice that $\lim_{N\to\infty} E[Z_n] = 1$.

The fact that $E[W_n]$ goes to infinity exponentially is consistent with the fact that the sample functions of W_n go to infinity exponentially.

The fact that $E[Z_n]$ goes to 1 is consistent with the asymptotic stationarity of the Z_n's.

5. 6.31, p. 393

Let $N(t)$ equal the number of calls that have arrived at or before time t. We are told $N(t)$ is a Poisson process with rate 10 calls/hour. The probability that no calls go unanswered by the secretary is the probability that no calls arrive in the first 15 minutes and no calls arrive in the last 15 minutes; i.e.

P(no calls unanswered) = P(N(.25) = 0 and N(1)-N(.75) = 0)

= P(N(.25) = 0) P(N(1)-N(.75) = 0), because the numbers of calls arriving in disjoint intervals are indepedent, for a Poisson process

= $\frac{(2.5)^0}{0!} e^{-2.5} \frac{(2.5)^0}{0!} e^{-2.5}$, because N(15) and N(60)-N(15) are Poisson random variables with mean $\lambda t = 10 \times .25 = 2.5$

= $\mathbf{e^{-5}}$ = **0.0673**

6. 6.53, p. 395. *X(t) = A cos ωt + B sin ωt*

(a) Show widesense stationarity: In Problem 3 we showed

$m_X(t) = 0$ and $C_X(t,s) = \sigma^2 \cos \omega(t-s)$.

Since the mean function is a constant and the covariance function depends only on time difference, we conclude that X(t) is widesense stationary.

(b) We will find a formula for $E[X^3(t)]$ and see that it depends on t. If X(t) were stationary, then $E[X^3(t)]$ would not depend on t because the density of X(t) would be the same for all t. Since for this process $E[X^3(t)]$ depends on t, it must be that X(t) is not stationary.

$E[X^3(t)] = E[(A \cos \omega t + B \sin \omega t)^3]$

$= E[A^3] \cos^3 \omega t + 3 E[A^2B] \cos^2 \omega t \sin \omega t + 3 E[AB^2] \cos \omega t \sin^2 \omega t + E[B^3] \sin^3 \omega t$

$= E[A^3] \cos^3 \omega t + 3 E[A^2] E[B] \cos^2 \omega t \sin \omega t + 3 E[A][B^2] \cos \omega t \sin^2 \omega t + E[B^3] \sin^3 \omega t$
since A and B are independent

$= E[A^3] \cos^3 \omega t + E[B^3] \sin^3 \omega t$ since A and B have zero means

$= E[A^3] (\cos^3 \omega t + \sin^3 \omega t)$ since A and B are identical.

This is easily seen to vary with t. For example, when $E[X^3(0)] = E[A^3]$, whereas $E[X^3(\pi/2\omega)] = E[A^3] (\cos^3(\pi/2) + \sin^3(\pi/2)) = E[A^3] ((1/\sqrt{2}))^3 + (2^{-1/2})^3) = E[A^3]/\sqrt{2}$.

7. 6.57, p. 396 *Z(t) = a X(t) + b Y(t), where {X(t)} and {Y(t)} are independendent WSS random processes. (The "independence" of random processes {X(t) and Y(t) means that every finite collection of X random variables are independent of every finite collection of Y random variables.)*

(a) *Is {Z(t)} WSS?*

mean function: $\mathbf{m_Z(t)}$ = E[Z(t)] = E[aX(t)+bY(t)] = a E[X(t)]+bE[Y(t)] **= 0**

autocorrelation function:

$\mathbf{R_Z(t,s)}$ = E[Z(t)Z(s)] = E[(aX(t) + bY(t))(aX(s)+bY(s))]

$= a^2 E[X(t)X(s)] + ab E[X(t) Y(s)] + ab E[X(s) Y(t)] + b^2 E[Y(s)Y(t)]$

$= a^2 R_X(t-s) + ab E[X(t)] E[Y(s)] + ab E[X(s)] E[X(t)] + b^2 R_Y(t-s)$
since X and Y are widesense stationary

$= a^2 R_X(t-s) + b^2 R_Y(t-s)$, since means of X and Y are zero.

Since Z's mean function is constant in time and its autocorrelation function depends only on time difference, Z(t) is widesense stationary.

(b) Since Z(t) is a linear combination of jointly Gaussian random variables, it is Gaussian. Therefore, its pdf is

$$\mathbf{f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi\sigma^2_{Z(t)}}} \exp\left\{- \frac{(z-m_Z(t))^2}{2\sigma^2_{Z(t)}}\right\}}$$

where $\mathbf{m_Z(t) = 0}$

and $\sigma^2_{Z(t)} = C_Z(t,t) = R_Z(t,t) - m^2_X(t) = a^2 R_X(0) + b^2 R_Y(0) - 0 = \mathbf{(a^2+b^2) R_X(0)}$,

where we used the fact that $R_X(\tau) = R_Y(\tau)$.

8. *Suppose $\{X_t\}$ is a wide sense stationary, continuous-time Gaussian random process with mean zero and autocorrelation function $R_X(\tau) = e^{-|\tau|}$.*

(a) *Find the probability that $|X(2) - X(5)| \le 2$.*
(Hint: Find the density of the random variable Y = X(2) - X(5). Make good use of the fact that X(2) and X(5) are jointly Gaussian.)

We will find $\Pr(|Y|\le 2) = \int_{-2}^{2} f_Y(y)\,dy$, where Y = X(2)-X(5). The random variable Y is Gaussian because it is a linear combination of Gaussian random variables. To find its density we need only find its mean and variance and substitute them into the Gaussian density formula.

$$E[Y] = E[X(2)-X(5)] = E[X(2)] - E[X(5)] = 0 - 0 = 0$$

$$\sigma^2_Y = E[Y^2] - (E[Y])^2 = E[Y^2] = E[(X(2)-X(5))^2] = E[X(2)^2] - 2\,E[X(2)X(5)] + E[X(5)^2]$$

$$= R_X(0) - 2\,R_X(3) + R_X(0) = 1 - 2\,e^{-3} + 1 = 2 - 2e^{-3} = 1.90$$

Therefore, $\mathbf{Pr(|Y|\le 2)} = \int_{-2}^{2} f_Y(y)\,dy = 1 - 2\int_{2}^{\infty} f_Y(y)\,dy$ since the Gaussian density is symmetric

$$= 1 - 2\int_{2}^{\infty} \frac{1}{\sqrt{2\pi\times 1.90}} \exp\left\{- \frac{y^2}{2\times 1.90})\right\}dy$$

$$= 1 - 2 \int_{2/\sqrt{1.9}} \frac{1}{\sqrt{2\pi}} \exp\left\{- \frac{u^2}{2}\right\} du \qquad \text{letting } u = \frac{y}{\sqrt{1.90}}$$

$$= 1 - 2\,Q(1.45) = 1 - 2 \times .07 = \mathbf{.86}$$

(b) *Find the covariance matrix for the random variables X(t), X(t+1), X(t+2).*

Let $Y_1 = X(t)$, $Y_2 = X(t+1)$ and $Y_3 = X(t+2)$. The covariance matrix is the 3 by 3 matrix C = $[c_{ij}]$ where

$$c_{ij} = \mathrm{cov}(Y_i, Y_j) = E[Y_iY_j] - E[Y_i]\,E[Y_j] = E[Y_iY_j] \quad \text{since the means are zero.}$$

Each of the covariances can be expressed in terms of the autocorrelation function of X. For example, $c_{12} = E[Y_1Y_2] = E[X(t)X(t+1)] = R_X(1)$. The complete covariance matrix is

$$\mathbf{C} = \begin{bmatrix} R_X(0) & R_X(1) & R_X(2) \\ R_X(1) & R_X(0) & R_X(1) \\ R_X(2) & R_X(1) & R_X(0) \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{e^{-1}} & \mathbf{e^{-2}} \\ \mathbf{e^{-1}} & \mathbf{1} & \mathbf{e^{-1}} \\ \mathbf{e^{-2}} & \mathbf{e^{-1}} & \mathbf{1} \end{bmatrix}$$