1. 6.14, p. 391

Since \( \{X(t)\} \) is a zero-mean Gaussian random process, any collection of its random variables has a jointly Gaussian density. In particular, \( X(t) \) and \( X(t+s) \) have a jointly Gaussian density, namely, from equation (4.79), p. 238:

\[
f_{X(t_1),X(t_2)}(x_1,x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{x_1^2}{\sigma_1^2} - 2\rho \frac{x_1 x_2}{\sigma_1 \sigma_2} + \frac{x_2^2}{\sigma_2^2} \right] \right\}
\]

where \( \sigma_1^2 = \text{var}(X(t_1)) = C_{X(t_1,t_1)} = \sigma^2 \) and \( \sigma_2^2 = \text{var}(X(t_2)) = C_{X(t_2,t_2)} = \sigma^2 \) and \( \rho = \frac{\text{cov}(X(t_1),X(t_2))}{\sigma_1 \sigma_2} = \frac{C_{X(t_1,t_2)}}{\sigma_1 \sigma_2} = e^{-|t_1-t_2|} \)

2. 6.15, p. 391. \( Z(t) = X(t) + Y \)

(a) mean function: \( m_Z(t) = E[Z(t)] = E[X(t) + Y] = tE[X] + E[Y] = t m_X + m_Y \)

autocovariance function: \( C_Z(t,s) = \text{cov}(Z(t),Z(s)) = E[(Z(t)-m_Z(t))(Z(s)-m_Z(s))] \)

\[
= E[(tX + Y - tm_X - m_Y)(sX + Y - sm_X - m_Y)] \\
= E[(tX - Y)(sX - Y)] \\
= ts E[(X-m_X)^2] + (t+s) E[(X-m_X)(Y-m_Y)] + E[(Y-m_Y)^2] \\
= ts \sigma_X^2 + (t+s) \rho_{XY} \sigma_X \sigma_Y + \sigma_Y^2
\]

(b) Given that \( X \) and \( Y \) are jointly Gaussian and that \( Z(t) \) is a linear combination of \( X \) and \( Y \), we may conclude immediately that \( Z(t) \) is Gaussian. Therefore, its pdf is

\[
f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi \sigma_{Z(t)}^2}} \exp\left\{ -\frac{(z-m_Z(t))^2}{2\sigma_{Z(t)}^2} \right\}
\]

where \( m_Z(t) = t m_X + m_Y \) and \( \sigma_{Z(t)}^2 = C_{Z(t,t)} = t^2 \sigma_X^2 + 2t \rho_{XY} \sigma_X \sigma_Y + \sigma_Y^2 \)

3. 6.16, a. p. 391. \( X(t) = A \cos \omega t + B \sin \omega t \)

(a) mean function: \( m_X(t) = E[X(t)] = E[A\cos \omega t + B\sin \omega t] = E[A] \cos \omega t + E[B] \sin \omega t = 0 \)

autocovariance function: \( C_X(t,s) = \text{cov}(X(t),X(s)) \)

\[
= E[(A\cos \omega t + B\sin \omega t) (A\cos \omega s + B\sin \omega s)], \quad \text{since } m_X(t)=m_X(s) = 0 \\
= E[A^2] \cos \omega t \cos \omega s + E[AB] (\cos \omega t \sin \omega s + \sin \omega t \cos \omega s) + E[B^2] \sin \omega t \sin \omega s \\
= \sigma^2 \cos \omega t \cos \omega s + \sigma^2 \sin \omega t \sin \omega s, \quad \text{since } E[AB] = E[A] E[B] = 0 \\
= \sigma^2 \cos \omega (t-s)
\]
4. 6.24, a,b p. 392  Notes:  $X_n$'s  are a Bernoulli random process.  Skip the question: “Is the sample mean meaningful ...?”

(a) Here's one possible sequence of $X_n$'s:

$$X_1, X_2, ..., X_{10} = 1, 0, 1, 0, 0, 1, 0, 1$$

Since $W_n = 2W_{n-1} + X_n$, the $W_n$ realization (i.e. sample function) resulting from this is:

$$W_1, W_2, ..., W_{10} = 1, 2, 5, 11, 22, 44, 88, 177, 354, 709$$

Since $Z_n = \frac{1}{2}Z_{n-1} + X_n$, the $W_n$ realization (i.e. sample function) resulting from this is:

$$Z_1, Z_2, ..., Z_{10} = 1, 1, 5, 13, 13, 141, 141, 653$$

Both $\{W_n\}$ and $\{Z_n\}$ are autoregressive random processes. Because the coefficient multiplying $W_{n-1}$ is larger than 1, the $W_n$’s are nonstationary, which is consistent with the fact that the $W_n$’s are increasing exponentially, and the sample mean $\frac{1}{n} \sum_{i=1}^{n} W_n \to \infty$, which the book seems to want to refer to as meaningless. My interpretation is that this is a nonstationary random process and since the probability distributions are changing, $E[W_n]$ does not have a great deal of significance.

On the other hand, since the coefficient multiplying $Z_{n-1}$ is less than 1, the $Z_n$’s approach a steady state behavior (they have a kind of asymptotic stationarity). The sample mean will approach the asymptotic mean value, namely, $\lim_{N \to \infty} E[Z_n]$ and the $Z_n$’s have a useful (meaningful?) sample mean. We’ll find $\lim_{N \to \infty} E[Z_n]$ in the next part.

(b)  

$$W_n = 2W_{n-1} + X_n = 2(2W_{n-2} + X_{n-1}) + X_n = ...$$

$$Z_n = \frac{1}{2}Z_{n-1} + X_n = \frac{1}{2}(\frac{1}{2}Z_{n-2} + X_{n-1}) + X_n = ...$$

$$\Rightarrow E[W_n] = 2^{n-1}E[X_1] + 2^{n-2}E[X_2] + ... + 4E[X_{n-2}] + 2E[X_{n-1}] + E[X_n]$$

$$= (2^{n-1})E[X] = \frac{1}{2}(2^n - 1)$$

$$Z_n = \frac{1}{2}Z_{n-1} + X_n = \frac{1}{2}Z_{n-2} + X_{n-1} = ...$$

$$= (1/2)^{n-1}X_1 + (1/2)^{n-2}X_2 + ... + (1/4)X_{n-2} + (1/2)X_{n-1} + X_n$$

$$\Rightarrow E[Z_n] = (1 + (1 - (1/2)^n-1))E[X] = (2 - (1/2)^{n-1})\frac{1}{2} = 1 - 2^{-n}$$

Notice that $\lim_{N \to \infty} E[Z_n] = 1$.

The fact that $E[W_n]$ goes to infinity exponentially is consistent with the fact that the sample functions of $W_n$ go to infinity exponentially.

The fact that $E[Z_n]$ goes to 1 is consistent with the asymptotic stationarity of the $Z_n$’s.

5. 6.31, p. 393

Let $N(t)$ equal the number of calls that have arrived at or before time $t$. We are told $N(t)$ is a Poisson process with rate 10 calls/hour. The probability that no calls go unanswered by the secretary is the probability that no calls arrive in the first 15 minutes and no calls arrive in the last 15 minutes; i.e.
\( P(\text{no calls unanswered}) = P( N(.25) = 0 \text{ and } N(1)-N(.75) = 0 ) \)

\[ = P( N(.25) = 0 )\ P( N(1)-N(.75) = 0 ), \text{ because the numbers of calls arriving in}\]
\[ \text{disjoint intervals are independent, for a Poisson process}\]
\[ = \left(\frac{2.5}{0!}\right)^0 e^{-2.5} \left(\frac{2.5}{0!}\right)^0 e^{-2.5}, \text{ because } N(15) \text{ and } N(60)-N(15) \text{ are Poisson random}\]
\[ \text{variables with mean } \lambda t = 10\times.25 = 2.5\]
\[ = e^{-5} = 0.0673\]

6. 6.53, p. 395.  \( X(t) = A \cos \omega t + B \sin \omega t \)

(a) Show wide sense stationarity: In Problem 3 we showed
\[ m_{X(t)} = 0 \text{ and } C_{X(t)} = \sigma^2 \cos \omega(t-s). \]
Since the mean function is a constant and the covariance function depends only on time difference, we conclude that \( X(t) \) is wide sense stationary.

(b) We will find a formula for \( E[X^3(t)] \) and see that it depends on \( t \). If \( X(t) \) were stationary, then \( E[X^3(t)] \) would not depend on \( t \) because the density of \( X(t) \) would be the same for all \( t \). Since for this process \( E[X^3(t)] \) depends on \( t \), it must be that \( X(t) \) is not stationary.

\[ E[X^3(t)] = E[ (A \cos \omega t + B \sin \omega t)^3 ] \]
\[ = E[A^3] \cos^3 \omega t + 3 E[A^2B] \cos^2 \omega t \sin \omega t + 3 E[AB^2] \cos \omega t \sin^2 \omega t + E[B^3] \sin^3 \omega t \]
\[ = E[A^3] \cos^3 \omega t + 3 E[A^2] E[B] \cos^2 \omega t \sin \omega t + 3 E[A] E[B^2] \cos \omega t \sin^2 \omega t + E[B^3] \sin^3 \omega t \]
\[ \text{since } A \text{ and } B \text{ are independent}\]
\[ = E[A^3] \cos^3 \omega t + E[B^3] \sin^3 \omega t \text{ since } A \text{ and } B \text{ have zero means}\]
\[ = E[A^3] (\cos^3 \omega t + \sin^3 \omega t) \text{ since } A \text{ and } B \text{ are identical.}\]
This is easily seen to vary with \( t \). For example, when \( E[X^3(0)] = E[A^3] \), whereas \( E[X^3(\pi/2\omega)] = E[A^3] \left( \cos^3(\pi/2) + \sin^3(\pi/2) \right) = E[A^3] (1/\sqrt{2})^3 + (2-1/2)^3 = E[A^3]/\sqrt{2}.\]

7. 6.57, p. 396  \( Z(t) = a X(t) + b Y(t) \), where \( \{X(t)\} \text{ and } \{Y(t)\} \text{ are independent WSS random processes. (The "independence" of random processes } \{X(t)\text{ and } Y(t)\text{ means that every finite collection of } X \text{ random variables are independent of every finite collection of } Y \text{ random variables.)}\)

(a) Is \( \{Z(t)\} \text{ WSS?}\)

mean function:  \( m_Z(t) = E[Z(t)] = E[ aX(t)+bY(t) ] = a E[X(t)]+bE[Y(t)] = 0 \)

autocorrelation function:
\[ R_Z(t,s) = E[Z(t)Z(s)] = E[ (aX(t)+bY(t))(aX(s)+bY(s)) ] \]
\[ = a^2 E[X(t)X(s)] + ab E[X(t) Y(s)] + ab E[X(s) Y(t)] + b^2 E[Y(s)Y(t)] \]
\[ = a^2 R_X(t-s) + ab E[X(t)] E[Y(s)] + ab E[X(s)] E[X(t)] + b^2 R_Y(t-s) \]
\[ \text{since } X \text{ and } Y \text{ are wide sense stationary}\]
\[ = a^2 R_X(t-s) + b^2 R_Y(t-s), \text{ since means of } X \text{ and } Y \text{ are zero.}\]
Since \( Z \)'s mean function is constant in time and its autocorrelation function depends only on time difference, \( Z(t) \) is wide sense stationary.
(b) Since \( Z(t) \) is a linear combination of jointly Gaussian random variables, it is Gaussian. Therefore, its pdf is

\[
f_{Z(t)}(z) = \frac{1}{\sqrt{2\pi\sigma^2_{Z(t)}}} \exp\left\{ -\frac{(z-m_{Z(t)})^2}{2\sigma^2_{Z(t)}} \right\}
\]

where \( m_{Z(t)} = 0 \)

and \( \sigma^2_{Z(t)} = C_{Z(t,t)} = R_Z(t,t) - n^2_{X(t)} = a^2 R_X(0) + b^2 R_Y(0) - 0 = (a^2+b^2) R_X(0) \),

where we used the fact that \( R_X(t) = R_Y(t) \).

8. Suppose \( \{X_t\} \) is a wide sense stationary, continuous-time Gaussian random process with mean zero and autocorrelation function \( R_X(t) = e^{-|t|} \).

(a) Find the probability that \( |X(2) - X(5)| \leq 2 \).

(Hint: Find the density of the random variable \( Y = X(2) - X(5) \). Make good use of the fact that \( X(2) \) and \( X(5) \) are jointly Gaussian.)

We will find \( \Pr(|Y| \leq 2) = \int_{-2}^{2} f_Y(y) \, dy \), where \( Y = X(2) - X(5) \). The random variable \( Y \) is Gaussian because it is a linear combination of Gaussian random variables. To find its density we need only find its mean and variance and substitute them into the Gaussian density formula.

\[
E[Y] = E[X(2)-X(5)] = E[X(2)] - E[X(5)] = 0 - 0 = 0
\]

\[
\sigma^2_Y = E[Y^2] - (E[Y])^2 = E[Y^2] = E[(X(2)-X(5))^2] = E[X(2)^2] - 2 E[X(2)X(5)] + E[X(5)^2]
\]

\[
= R_X(0) - 2 R_X(3) + R_X(0) = 1 - 2e^{-3} + 1 = 2 - 2e^{-3} = 1.90
\]

Therefore, \( \Pr(|Y| \leq 2) = \int_{-2}^{2} f_Y(y) \, dy = 1 - 2 \int_{-\infty}^{\infty} f_Y(y) \, dy \) since the Gaussian density is symmetric.

\[
= 1 - 2 \int_{-2}^{2} \frac{1}{\sqrt{2\pi \cdot 1.90}} \exp\left\{ -\frac{y^2}{2\times 1.90} \right\} \, dy
\]

\[
= 1 - 2 \int_{-\sqrt{1.90}}^{\sqrt{1.90}} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\} \, du \quad \text{letting} \quad u = \frac{y}{\sqrt{1.90}}
\]

\[
= 1 - 2 Q(1.45) = 1 - 2 \times 0.07 = .86
\]

(b) Find the covariance matrix for the random variables \( X(t), X(t+1), X(t+2) \).

Let \( Y_1 = X(t), Y_2 = X(t+1) \) and \( Y_3 = X(t+2) \). The covariance matrix is the 3 by 3 matrix \( C = [c_{ij}] \) where

\[
c_{ij} = \text{cov}(Y_i, Y_j) = E[Y_i Y_j] - E[Y_i] E[Y_j] = E[Y_i Y_j] \quad \text{since the means are zero}
\]

Each of the covariances can be expressed in terms of the autocorrelation function of \( X \). For example, \( c_{12} = E[Y_1 Y_2] = E[X(t)X(t+1)] = R_X(1) \). The complete covariance matrix is

\[
C = \begin{bmatrix}
R_X(0) & R_X(1) & R_X(2) \\
R_X(1) & R_X(0) & R_X(1) \\
R_X(2) & R_X(1) & R_X(0)
\end{bmatrix}
= \begin{bmatrix}
1 & e^{-1} & e^{-2} \\
e^{-1} & 1 & e^{-1} \\
e^{-2} & e^{-1} & 1
\end{bmatrix}
\]