Problems from Leon-Garcia's book

1. 1-11, p. 22. See Problems 1-7 and 1-9 for definitions of sample mean and sample variance.

   a) Sample mean: \( <x>_{20} = 8.55 \), because \( n = 20 \) and \( <x>_n = \frac{1}{n} \sum_{i=1}^{n} x_i \) where \( x_1, \ldots, x_n \) is the data.

   Sample variance: \( <v^2>_{20} = 29.05 \), because \( <v^2>_n = \frac{1}{n} \sum_{i=1}^{n} (x_i - <x>_m)^2 \)

   b) The event "interarrival time is greater than 10 ms" occurs 8 times out of 20.

   Relative frequency \( = \frac{2}{5} \)

2. 2-1, p. 73

   a) sample space: \( S = \{1,2,3,4,5,6\} \)

   b) \( A = \{2,4,6\} \)

   c) \( A^c = \{1,3,5\} \). This is the event "odd number of dots".

   d) Probability law: \( P(A) = \frac{|A|}{6} \), for any subset \( A \) of \( S \), where \( |A| \) denotes the size of \( A \).

   Another sufficient answer is: \( P(i) = \frac{1}{6} \), \( i = 1, \ldots, 6 \)

3. 2-3, p. 73

   a) sample space: \( S = \{2,3,4,5,6,7,8,9,10,11,12\} \)

   b) \( A = \{2,4,6,8,10,12\} \)

   c) event "sum = 2" corresponds to \( \{(1,1)\} \)

   event "sum = 3" corresponds to \( \{(1,2),(2,1)\} \)

   event "sum = 4" corresponds to \( \{(1,3),(3,1),(2,2)\} \)

   and in general for \( 1 < k \leq 12 \)

   event "sum = k" corresponds to \( \bigcup_{j=1}^{\min(6,k-1)} \{(j,k-j)\} \)

4. 2-4, p. 73

   a) sample space \( S = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \)

   b) \( \{(4,1), (4,2), (4,3), (4,4)\} \)

   c) \( \{(3,3), (4,3), (5,3), (6,3)\} \)

   d) \( \{(6,6)\} \)
5. 2-6, p. 73
   a) sample space: \( S = \{ (F,F), (F,R), (F,K), (R,F), (R,R), (R,K), (K,F), (K,R), (K,K) \} \)
   b) event = \( \{ (F,F), (F,R), (R,F), (R,R) \} \)

6. 2-7, p. 73
   a) sample space: \( S = \{ (1,2,3), (2,1,3), (3,1,2), (1,3,2), (2,3,1), (3,2,1) \} \)
   b) \( A_1 = \{ (1,2,3), (1,3,2) \} \)
   c) \( A_2 = \{ (1,2,3), (3,2,1) \} \)
   d) \( A_3 = \{ (1,2,3), (2,1,3) \} \)
   e) \( A_1 \cap A_2 \cap A_3 = \{ (1,2,3) \} \)
   f) \( A_1 \cup A_2 \cup A_3 = \{ (1,2,3), (1,3,2), (3,2,1), (2,1,3) \} \)

7. 2-15, p. 74 (Use the axioms, not Venn diagrams, though Venn diagram may suggest what to do.)
   a) sample space: \( S = \{ (T_1, T_2) : 0 \leq T_1 < T_2 \leq 24 \} \)
   b) \( A = \{ \text{awake at 9} \} = \{ T_1 \leq 9 \text{ and } T_2 \geq 9 \} = \{ \text{wake up before 9} \} \cap \{ \text{go to sleep after 9} \} = \{ (T_1, T_2) : 0 \leq T_1 < 9 \text{ and } 9 < T_2 \leq 24 \} \)
   c) the amount of time the student is awake is \( T_2 - T_1 \). the amount of time the student is asleep is \( 24-(T_2-T_1) \). Therefore,
   \[ B = \{ \text{student asleep more than he is awake} \} = \{ (T_1, T_2) \in S : 24-(T_2-T_1) > T_2-T_1 \} = \{ (T_1, T_2) \in S : T_2-T_1 < 12 \} = \{ (T_1, T_2) \in S : T_2 < T_1 + 12 \} \]
   d) \( A^c = \{ (T_1, T_2) \in S : (T_1, T_2) \notin A \} = \{ (T_1, T_2) : 0 \leq T_2 < T_2 \leq 24, \text{ and } (T_1 > 9 \text{ or } T_2 < 9) \} \)
   e) \( A^c \cap B = B - A = \{ \text{student is asleep at 9 and sleeps for more than 12 hours} \} \)
8. Find a probability model for the experiment described in Prob. 2-17, p. 75. 
Then do Problem 2-17

The sample space is \( S = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(2,6),(3,1),\ldots,(6,6)\} \), and the probability of each elementary event \( \{(i,j)\} \) is \( \frac{1}{36} \).

a) \( A_2 = \{(1,1)\} \), \( P(A_2) = P(\{(1,1)\}) = \frac{1}{36} \)

\[ A_3 = \{(1,2), (2,1)\}, \quad P(A_3) = P(\{(1,2)\}) + P(\{2,1\}) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36} \]

\[ A_4 = \{(1,3), (3,1), (2,2)\}, \quad P(A_4) = 3 \times \frac{1}{36} = \frac{3}{12}, \quad P(A_5) = 4 \times \frac{1}{36} = \frac{4}{36} \]

Similarly,

\[ P(A_6) = \frac{5}{36}, \quad P(A_7) = \frac{6}{36}, \quad P(A_8) = \frac{5}{36}, \quad P(A_9) = \frac{4}{36}, \quad P(A_{10}) = \frac{3}{36}, \quad P(A_{11}) = \frac{2}{36}, \quad P(A_{12}) = \frac{1}{36} \]

b) \( B = \{\text{two outcomes are different}\} \). We could list the members of this set, but it's easier to list the elements of the complement:

\[ B^c = \{\text{two outcomes are the same}\} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\} \]

\[ P(B^c) = \frac{6}{36}. \quad \text{And therefore} \quad P(B) = 1 - P(B^c) = 1 - \frac{6}{36} = \frac{30}{36} \]

9. 2-19, p. 75

There are many ways to do this problem. Here's one.

\{exactly one of the events \( A \) or \( B \) occurs\} = \(\{x \in S : (x \text{ is in } A \text{ but not } B) \text{ or } (x \text{ is in } B \text{ but not } A)\} \)

\[ = (A - B) \cup (B - A) \].

\[ P(A-B) = P(A) - P(A \cap B) \quad \text{(because } A-B, A \cap B \text{ are disjoint, and } (A-B) \cup (A \cap B) = A, \text{ so by Axiom 3: } P(A-B)+P(A \cap B) = P(B)) \]

Similarly, \( P(B-A) = P(B) - P(A \cap B) \). Finally, since \( A - B \) and \( B - A \) are disjoint

\[ P((A - B) \cup (B - A)) = P(A - B) + P(B - A) = P(A) - P(A \cap B) + P(B) - P(A \cap B) \]

\[ = P(A) + P(B) - 2P(A \cap B) \]

10. 2-29, p. 76

Let \( T \) denote the lifetime. Then \( A = \{T > 5\} \) and \( B = \{T > 10\} \).

a) \( P(A \cap B) = P(\{T > 5\} \cap \{T > 10\}) = P(\{T > 10\}) = e^{-10} \)

\[ P(A \cup B) = P(\{T > 5\} \cup \{T > 10\}) = P(\{T > 5\}) = e^{-5} \]

b) \{lifetime greater than 5 but less than or equal to 10\} = \{5 < T \leq 10\}

Now, \( \{T > 5\} = \{5 < T \leq 10\} \cup \{T > 10\} \) and the two events on the righthand side are disjoint, so

\[ P(\{T > 5\}) = P(\{5 < T \leq 10\}) + P(\{T > 10\}). \]

Therefore,

\[ P(\{5 < T \leq 10\}) = P(\{T > 5\}) - P(\{T > 10\}) = e^{-5} - e^{-10} \]