Homework Set 2 Solutions **EECS 401** **Friday, Jan. 28, 2000,**

This is the "final" version. Revisions to the 1/21/00 draft are marked with asterisks: *****.

1. *Find a probability model for the experiment of problem 2-4, p. 73. (This was one of the cancelled parts of the previous assignment.)*

   The probability model consists of:

   Variable name: $\underline{N} = (N_1,N_2)$

   Sample space: sample space

   $$S = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

*** Probability law: $P((i,j)) = \frac{1}{6i}$, **for** $i = 1,\ldots,6$ **and** $j = 1,\ldots,i$

   As mentioned in class, for a discrete-valued experiment like this, it is enough to list the probability of each individual outcome. The reason for this probability law is:

   (1) Probability of the event "first dice equals i" = $\frac{1}{6}$ = P({(i,1), (i,2), ...,(i,i)})

   (2) Each outcome within the set {(i,1), (i,2), ...,(i,i)} has equal probability. Therefore $P((i,j)) = \frac{1}{6i}$.

2. *2-20, p. 75.*

   Since it's hard to draw Venn diagrams with the word processor, so I'll do these by formulas:

   $\mathbf{P(A^c \quad B^c)} = P((A\cap B)^c)$ by DeMorgan's law

   $= 1 - P(A\cap B) = \mathbf{1 - z}$

   $\mathbf{P(A\cap B^c)} = P(A) - P(A\cap B) = \mathbf{x - z}$

   $\mathbf{P(A^c \quad B)} = 1 - P((A^c \quad B)^c) = 1 - P(A\cap B^c) = 1 - (x-z) = \mathbf{1 - x + z}$

   $\mathbf{P(A^c\cap B^c)} = 1 - P((A^c\cap B^c)^c) = 1 - P(A \quad B) = 1 - (P(A) + P(B) - P(A\cap B)) = \mathbf{1 - x - y + z}$

3. *Find a probabilitiy model for the experiment for problem 2-23, p. 75. Then do the problem.*

   Probability model:

   Variable: $\underline{X} = (X_1,X_2,X_3,X_4)$

   Sample space: S = { HHHH, HHHT, HHTH, HHTT, ... , TTTT}

   = set of all sequences of length 4 where each term is H or T

   Probability law: $P(\underline{x}) = \frac{1`}{16}$ for any $\underline{x}$ S

   $\mathbf{P(A_2)}$ = P({HHHH, HHHT, HHTH, HHTT, THHH, THTH, THHT}) $= 8 \times \frac{1}{16} = \mathbf{\frac{1}{2}}$

P($A_1 \cap A_3$) = P({HHHH, HHHT, HTHH, HTHT}) = $4 \times \frac{1}{16}$ = $\mathbf{\frac{1}{4}}$

P($A_1 \cap A_2 \cap A_3 \cap A_4$) = P({HHHH}) = $\mathbf{\frac{1}{16}}$

P($A_1 \quad A_2 \quad A_3 \quad A_4$) = P({TTTT}^{c}) = 1 - P({TTTT}) = $1 - \frac{1}{16}$ = $\mathbf{\frac{15}{16}}$

4. *2-24, p. 75, except assume that A = {k > 2} and B = {k>4} (Hint: The experiment described here is closely related to that in the previous problem.)*

   Consider the sample space of all possible outcomes of four tosses:

   S = { HHHH, HHHT, HHTH, HHTT, ..., TTTT}

   Each of these outcomes has equal probability, namely, 1/16.

   While the experiment that is described in the problem can involve many more than four tosses, we observe that that the events A = {k>2} occurs precisely when the event A' = {TTHT, TTHH, TTTH, TTTT} of the sample space S occurs. Therefore,

   P(A) = P(A') = $\frac{4}{16}$ = $\mathbf{\frac{1}{4}}$

   Similarly, the event B = {k > 4} occurs precisely when the event B' = {TTTT} occurs. So

   P(B) = P(B') = $\mathbf{\frac{1}{16}}$ and **P(B^c)** = 1 - P(B) = $\mathbf{\frac{15}{16}}$.

   Also, **P(A$\cap$B)** = P(A'$\cap$B') = P({TTTT}) = $\mathbf{\frac{1}{16}}$,

   and **P(A B)** = P(A' B') = P({TTHT, TTHH, TTTH, TTTT}) = P(A') = $\mathbf{\frac{1}{4}}$

5. 2-29, p. 76. (this problem was cancelled because it was part of the previous homework set)

6. Which of the following are true statements?

   (a) If E F, then F^c E^c. **TRUE**

   (b) F = F$\cap$E F$\cap$E^c. **TRUE**

   (c) E F = E F$\cap$E^c. **TRUE**

   (d) If E$\cap$F = φ and F$\cap$G = φ, then E$\cap$G = φ. **FALSE**

   counter example: E = (0,3), F = (5,7), G = (2,4)

   (e) If E F, then E $\cap$ G F $\cap$ G. **TRUE**

   (f) If P(A B) = P(A) + P(B), then A and B are disjoint. **FALSE**

   this does imply P(A$\cap$B) = 0, but the set A$\cap$B need not be empyt.

7. *A 5-sided die is tossed and the outcome is the number facing down. We are given the probabilities of the following events.*

P(odd) = .7, P({1,5}) = .4,

Find the probabilities of as many of the following events as possible.

{1}, {2}, {3}, {4}, {5}, {1,3}, {2,4}, {2,5}

P({3}) = P(odd) - P({1,5}) = .7 - .4 = .3

P({2,4}) = P(even) = 1 - P(odd) = 1 - .7 **= .3**

These are the only ones we can find. (We can't assume the dice is "fair".)

8. *Suppose A and B are subsets of a sample space S and P(A) = 1. Use the axioms of probability to prove the following:*

(a) *P(A   B) = 1.*

A good way to prove this is to separately prove the two statements: P(A   B) ≤ 1 and P(A   B) ≥ 1. The first follows directly from the corollary that says that the probability of any set is less than or equal to 1. The second statement can be proved as follows:

P(A   B) ≥ P(A) = 1 because A   B and a corollary says P(E) ≤ P(F) when E   F

Since P(A   B) ≤ 1 and P(A   B) ≥ 1, it follows that P(A   B) = 1.

(b) *P(A∩B) = P(B).*

We use the same approach. P(A∩B) ≤ P(B) because A∩B   B and because of the corollary mentioned above. By another corollary

P(A∩B) = P(A) + P(B) - P(A   B)   ≥ 1 + P(B) - 1 = P(B), since P(A) = 1 & P(A   B) ≤ 1.

Since P(A∩B) ≤ P(B) and P(A∩B) ≥ P(B), it follows that P(A∩B) = P(B).

These facts will be useful later in the course.

9. *Suppose A and B are subsets of a sample space S and P(A) = 0. Use the axioms of probability to prove the following:*

(a) *P(A   B) = P(B).*

We use the same approach. P(A   B) ≥ P(B), because B   A   B and because of the corollary.

P(A   B) = P(A) + P(B) - P(A∩B) by a corollary

≤ 0 + P(B) - 0 = P(B), because P(A) = 0 & because the 1st axiom says P(A∩B) ≥ 0

Since P(A   B) ≤ P(B) and P(A   B) ≥ P(B), it follows that P(A   B) = P(B).

(b) *P(A∩B) = 0.*

We use the same approach. P(A∩B) ≥ 0 by the 1st axiom of probability.

P(A∩B) ≤ P(A) = 0, because A∩B   A and because of the aforementioned corollary.

Since P(A∩B) ≤ 0 and P(A∩B) ≥ 0, it follows that P(A∩B) = 0.

These facts will be useful later in the course.

10. *A certain store accepts either the American Express or the VISA credit card. It finds that when a customer enters the store, the probability is .24 that he/she carries an American Express card, .61 that he/she carries a VISA card, .11 that he/she carries both, and .45 that he she/carries a Master card. What is the probability that a customer carries a credit card that the store will accept?*

From the problem statement we deduce

A = event customer carries AmEx,	P(A) = .24,
V = event customer carries Visa,	P(V) = .61,
AV = event customer carries AmEx and Visa,	P(AV) = .11.
M = event customer carries MC,	P(MC) = .45

Also we deduce that $AV = A \cap V$

We want to find the probability of the event that customer carries AmEX or Visa. We recognize this as the event A V, and we find

P(A V) = P(A) + P(V) - P(A ∩ V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 **= .74**

11. *You are on a trip from Detroit to Denver with a change of planes in Chicago. The plane from Detroit to Chicago arrives randomly between 12:00 noon and 1:00 PM. The connecting plane form Chicago leaves randomly between 1:00 PM and 2:00 PM. To make the connection, at least 30 minutes are required between planes. What is the probability that you will miss your connection? (Hint: It might be wise to make a probability model before attempting to find the answer.)*

Probability Model: Variable (A,D) where A = time you arrive, D = time plane departs.

Sample space: $S = \{ (a,d) : 0 \le a \le 1,\ 1 \le d \le 2 \}$, (0 = noon)

Probability law (this is a key part): for any subset B of S, $P(B) = \frac{|B|}{|S|}$ = area of B

Why this law? because all sets of the same area should have the same probability

Pr(miss your connection) = $P(\{(a,d) : d < a + .5,\ 0 \le a \le 0,\ 1 \le d \le 2\})$

= area of shaded region below = $\mathbf{\frac{1}{8}}$