Homework Set 2 Solutions *DRAFT* **EECS 401** **Friday, Jan. 21, 2000,**

1. *Find a probability model for the experiment of problem 2-4, p. 73. (This was one of the cancelled parts of the previous assignment.)*

   The probability model consists of:

   Variable name: $\underline{\mathbf{N}} = (\mathbf{N_1},\mathbf{N_2})$

   Sample space: sample space

   $$\mathbf{S} = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,1), (5,2), (5,3), (5,4), (5,5), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

   Probability law: $\mathbf{P((i,j)) = \frac{1}{i}}$ **for** $\mathbf{j = 1,...,i}$

   (as mentioned in class, for a discrete-valued experiment like this, it is enough to list the probability of each individual outcom.)

2. *2-20, p. 75.*

   Since it's hard to draw Venn diagrams with the word processor, so I'll do these by formulas:

   $\mathbf{P(A^c \cup B^c)} = P((A\cap B)^c)$ by DeMorgan's law

   $= 1 - P(A\cap B) = \mathbf{1 - z}$

   $\mathbf{P(A\cap B^c)} = P(A) - P(A\cap B) = \mathbf{x - z}$

   $\mathbf{P(A^c\cup B)} = 1 - P((A^c\cup B)^c) = 1 - P(A\cap B^c) = 1 - (x-z) = \mathbf{1 - x + z}$

   $\mathbf{P(A^c\cap B^c)} = 1 - P((A^c\cap B^c)^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A\cap B)) = \mathbf{1 - x - y + z}$

3. *Find a probabilitiy model for the experiment for problem 2-23, p. 75. Then do the problem.*

   Probability model:

   Variable: $\underline{X} = (X_1,X_2,X_3,X_4)$

   Sample space: S = { HHHH, HHHT, HHTH, HHTT, ... , TTTT}

   = set of all sequences of length 4 where each term is H or T

   Probability law: $P(\underline{x}) = \frac{1`}{16}$ for any $\underline{x} \in S$

   $\mathbf{P(A_2)}$ = P({HHHH, HHHT, HHTH, HHTT, THHH, THTH, THHT}) $= 8 \times \frac{1}{16} = \mathbf{\frac{1}{2}}$

   $\mathbf{P(A_1 \cap A_3)}$ = P({HHHH, HHHT, HTHH, HTHT}) $= 4 \times \frac{1}{16} = \mathbf{\frac{1}{4}}$

   $\mathbf{P(A_1 \cap A_2 \cap A_3 \cap A_4)}$ = P({HHHH}) $= \mathbf{\frac{1}{16}}$

   $\mathbf{P(A_1 \cup A_2 \cup A_3 \cup A_4)}$ = P({TTTT}^{c}) = 1 - P({TTTT}) $= 1 - \frac{1}{16} = \mathbf{\frac{15}{16}}$

4. *2-24, p. 75, except assume that $A = \{k > 2\}$ and $B = \{k>4\}$ (Hint: The experiment described here is closely related to that in the previous problem.)*

Consider the sample space of all possible outcomes of four tosses:

$S = \{$ HHHH, HHHT, HHTH, HHTT, ..., TTTT$\}$

Each of these outcomes has equal probability, namely, 1/16.

While the experiment that is described in the problem can involve many more than four tosses, we observe that that the events $A = \{k>2\}$ occurs precisely when the event $A' = \{TTHT, TTHH, TTTH, TTTT\}$ of the sample space S occurs. Therefore,

$$\mathbf{P(A)} = P(A') = \frac{4}{16} = \mathbf{\frac{1}{4}}$$

Similarly, the event $B = \{k > 4\}$ occurs precisely when the event $B' = \{TTTT\}$ occurs. So

$$\mathbf{P(B)} = P(B') = \mathbf{\frac{1}{16}} \text{ and } \mathbf{P(B^c)} = 1 - P(B) = \mathbf{\frac{15}{16}} .$$

Also, $\mathbf{P(A \cap B)} = P(A' \cap B') = P(\{TTTT\}) = \mathbf{\frac{1}{16}}$,

and $\mathbf{P(A \cup B)} = P(A' \cup B') = P(\{TTHT, TTHH, TTTH, TTTT\}) = P(A') = \mathbf{\frac{1}{4}}$

5. 2-29, p. 76.

Probability model (it's not required but it's a good idea):

Variable T = life time, Sample space $S = (0,\infty)$,

Prob. Law: $P((t,\infty)) = e^{-t}$, for any $t \geq 0$. This implies $P((a,b]) = e^{-a} - e^{b}$, $0 \leq a \leq b < \infty$.

$A = (5,\infty)$, $B = (10,\infty)$

a. $\mathbf{P(A \cap B)} = P((5,\infty) \cap (10,\infty)) = P((10,\infty)) = \mathbf{e^{-10}}$

$\mathbf{P(A \cup B)} = P((5,\infty) \cup (10,\infty)) = P((5,\infty)) = \mathbf{e^{-5}}$

b. **Pr(lifetime greater than 5 but less than or equal to 10)** $= P((5,10]) = e^{-5} - e^{-10}$

6. Which of the following are true statements?

(a) If $E \subset F$, then $F^c \subset E^c$. **TRUE**

(b) $F = F \cap E \cup F \cap E^c$. **TRUE**

(c) $E \cup F = E \cup F \cap E^c$. **TRUE**

(d) If $E \cap F = \phi$ and $F \cap G = \phi$, then $E \cap G = \phi$. **FALSE**

counter example: $E = (0,3)$, $F = (5,7)$, $G = (2,4)$

(e) If $E \subset F$, then $E \cap G \subset F \cap G$. **TRUE**

(f) If $P(A \cup B) = P(A) + P(B)$, then A and B are disjoint. **FALSE**

this does imply $P(A \cap B) = 0$, but the set $A \cap B$ need not be empyt.

7. *A 5-sided die is tossed and the outcome is the number facing down. We are given the probabilities of the following events.*

P(odd) = .7, P({1,5}) = .4,

Find the probabilities of as many of the following events as possible.

{1}, {2}, {3}, {4}, {5}, {1,3}, {2,4}, {2,5}

P({3}) = P(odd) - P({1,5}) = .7 - .4 = .3

P({2,4}) = P(even) = 1 - P(odd) = 1 - .7 **= .3**

These are the only ones we can find. (We can't assume the dice is "fair".)

8. *Suppose A and B are subsets of a sample space S and P(A) = 1. Use the axioms of probability to prove the following:*

(a) *$P(A \cup B) = 1$.*

A good way to prove this is to separately prove the two statements: $P(A \cup B) \leq 1$ and $P(A \cup B) \geq 1$. The first follows directly from the corollary that says that the probability of any set is less than or equal to 1. The second statement can be proved as follows:

$P(A \cup B) \geq P(A) = 1$ because $A \subset B$ and a corollary says $P(E) \leq P(F)$ when $E \subset F$

Since $P(A \cup B) \leq 1$ and $P(A \cup B) \geq 1$, it follows that $P(A \cup B) = 1$.

(b) *$P(A \cap B) = P(B)$.*

We use the same approach. $P(A \cap B) \leq P(B)$ because $A \cap B \subset B$ and because of the corollary mentioned above. By another corollary

$P(A \cap B) = P(A) + P(B) - P(A \cup B) \quad \geq 1 + P(B) - 1 = P(B)$, since $P(A) = 1$ & $P(A \cup B) \leq 1$.

Since $P(A \cap B) \leq P(B)$ and $P(A \cap B) \geq P(B)$, it follows that $P(A \cap B) = P(B)$.

These facts will be useful later in the course.

9. *Suppose A and B are subsets of a sample space S and P(A) = 0. Use the axioms of probability to prove the following:*

(a) *$P(A \cup B) = P(B)$.*

We use the same approach. $P(A \cup B) \geq P(B)$, because $B \subset A \cup B$ and because of the corollary.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ by a corollary

$\leq 0 + P(B) - 0 = P(B)$, because $P(A) = 0$ & because the 1st axiom says $P(A \cap B) \geq 0$

Since $P(A \cup B) \leq P(B)$ and $P(A \cup B) \geq P(B)$, it follows that $P(A \cup B) = P(B)$.

(b) *$P(A \cap B) = 0$.*

We use the same approach. $P(A \cap B) \geq 0$ by the 1st axiom of probability.

$P(A \cap B) \leq P(A) = 0$, because $A \cap B \subset A$ and because of the aforementioned corollary.

Since $P(A \cap B) \leq 0$ and $P(A \cap B) \geq 0$, it follows that $P(A \cap B) = 0$.

These facts will be useful later in the course.

10. *A certain store accepts either the American Express or the VISA credit card. It finds that when a customer enters the store, the probability is .24 that he/she carries an American Express card, .61 that he/she carries a VISA card, .11 that he/she carries both, and .45 that he she/carries a Master card. What is the probability that a customer carries a credit card that the store will accept?*

From the problem statement we deduce

A = event customer carries AmEx,	P(A) = .24,
V = event customer carries Visa,	P(V) = .61,
AV = event customer carries AmEx and Visa,	P(AV) = .11.
M = event customer carries MC,	P(MC) = .45

Also we deduce that $AV = A \cap V$

We want to find the probability of the event that customer carries AmEX or Visa. We recognize this as the event $A \cup V$, and we find

$$\mathbf{P(A \cup V)} = P(A) + P(V) - P(A \cap V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = \mathbf{.74}$$

11. *You are on a trip from Detroit to Denver with a change of planes in Chicago. The plane from Detroit to Chicago arrives randomly between 12:00 noon and 1:00 PM. The connecting plane form Chicago leaves randomly between 1:00 PM and 2:00 PM. To make the connection, at least 30 minutes are required between planes. What is the probability that you will miss your connection? (Hint: It might be wise to make a probability model before attempting to find the answer.)*

Probability Model: Variable (A,D) where A = time you arrive, D = time plane departs.

Sample space: $S = \{ (a,d) : 0 \le a \le 1,\ 1 \le d \le 2 \}$, (0 = noon)

Probability law (this is a key part): for any subset B of S, $P(B) = \frac{|B|}{|S|}$ = area of B

Why this law? because all sets of the same area should have the same probability

Pr(miss your connection) = $P(\{(a,d) : d < a + .5,\ 0 \le a \le 0,\ 1 \le d \le 2\})$

= area of shaded region below = $\mathbf{\frac{1}{8}}$