Homework 3 Solutions *DRAFT* **EECS 401** **Friday, Jan. 28, 2000,**

1. 2-32, p. 76, L-G

   The number of distinct ordered triplets = 60 × 60 × 60 = $\mathbf{60^3}$ = **216,000**

2. 2-35, p. 76, L-G

   The number of distinct 7-tuples = 8 × 10 × 10 × 10 × 10 × 10 × 10 = 8 × $\mathbf{10^6}$

3 2-38, p. 76, L-G

There are 3! permutations (possible orderings of 3 books) of which only one corresponds to the correct order. Assuming all permutations are equiprobable,

$$\mathbf{P(correct\ order) = \frac{1}{3!} = \frac{1}{6}}$$

4. 2-40, p. 77, L-G

   number of ways of picking 1 out of 5 = $\binom{5}{1}$ = **5**

   number of ways of picking 2 out of 5 = $\binom{5}{2}$ = **10**

   number of ways of picking none, some or all 5 = $\sum_{i=1}^{5} \binom{5}{i}$ = $\mathbf{2^5}$ (by binomial thm)

5. 2-43, p. 77, L-G

   There are $\binom{49}{6}$ ways of choosing six numbers. Assuming each is equiprobable, the probability of winning is

$$\frac{1}{\binom{49}{6}} = \mathbf{7.15 \times 10^{-8}}$$

6. 2-54, p. 78, L-G (Note: Corollary 1 is in Section 2.2.)

The situation described is equivalent to the experiment of choosing twenty numbers in sequence from the set {1,2,...,365} with replacement and recording the 20 numbers in order.
A probability model for the experiment is:

Variable representing the outcome: $\underline{X} = (X_1,\ldots,X_{20})$.

Sample space: $S = \{\underline{x} = (x_1,\ldots,x_{20}) : x_i \quad \{1,\ldots,365\}\ \}$

Probability law: $P(\underline{x}) = \frac{1}{365^{20}}$ for $\underline{x} \quad S$

(therefore, for any set A, $P(A) = |A| \times \frac{1}{365^{20}}$)

We are interested in the probability of the event A that 2 or more students have the same birthday. As suggested by the hint, it is easier to findthe probability of the complement A^c, which is the event that no two students have the same birthday.

$A^c = \{\underline{x} = (x_1,\ldots,x_{20}) : x_i \quad \{1,\ldots,365\},\ x_i \neq x_j\ ,\ i = 1,\ldots,20,\ j = 1,\ldots,20\}$

The number of elements in A^c is $365 \times 364 \times \ldots \times 346 = \dfrac{365!}{345!} = 1.037 \times 10^{51}$

And so $P(A^c) = |A| \times \dfrac{1}{365^{20}} = \dfrac{365!}{345!} \times \dfrac{1}{365^{20}} = .589$

$P(A) = 1 - P(A^c) = .411$

7. *A circular pan of water with radius 10 inches is stirred and the position of a floating spec is recorded after the water has come to rest.*

(a) *Determine the conditional probability that the spec is more than 1 inch from the center given that it is less than the 3 inces from the center.*

We begin with a probability model:

Variable representing outcome: (X,Y) where X and Y represent the horizontal and vertical positions, respectively, of the spec after it comes to rest.

Sample space: $S = \{ (x,y) : x^2 + y^2 \leq 10 \}$

Probability law: $P(A) = \dfrac{\text{area } A}{\text{area } S} = \dfrac{\text{area } A}{100\pi}$

(here we have given a general formula for the probability of any set)

We need to find $P(A|B)$ where $A = \{ (x,y) : 1 < x^2 + y^2 \leq 10 \}$ and $B = \{ (x,y) : x^2 + y^2 < 3 \}$

From the definition of conditional probability

$$\mathbf{P(A|B)} = \frac{P(A\cap B)}{P(B)} = \frac{\text{area}(A\cap B)/100\pi}{\text{area}(B)/100\pi} = \frac{\text{area}(A\cap B)}{\text{area}(B)} = \frac{\pi 3^2 - \pi 1^2}{\pi 3^2} = \mathbf{\frac{8}{3}}$$

(b) *Determine the conditional probability that the spec is both to the north and east of the landing point, given that it is less than 5 inches from the center.*

We need to find $P(A|B)$ where $A = \{(x,y): x^2 + y^2 \leq 10, x > 0, y > 0\}$ and $B = \{(x,y): x^2 + y^2 < 5\}$

From the definition of conditional probability

$$\mathbf{P(A|B)} = \frac{P(A\cap B)}{P(B)} = \frac{\text{area}(A\cap B)/100\pi}{\text{area}(B)/100\pi} = \frac{\text{area}(A\cap B)}{\text{area}(B)} = \frac{\pi 5^2/4}{\pi 5^2} = \mathbf{\frac{1}{4}}$$

8. 2-49, p. 77, L-G.

Let B be a set such that $P(B) > 0$, so the axioms of probability are satisfied. Then,

Axiom 1: $P(A|B) \geq 0$ for every set A, because $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ and both numerator and denominator are never negative.

Axiom 2: $P(S|B) = 1$, because $P(S|B) = \dfrac{P(S\cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$.

Axiom 3: $P(A_1 \quad A_2|B) = P(A_1|B) + P(A_2|B)$ if A_1 and A_2 are disjoint because

$$P(A_1 \quad A_2|B) = \frac{P((A_1 \quad A_2)\cap B)}{P(B)} = \frac{P((A_1\cap B) \quad (A_2\cap B))}{P(B)}$$

$$= \frac{P(A_1\cap B)+P(A_2\cap B)}{P(B)} \quad \text{since } A_1\cap B \text{ and } A_2\cap B \text{ are disjoint}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} = P(A_1|B) + P(A_2|B)$$

Axiom 4: $P(A_1 \quad A_2 \quad \ldots |B) = P(A_1|B) + P(A_2|B) + \ldots$ if A_1, A_2, ... is sequence of disjoint events, because

$$P(A_1 \quad A_2 \quad \ldots |B) = \frac{P((A_1 \quad A_2 \quad \ldots) \cap B)}{P(B)} = \frac{P((A_1 \cap B) \quad (A_2 \cap B) \quad \ldots)}{P(B)}$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B) + \ldots}{P(B)}$$ since $A_1 \cap B$, $A_2 \cap B$, ... are disjoint

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} + \ldots = P(A_1|B) + P(A_2|B) + \ldots$$

9. 2-50, p. 77, L-G.

$P(A \cap B \cap C) = P(A \cap D)$ where $D = B \cap C$

$= P(A|D)\, P(D)$ by the definition of conditional probability

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$= P(A|B \cap C)\, P(B|C)\, P(C)$ by the definition of conditional probability

10. 2-51, p. 77, L-G.

Probability model for the experiment:

(N_1,N_2), where N_1 and N_2 are outcomes of first and second dice, respectively

$S = \{ (1,1), (1,2), \ldots, (1,6), (2,1),\ldots,(6,6)\} = \{ (i,j) : i \quad \{1,2,\ldots,6\},\ j \quad \{1,2,\ldots,6\} \}$

$P((i,j)) = \frac{1}{36}$

A = "N_1+N_2 is even" = $\{(i,j) : i \quad \{1,2,\ldots,6\},\ j \quad \{1,2,\ldots,6\},\ i + j \quad \{2,4,\ldots,12\} \}$

B = "N_1 is even and N_2 is even" = $\{ (i,j) : i \text{ is even}, j \text{ is even}\}$

notice that $B \quad A$

(a) $\mathbf{P(A|B)} = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)}$

(b) $\mathbf{P(B|A)} = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{|B|/36}{|A|/36} = \frac{|B|}{|A|} = \frac{3\times3}{6\times3} = \mathbf{\frac{1}{2}}$

11. 2-56, p. 78, L-G

Probability model:

T = time of arrival
$S = (8,9)$ (the interval)

$P(A) = \frac{\text{length of A in hours}}{\text{1 hour}}$ = length of A in hours

Let B = event she has not arrived by 8:30 = event she arrives after 8:30 = (8.5,9) (the interval)

Let A = event she arrives in between 8:30 and 8:31 = (8.5, 8.5+1/60) .

Then $\mathbf{P(A|B)} = \frac{P(A\cap B)}{P(B)} = \frac{P((8.5,8.5+1/60))}{P((8.5,9))} = \frac{1/60}{1/2} = \mathbf{\frac{1}{30}}$

Let B = event she has not arrived by 8:50 = event she arrives after 8:50 = (8 5/6,9)

Let A = event she arrives in between 8:50 and 8:31 = (8 5/6, 8 5/6+1/60) .

Then $\mathbf{P(A|B)} = \frac{P(A\cap B)}{P(B)} = \frac{P((8\ 5/6,8\ 5/6+1/60))}{P((8\ 5/6,9))} = \frac{1/60}{1/6} = \mathbf{\frac{1}{10}}$

Knowledge that the professor arrives by 9 AM implies that the probability of imminent arrival approaches 1 as time approaches 9 AM.

12. *A store opens at time 0 and never closes. The probability that the first customer to enter the store arrives between times t_1 and t_2 is*

$$\int_{t_1}^{t_2} e^{-t}\,dt\ ,\ \text{where } 0 \le t_1 \le t_2 < \infty\ .\ \text{(Time is measured continuously from 0.)}$$

(a) *Find the probability that the first customer arrives between times t and t+1 given that no customer arrive at or before time t.*

Let A = event that first customer arrives between t and t+1 = (t,t+1)

Let B = event that no customer arrives before time t

= event that first customer arrives at time t or later = $[t,\infty)$

$$\mathbf{P(A|B)} = \frac{P(A\cap B)}{P(B)} = \frac{P(\ (t,t+1) \cap [t,\infty)\ )}{P([t,\infty))} = \frac{P(\ (t,t+1)\ )}{P([t,\infty))} = \frac{\int_t^{t+1} e^{-u}\,du}{\int_t^{\infty} e^{-u}\,du}$$

$$= \frac{e^{-t}-e^{-t-1}}{e^{-t}} = \mathbf{1 - e^{-1}}$$

(b) *Discuss the dependence of your answer on t.*

Interestingly the answer does not depend on t. That is, the probability of arrival in the next minute does not get larger as time passes. This is much different than the situation in the previous problem.

13. 2-57a., p. 78, L-G

Let X denote the input and Y the output.

(a) $\mathbf{P(Y=0)}$ = P(Y=0|X=0) P(X=0) + P(Y=0|X=1) P(X=1) by the law of total probability

$$= \mathbf{(1-\varepsilon_1)\,\frac{1}{2}} + \varepsilon_2\,\mathbf{\frac{1}{2}} = \mathbf{\frac{1}{2}\,(1 - \varepsilon_1 + \varepsilon_2)}$$