Homework Set 4 Solutions *DRAFT* **EECS 401** **Feb. 7, 2000**

1. *A total of 46 percent of the eligible voters in a certain city classify themselves as Independents, whereas 30 percent classify themselves as Liberals, and 24 percent as Conservatives. In a recent local election, 35 percent of the Independents, 62 percent of the Liberals, and 58 percent of the Conservatives actually voted. An eligible voter is chosen at random.*

   (a) Find the probability that this voter actually voted.

   By the law of total probability

$$\mathbf{Pr(V)} = Pr(V|I)\,Pr(I) + Pr(V|L)\,Pr(L) + Pr(V|C)\,Pr(C)$$
$$= .35 \times .46 + .62 \times .30 + .58 \times .24 = \mathbf{.486}$$

   (b) *Given that this voter voted in the election, what is the probability that he or she is an Independent?*

   Using Bayes rule: $\mathbf{Pr(I|V)} = \dfrac{Pr(V|I)\,Pr(I)}{Pr(V)} = \dfrac{.35 \times .46}{.486} = \mathbf{.331}$.

2. 2.57 b, p. 78

   Let X denote the input and Y the output.
   Recall that in the previous homework set we solved part (a) and found

$$P(Y=0) = \frac{1}{2}(1 - \varepsilon_1 + \varepsilon_2)$$

   (b) Now using Bayes rule we find

$$\mathbf{P(X=0|Y=1)} = \frac{P(Y=1|X=0)\,P(X=0)}{P(Y=1)} = \frac{\varepsilon_1\ 1/2}{1\text{-}P(Y=0)} = \frac{\varepsilon_1\ 1/2}{1/2(1+\varepsilon_2-\varepsilon_2)} = \frac{\boldsymbol{\varepsilon_1}}{\mathbf{1-\boldsymbol{\varepsilon_2}+\boldsymbol{\varepsilon_1}}}$$

$$\mathbf{P(X=1|Y=1)} = 1 - P(X=0|Y=1) = 1 - \frac{\varepsilon_1}{1-\varepsilon_2+\varepsilon_1} = \frac{\mathbf{1-\boldsymbol{\varepsilon_2}}}{\mathbf{1-\boldsymbol{\varepsilon_2}+\boldsymbol{\varepsilon_1}}}$$

   Which input is more probable? This question is somewhat vague. We were given that the inputs are equiprobable. So this question must really be asking which input is more probable given Y = 1. In this case, X = 0 is more probable if $\varepsilon_1 > 1 - \varepsilon_2$ and X = 1 is more probable if otherwise.

3. 2.62, p. 79 just for "A and B^c"

   We are given that A and B are independent. Hence, $P(A\cap B) = P(A)\,P(B)$. It's also true that $P(A|B) = P(A)$ and $P(B|A) = P(B)$, but it's usually easier to work with the relation $P(A\cap B) = P(A)\,P(B)$.

   To show A and B^c are independent it suffices to show $P(A\cap B^c) = P(A)\,P(B^c)$.

   By the law of total probability, $P(A\cap B^c) + P(A\cap B) = P(A)$. Therefore,

$$P(A\cap B^c) = P(A) - P(A\cap B) = P(A) - P(A)\,P(B) \text{ by independence of A and B}$$
$$= P(A)\,(1 - P(B)) = P(A)\,P(B^c) \text{ which is what we needed to show.}$$

4. 2.66, (a) and (c), p. 79

   A, B and C are events.

   (a) **P(exactly one of these three events occurs)**

$$= P(\,(A\cap B^c\cap C^c) \quad (A^c\cap B\cap C^c) \quad (A^c\cap B^c\cap C)\,)$$
$$= P(A\cap B^c\cap C^c) + P(A^c\cap B\cap C^c) + P(A^c\cap B^c\cap C\,) \text{ since these three events are disjoint}$$

$$= \mathbf{P(A)\, P(B^c)\, P(C^c) \;+\; P(A^c)\, P(B)\, P(C^c) \;+\; P(A^c)\, P(B^c)\, P(C)}$$

because A, B and C are independent, and so is any combination of their complements

(c) **P(one or more of these occurs)** = 1 - P(none occurs) = 1 - $P(A^c \cap B^c \cap C^c)$

$$= \mathbf{1 - P(A^c)\, P(B^c)\, P(C^c)}$$

5. 2.70, p. 79

We need to find a value of ε such that A_0 is independent of B_0, A_1 is independent of B_1, A_0 is independent of B_1 and A_1 is independent of B_0. From the derivations in Example 2.23, we know

$P(A_0 \cap B_0) = (1-p)(1-\varepsilon)$	$P(A_0) = 1-p$
$P(A_0 \cap B_1) = (1-p)\,\varepsilon$	$P(A_1) = p$
$P(A_1 \cap B_0) = p\varepsilon$	$P(B_0) = P(A_0 \cap B_0) + P(A_1 \cap B_0) = 1-p - \varepsilon + 2p\varepsilon$
$P(A_1 \cap B_1) = p\,(1-\varepsilon)$	$P(B_1) = p + \varepsilon - 2p\varepsilon$

To make A_0 is independent of B_0, we equate $P(A_0 \cap B_0) = P(A_0)\,P(B_0)$, which yields

$(1-p)(1-\varepsilon) = (1-p)(1-p - \varepsilon + 2p\varepsilon) \qquad \varepsilon = \mathbf{\frac{1}{2}}$

One may check that this makes the other three pairs of events indpendent as well.

Such a channel is useless for communications precisely because the outputs are independent of the inputs.

6. 2.71 (hint: there are two approaches)

It is easiest to find the the probability of one, two or three errors and then subtract these probabilities from one. .

The probability of no errors is $(1-p)^{100}$.

The probability of one error is $100\,p(1-p)^{99}$ because the probability of any particular way of getting one error is $p(1-p)^{99}$ and there are 100 different ways of getting 1 error.

The probability of two errors is $\binom{100}{2} p^2(1-p)^{98}$ because the probability of any particular way of getting two errors is $p^2(1-p)^{98}$ and there are $\binom{100}{2}$ ways of getting two errors.

Probability of 3 or more errors $= 1 - (1-p)^{100} - 100\,p(1-p)^{99} - \binom{100}{2} p^2(1-p)^{98}$

$= 1 - .99985 = \mathbf{.00015}$

7. 2.76 (a) and (b), p. 80

(a) **P(k errors in n operations)** $= \mathbf{\binom{n}{k} p^k (1-p)^{n-k}}$

(b) **P($\mathbf{k_1}$ type 1 errors in n operations)** $= \mathbf{\binom{n}{k_1} (pa)^{k_1} (1-pa)^{n-k_1}}$

where pa = probability of a type 1 error = prob of error × prob. type 1 error given error

8. 2.91 (a), p. 82

P(the two people get the same number of heads in 3 tosses)

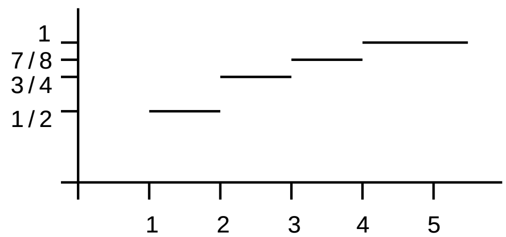
$$= \sum_{i=0}^{3} \text{P(each gets i heads in 3 tosses)}$$

$$= \sum_{i=0}^{3} \text{P(person one gets i heads in 3 tosses) P(person two gets i heads in 3 tosses)}$$

$$= \sum_{i=0}^{3} \binom{3}{i}\frac{1}{8}\binom{3}{i}\frac{1}{8} = \frac{20}{64} = \mathbf{\frac{5}{16}}$$

9. 3.7, p. 175

Y is a discrete random variable with cdf $F_Y(y)$ shown below:

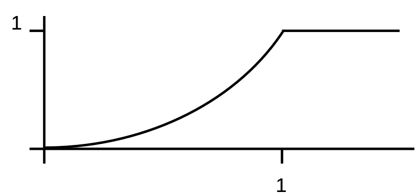


10. 3.9, p. 175; in addition give the formula for the cdf that you find.

If $y \geq 0$, $F_Y(y) = P(Y \leq y) = \frac{\text{area of circle of radius y}}{\text{area of circle of radius 1}} = \frac{\pi y^2}{\pi 1^2} = y^2$

If $y < 0$, $F_Y(y) = P(Y \leq y) = 0$

In summary, $\mathbf{F_Y(y)} = \begin{cases} \mathbf{0}, & \mathbf{y < 0} \\ \mathbf{y^2}, & \mathbf{y \geq 0} \\ \mathbf{1}, & \mathbf{y > 1} \end{cases}$ as illustrated here

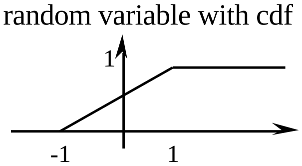


Y is a continuous random variasble.

11. 3.12, p. 175

A uniform random variable is a continuous random variable with cdf

$$F_U(u) = \begin{cases} 0, & u<-1 \\ \frac{u+1}{2}, & -1 \leq u \leq 1 \\ 1, & u>1 \end{cases}$$



and we use the fact that

$$\Pr(a \leq U \leq b) = \Pr(a<U<b) = \Pr(a \leq U<b) = \Pr(a<U \leq b) = F_U(b)-F_U(a)$$

$$= \frac{b-a}{2} \quad \text{if } -1 \leq a \leq b \leq 1, \text{ then}$$

Pr(U>0) $=$ Pr($0 \le U \le 1$) $=$ $\mathbf{\frac{1}{2}}$

Pr(|U|<1/3) $=$ Pr($-1/3 \le U \le 1/3$) $=$ $\mathbf{\frac{1}{3}}$

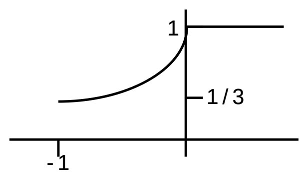
Pr(|U|≥3/4) $=$ Pr($3/4 \le U \le 1$) + P($-1 \le U \le -3/4$) $=$ $\mathbf{\frac{1}{4}}$

Pr(U≤5) $=$ Pr($-1 \le U \le 1$) $=$ **1**

Pr(1/3 < U < 1/2) $=$ $\mathbf{\frac{1}{12}}$

12. 3.13, p. 175

$$F_X(x) = \begin{cases} 1/3 + (2/3)(x+1)^2 , & -1 \le x \le 0 \\ 0 , & x < -1 \\ 1 , & x > 0 \end{cases}$$



Notice that P($-1 \le X \le 0$) = 1 and P(X=-1) = 1/3

P(A) $=$ P(X>1/3) $=$ **0**

P(B) $=$ P(|X|≥1) = P(X=-1) = $\mathbf{\frac{1}{3}}$

P(C) $=$ $P(|X-\frac{1}{3}| < 1) = P(-\frac{2}{3} < X < \frac{4}{3}) = F_X(\frac{4}{3}) - F_X(\frac{-2}{3}) - P(X=\frac{4}{3}) + P(X=-\frac{2}{3})$

$= 1 - (\frac{1}{3} + \frac{2}{3}(\frac{1}{3})^2 - 0 + 0 = \mathbf{\frac{16}{27}}$

P(D) $=$ $P(X<0) = P(X \le 0) - P(X=0) = P(X \le 0) = F_X(0) =$ **1**

13. 3.20, p. 176

$$f_X(x) = \begin{cases} cx(1-x) , & 0 \le x \le 1 \\ 0 , & \text{else} \end{cases} ,$$

(a) We choose c so that $f_X(x)$ integrates to one; i.e.

$$1 = \int_{-1}^{1} f_X(x)\,dx = c \int_{-1}^{1} x(1-x^4)\,dx = c\left[x-\frac{x^5}{5}\right]_{-1}^{1} = \frac{8}{5}c \qquad \mathbf{c = \frac{5}{8}}$$

(b) $\mathbf{F_X(x) = 0,\ x < -1}$; $\mathbf{F_X(x) = 1,\ x > 1}$, and

$$\mathbf{F_X(x)} = \int_{-1}^{x} \frac{5}{8}(1-x'^4)\,dx' = \mathbf{\frac{1}{2} + \frac{1}{8}(5x-x^5)} \quad \text{for } -1 \le x \le 1$$

(c) $\mathbf{P(|X|<\frac{1}{2})} = P(-\frac{1}{2} < X < \frac{1}{2}) = F_X(\frac{1}{2}) - F_X(-\frac{1}{2}) - P(X=\frac{1}{2}) = \frac{79}{8 \times 16} = \mathbf{.617}$

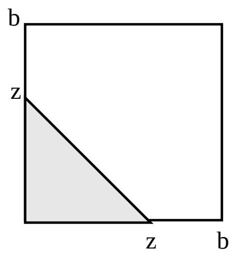
14. 3.22, p. 177

(a) We'll find the pdf by first finding the cdf.

$$F_Z(z) = \Pr(Z \le z) = \frac{\text{area of region of dart board causing } Z \le z}{\text{area of dart board}} .$$

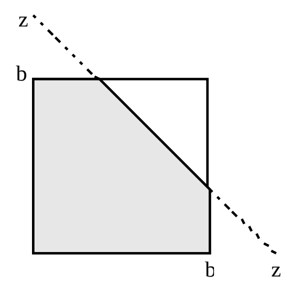
If $0 \le z \le b$, this is

$$F_Z(z) = \frac{z^2}{2b^2} =$$



If $b \le z \le 2b$, this is

$$F_Z(z) = 1 - \frac{(2sb-z)^2}{2b^2} = 2\frac{z}{b} - \frac{z^2}{2b^2} - 1 =$$



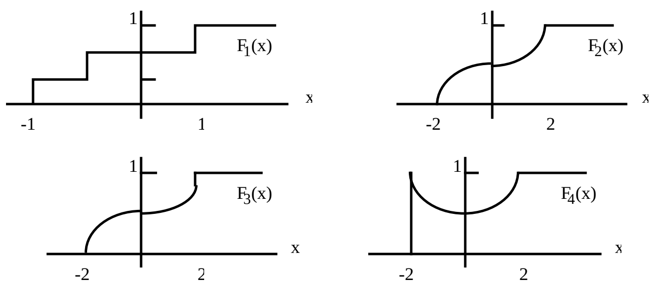
while if ≤ 0, $F_Z(z) = 0$ and if $z \ge 2b$, $F_Z(z) = 1$.

Taking the derivative of the cdf gives the density:

$$\mathbf{f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 0 , & z \le 0 \\ z/b^2 , & 0 \le z \le b \\ 2/b - z/b^2 , & b \le z \le 2b \\ 0 , & z \ge 2b \end{cases}}$$

(b) $\mathbf{Pr(Z>1.5b)} = \int_{1.5b}^{\infty} f_Z(z)\,dz = \int_{1.5b}^{2b} \left(\frac{2}{b} - \frac{z}{b^2}\right) dz = \mathbf{\frac{b^2}{8}}$

15. Which of the functions shown below are valid cumulative distribution functions. For each that is valid, what type of random variable does it represent?



(a) **All but the one on the lower right are valid.**

(b) **The upper left is a discrete random variable.**

The upper right is a continuous random variable.

The lower left is a mixed random variable.