Homework Set 5 Solutions **EECS 401** **Feb. 14, 2000**

In this version, the answer to problem 4b has been corrected.

1. 3.25, p. 177

Assuming $p = 1/2$

$$\mathbf{f_X(x) = \sum_{i=0}^{8} \binom{8}{i} 2^{-8} \delta(x-i)}$$

2. 3.34 (a) and (c), p. 178

A geometric random variable has probability mass function of the form

$p_N(n) = (1-p)^{n-1} p, \; n = 1,2,...$ where p is a parameter, $0<p<1$

or

$p_N(n) = (1-p)^{n} p, \; n = 0,1,...$ where p is a parameter, $0<p<1$

Both are valid.

(a) Assuming the former

$$\mathbf{Pr(N>k)} = \sum_{n=k+1}^{\infty} (1-p)^{n-1} p \;, \quad \text{let } m = n-k$$

$$= \sum_{m=1}^{\infty} (1-p)^{m+k-1} p = (1-p)^k \sum_{m=1}^{\infty} (1-p)^{m-1} p$$

$$= \mathbf{(1-p)^k} \quad \text{since} \sum_{m=1}^{\infty} (1-p)^{m-1} p = \sum_{m=1}^{\infty} p_N(m) = 1$$

Or, assuming the latter

$$\mathbf{Pr(N>k)} = \sum_{n=k+1}^{\infty} (1-p)^{n} p \;, \quad \text{let } m = n-k-1$$

$$= \sum_{m=0}^{\infty} (1-p)^{m+k+1} p = (1-p)^{k+1} \sum_{m=0}^{\infty} (1-p)^{m} p$$

$$= \mathbf{(1-p)^{k+1}} \quad \text{since} \sum_{m=0}^{\infty} (1-p)^{m} p = \sum_{m=0}^{\infty} p_N(m) = 1$$

(c) Assuming the former

$$\mathbf{Pr(N \text{ is even})} = \Pr(N = 2 \text{ or } 4 \text{ or } 6 \text{ or } ...) = \sum_{n=1}^{\infty} p_N(2n) = \sum_{n=1}^{\infty} (1-p)^{2n-1} p$$

$$= p\,(1-p)^{-1} \sum_{n=1}^{\infty} \left((1-p)^2\right)^n$$

$$= p\,(1-p)^{-1} \left(\frac{1}{1-(1-p)^2} - 1 \right) \quad \text{using} \sum_{i=1}^{\infty} \alpha^i = \frac{1}{1-\alpha} - 1$$

$$= p\,(1-p)^{-1} \frac{1-2p+p^2}{2p-p^2} = \mathbf{\frac{p}{2}}$$

Or, assuming the latter

$$\mathbf{Pr(N \text{ is even})} = \Pr(N = 0 \text{ or } 2 \text{ or } 4 \text{ or } 6 \text{ or } ...) = \sum_{n=0}^{\infty} p_N(2n) = \sum_{n=0}^{\infty} (1-p)^{2n} p$$

$$= p \sum_{n=1}^{\infty} ((1-p)^2)^n = p \left(\frac{1}{1-(1-p)^2} \right) \quad \text{using} \quad \sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$$

$$= \frac{p}{2p-p^2} = \mathbf{\frac{1}{2-p}}$$

3. 3.35, p. 178

A geometric random variable X has probability mass function

$$p_X(x) = p\,(1-p)^k, \quad k = 0,1,2,... \quad \text{where } p \text{ is some parameter, } 0 < p < 1.$$

(One can also solve this problem assuming the other form of the geom. pmf.)

If $k > 0$ and $j > 0$, then

$$P(M \geq k) = \sum_{i=k}^{\infty} p\,(1-p)^{i-1} = \sum_{i=0}^{\infty} p\,(1-p)^{i+k-1}$$

(since both summations start with $p\,(1-p)^k + p\,(1-p)^{k+1} + \ldots$)

$$= (1-p)^{k-1} \sum_{i=0}^{\infty} p\,(1-p)^i = (1-p)^{k-1}\, p \frac{1}{1-(1-p)} \quad \left(\text{because } \sum_{i=0}^{\infty} q^i = \frac{1}{1-q}\right)$$

$$= (1-p)^{k-1}$$

$$P(M \geq k+j | M > j) = \frac{P(M \geq k+j,\ M > j)}{P(M > j)} = \frac{P(M \geq k+j)}{P(M > j)} = \frac{\sum_{i=k+j}^{\infty} p(1-p)^{i-1}}{\sum_{i=j+1}^{\infty} p(1-p)^{i-1}} = \frac{\sum_{i=k+j}^{\infty} (1-p)^{i-1}}{\sum_{i=j+1}^{\infty} (1-p)^{i-1}}$$

$$= \frac{\sum_{i=j+1}^{\infty} (1-p)^{i-1}\,(1-p)^{k-1}}{\sum_{i=j+1}^{\infty} (1-p)^{i-1}} = (1-p)^{k-1} = P(M \geq k)$$

We think of M as the time occurrence of an event. We say M is *memoryless* because if we have been waiting j time units without the event occurring (i.e. if $M > j$), then the probability that the event does not occur in the next $k-1$ time units (i.e. that $M \geq k+j$) is independent of the amount of time j that we have so far waited. There is no "memory" of the fact that we have so far waited j time units.

4. 3.37, p. 178

Let X be a random variable representing the number of messages that arrive in a 1 second interval. The problem tells us that X is a Poisson random variable with pmf

$$p_X(k) = \frac{15^k}{k!} e^{-15}, \quad k = 0, 1, 2, \ldots$$

(a) **Pr(no messages arrive in 1 sec)** = $\Pr(X=0) = p_X(0) = \mathbf{e^{-15} = 3.06 \times 10^{-7}}$

*** (b) **Pr(more than 10 messages arrive in 1 sec)** = $\Pr(X>10) = 1 - \Pr(X \leq 10)$

$$= 1 - \sum_{i=0}^{10} p_X(k) = 1 - \sum_{i=0}^{10} \frac{15^k}{k!} e^{-15} = \mathbf{.8815}$$

5. 3.38, p. 178, just for k = 0 and 3.

	n=10, p=.1		n=20, p=.05		n=100, p=.01	
k =	0	3	0	3	0	3
binomial	.3487	.0574	.3585	.06	.366	.061
Poisson	.3679	.0613	.3679	.0613	.3679	.0613

6. 3.40, p. 178. Insert the words "at the end of the day" after "The number of orders waiting". *and next time change 90% to 10%, because the question is much more interesting and the answer is much more interesting*

We want to choose n so that P(number of orders waiting > 4) < 0.9 . Notice that

$$P(\text{number of orders waiting} > 4) = 1 - P(\text{number orders waiting} \le 4)$$

$$= 1 - \sum_{k=0}^{4} P(\text{k orders are waiting}) = 1 - \sum_{k=0}^{4} \frac{\alpha^k}{k!} e^{-\alpha} = 1 - \sum_{k=0}^{4} \frac{(3/n)^k}{k!} e^{-3/n}$$

Let's try n = 1, then n = 2, then n = 3, etc. It turns out that n =1 gives

$$P(\text{number of order waiting} > 4) = 1 - e^{-3} - 3\, e^{-3} - \frac{9}{2} e^{-3} - \frac{9}{2} e^{-3} - \frac{27}{8} e^{-3}$$

$$= 1 - e^{-3} \left(1 + 3 + 9 + \frac{27}{8}\right) = .185$$

Since this is less than .9 , **one employee is sufficient**.

Assuming n = 1 employee,

$$\textbf{P(no orders waiting)} = \frac{(3/n)^k}{k!} e^{-3/n} \quad \text{with } k = 0$$

$$= e^{-3} = \mathbf{0.050}$$

7. 3.44, p. 179 just P(X<m) and P(|X-m| > kσ) for k = 1,3,5

Since X is Gaussian with mean m and variance σ^2, it is a continuous random variable with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{- \frac{(x-m)^2}{2\sigma^2}\right\}$$

Note that since X is a continuous random variable we know P(X=x) = 0 for all x. So P(X≤x) = P(X< x) which simplifies things a bit.

(a) $\textbf{P(X<m)} = \int_{-\infty}^{m} f_X(x)\, dx = \mathbf{\frac{1}{2}}$ because $f_X(x)$ is symmetric about the point x=m; i.e. so it has half its probability to the left of m.

(b) $P(|X-m| > k\,\sigma) = P(X < m-k\sigma \text{ or } X > m+k\sigma) = P(X < m-k\sigma) + P(X > m+k\sigma)$

$$= \int_{-\infty}^{m-k\sigma} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{- \frac{(x-m)^2}{2\sigma^2}\right\} dx + \int_{m+k\sigma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{- \frac{(x-m)^2}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{-k} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{- \frac{u^2}{2}\right\} du\, \sigma + \int_{k}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{- \frac{u^2}{2}\right\} du\, \sigma$$

where we let u = (x-m)/σ when changing variables in the integrals

$$= \int_{-\infty}^{-k} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du + \int_{k}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du$$

$$= 2 \int_{k}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du \quad \text{by the symmetry of the integrand}$$

$= 2\,Q(k)$ by definition of the Q function

So now using Table 3.3 on p. 116, we obtain the following table for $P(|X-m| > k\,\sigma) = 2\,Q(k)$

k =	1	2	3	4	5
2 Q(k)	.318	.0456	.0027	.0000634	5.74×10^{-7}

(Only the answers for k = 1,3,5 are required.)

This table shows, for example, that the probability of X falling farther than σ from the mean is .318.

8. 3.45, p. 179

We need to find P(receiver makes error | 0 sent). Given that 0 is sent, the event

{receiver makes error} ⇔ {receiver decides 1} ⇔ {Y > 0} ⇔ {-1 + N > 0 } ⇔ {N > 1}

where "⇔" means "is equivalent to" or "happens if and only if". Therefore

P(receiver makes error | 0 sent) = P(N > 1) = Q(1) **= 0.159**

where Q() denotes the Q-function, which is defined on p. 115 and tabulated on p. 116.

9. *Consider a random variable whose cdf is shown in Figure P3.1 on p. 175.*

(a) *Find its pdf.*

X is a mixed random variable. We see that $Pr(X=0) = \frac{1}{4}$ and $Pr(X=1) = \frac{1}{2}$. All other points have probability zero. The slope of $F_X(x)$ is 1/4 for $0 < x < 1$. Therefore,

$$\mathbf{f_X(x) = \frac{1}{4}\,\delta(x) + \frac{1}{2}\,\delta(x-1) + \begin{cases} \frac{1}{4}, & 0 < x < 1 \\ 0, & \text{else} \end{cases}}$$

(b) Find $Pr(1/2 \le X \le 1.5)$ by integrating the pdf.

$$\mathbf{Pr(1/2 \le X \le 1.5)} = \int_{.5}^{1.5} f_X(x)\,dx$$

$$= \int_{.5}^{1.5} \frac{1}{4}\,\delta(x)\,dx + \int_{.5}^{1.5} \frac{1}{2}\,\delta(x-1)\,dx + \int_{.5}^{1} \frac{1}{4}\,dx$$

$$= 0 + \frac{1}{2} + \frac{1}{8} = \mathbf{\frac{5}{8}}$$