Homework Set 6 Solutions *DRAFT* **EECS 401** **Feb. 25, 2000**

1. 3.51, p. 180 (Hint: use symmetry where possible.)

   The possible values of Y are -3.5d, - 2.5d, -1.5d, -.5d, .5d, 1.5d, 2.5d, 3.5d. So Y is a discrete random variable. To find the pmf of Y, we need to find the probability of every possible value of Y

   (a) The pmf of Y is:

   $$\mathbf{p_Y(-3.5d)} = P(-\infty \le X \le -3d) = \int_{-\infty}^{-3d} f_X(x)\,dx = \int_{-\infty}^{-3d} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} e^{-3ad}}$$

   by symmetry $\mathbf{p_Y(3.5d)} = p_Y(-3.5d) = \mathbf{\frac{1}{2} e^{-3ad}}$

   $$\mathbf{p_Y(-2.5d) = p_Y(2.5d)} = P(-3d \le X \le -2d) = \int_{-3d}^{-2d} f_X(x)\,dx = \int_{-3d}^{-2d} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} (e^{-2ad}-e^{-3ad})}$$

   $$\mathbf{p_Y(-1.5d) = p_Y(1.5d)} = P(-2d \le X \le -d) = \int_{-2d}^{-d} f_X(x)\,dx = \int_{-2d}^{-d} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} (e^{-ad}-e^{-2ad})}$$

   $$\mathbf{p_Y(-.5d) = p_Y(.5d)} = P(-d \le X \le 0) = \int_{-d}^{0} f_X(x)\,dx = \int_{-d}^{0} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} (1-e^{-ad})}$$

   and finally $\mathbf{p_Y(y) = 0}$ **for all other values of y**

   (b) $$\mathbf{P(|X|>4d)} = 2\int_{4d}^{\infty} f_X(x)\,dx = \int_{4d}^{\infty} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{e^{-4ad}}$$

2. 3.53, p. 180

   X, the random variable representing the grades, is Gaussian with mean m and standard deviation σ'.

   Its pdf is $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

   Let Y = aX + b. We wish to choose a and b so that Y is Gaussian with mean m' and standard deviation σ'.

   Solution 1: $F_Y(y) = \Pr(Y\le y) = \Pr(aX+b\le y) = \Pr(X\le \frac{y-b}{a}) = F_X(\frac{y-b}{a})$.

   $$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(\frac{y-b}{a}) = \frac{d}{dx}F_X(x)\Big|_{x=(y-b)/a} \frac{d}{dy}\frac{y-b}{a} = f_X(\frac{y-b}{a})\frac{1}{a}$$

   $$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{((y-b)/a - m)^2}{2\sigma^2}} \frac{1}{a}$$

   $$= \frac{1}{\sqrt{2\pi}a\sigma} e^{-\frac{(y-am-b)^2}{2a^2\sigma^2}}.$$

   We recognize the above as a Gaussian density with mean am+b and standard deviation aσ.

   Equation m' = am + b and σ' = a σ yields $\mathbf{a} = \frac{\sigma'}{\sigma}$, $\mathbf{b} = m' - am = \mathbf{m' - \frac{\sigma'}{\sigma} m}$

Solution 2: $m' = E[Y] = E[aX+b] = a\,E[X] + b$ by linearity of expectation

$$= am + b$$

$$(\sigma')^2 = var(Y) = E\,(Y-E[Y])^2 = E\,(aX+b - (a\,E[X] + b))^2 = E(a(X-E[X]))^2$$

$$= a^2\,E\,(X-E[X])^2 \quad \text{by linearity of expectation}$$

$$= a^2\,\sigma^2$$

Solving for a and b yields, $\mathbf{a = \frac{\sigma'}{\sigma}}$, $\mathbf{b = m' - \frac{\sigma'}{\sigma}\,m}$

3. 3.56, p. 180

$P = R\,X^2$ where X is Gaussian with $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$.

First we find the cdf:

for $y \geq 0$, $F_P(y) = P(P \leq y) = P(RX^2 \leq y) = P(-\sqrt{y/R} \leq X \leq \sqrt{y/R})$

$$= F_X(\sqrt{y/R}) - F_X(-\sqrt{y/R})$$

and for $y < 0$, $F_P(y) = 0$

Then we take the derivative of the cdf to obtain the pdf:

for $y \geq 0$,

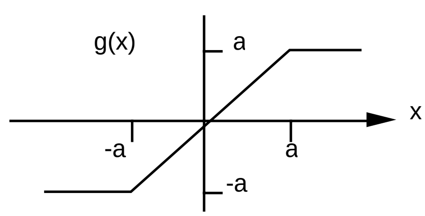
$$\mathbf{f_P(y)} = \frac{d}{df} F_P(y) = f_X(\sqrt{y/R})\,(1/2)\,(y/R)^{-1/2}\,(1/R) - f_X(-\sqrt{y/R})\,(-1/2)\,(y/R)^{-1/2}\,(1/R)$$

$$= \frac{1}{2\sqrt{yR}}\left(f_X(\sqrt{y/R}) + f_X(-\sqrt{y/R})\right) = \frac{1}{2\sqrt{yR}}\,2\,\frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{y}{R}\frac{1}{2\sigma^2}\}$$

$$= \mathbf{\frac{1}{\sqrt{2\pi\sigma^2 y R}}\ \exp\{-\frac{y}{2\sigma^2 R}\}}$$

and **for $y < 0$, $f_P(y) = 0$**

4. 3.57 a, p. 180 (Assume X is a continuous random variable. Express your answers in terms of the cdf and/or pdf of X.)



(a) cdf: **for $y < -a$, $F_Y(y)$** $= P(Y \leq y) =$ **0**,

for $-a \leq y < a$, $F_Y(y)$ $= P(Y \leq y) = P(g(X) \leq y) = P(X \leq y)$ **$= F_X(y)$**

for $y \geq a$, $F_Y(y)$ $= P(Y \leq y)$ **= 1**

pdf: **for $y < -a$ or $y > a$, $f_Y(y) = 0$**

$$\textbf{for } -a \le y \le a,\ f_Y(y) = \frac{d}{dy} F_Y(y) = P(X \le -a)\ \delta(y+a) + f_X(y) + P(X \ge a)\ \delta(y-a)$$

$$= F_X(-a)\ \delta(y+a) + f_X(y) + (1 - F_X(a) + P(X=a))\ \delta(y-a)$$

(c) cdf: $\textbf{for } y < -a,\ F_Y(y) = 0$

$$\textbf{for } -a \le y < a,\ F_Y(y) = F_X(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(z-m)^2}{2\sigma^2}\}\ dz$$

$$\textbf{for } y \ge a,\ F_Y(y) = 1$$

pdf: $\textbf{for } y < -a \textbf{ or } y > a,\ f_Y(y) = 0$

$$\textbf{for } -a \le y \le a,\ f_Y(y) = \left(\int_{-\infty}^{-a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(z-m)^2}{2\sigma^2}\}\ dz\right) \delta(y+a)$$

$$+ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y-m)^2}{2\sigma^2}\} + \left(\int_{a}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(z-m)^2}{2\sigma^2}\}\ dz\right) \delta(y-a)$$

5. 3.65, p. 181

This problem is about a discrete random variable with outcomes {1,2, ..., n} and pmf

$$p_X(i) = \begin{cases} \frac{1}{n}, & i \text{ in } \{1,2,\ldots,n\} \\ 0, & \text{otherwise} \end{cases}$$

The mean

$$E[X] = \sum_{i=1}^{n} i\, p_X(i) = \sum_{i=1}^{n} i \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$$

Note that an easy way to derive the fact that $\sum_{i=1}^{n} i = (n+1)/2$ is that

$$2 \sum_{i=1}^{n} i = (1 + 2 + \ldots + n) + (n + n\text{-}1 + \ldots + 1) = (n+1) + (n+1) + \ldots + (n+1) = n(n+1)$$

To compute the variance we use: $\text{var}(X) = E[X^2] - (E[X])^2$

So we need to find

$$\mathbf{E[X^2]} = \sum_{i=1}^{n} i^2 p_X(i) = \sum_{i=1}^{n} i^2 \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \mathbf{\frac{(n+1)(2n+1)}{6}}$$

Then

$$\mathbf{var(X)} = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \mathbf{\frac{n^2-1}{12}}$$

6. 3.75 (a), p. 182

The net reward is $aX^2 + bX - nd$, where X is a binomial random variable with pmf

$$p_X(x) = \begin{cases} \binom{n}{x} 2^{-n}, & x = 0, 1, \ldots, n \\ 0, & \text{else} \end{cases}$$

The expected net reward is

$$E[aX^2 + bX - nd] = a\,E[X^2] + b\,E[X] - nd$$

From Table 3.1, $E[X] = n/2$ and $\text{var}(X) = n/4$.

Therefore, $E[X^2] = \text{var}(X) + (E[X])^2 = n^2/4 + n^2/4 = n^2/2$. Finally,

$$\mathbf{E[\ aX^2 + bX - nd\ ] = a\frac{n^2}{2} + b\frac{n}{2} - nd}$$

7. 3.79, p. 182

X is a discrete random variable with pmf $p_X(x) = \begin{cases} \frac{1}{n}, & x = 1, \ldots, n \\ 0, & \text{else} \end{cases}$

$\mathbf{E[Y]} = E[K+LX] = K + L\,E[X] = \mathbf{K + L\frac{n+1}{2}}$ from Problem 3

$\mathbf{var(Y)} = L^2\,\text{var}(X)$ from Problem 1

$= \mathbf{L^2\frac{n^2-1}{12}}$ from Problem 3

8. The "game show problem" is due with the next assignment.