1. 3.51, p. 180 (Hint: use symmetry where possible.)

The possible values of \( Y \) are -3.5d, -2.5d, -1.5d, -0.5d, 0.5d, 1.5d, 2.5d, 3.5d. So \( Y \) is a discrete random variable. To find the pmf of \( Y \), we need to find the probability of every possible value of \( Y \).

(a) The pmf of \( Y \) is:

\[
p_Y(-3.5d) = P(-\infty \leq X \leq -3d) = \int_{-\infty}^{-3d} \frac{\alpha}{2} e^{-\frac{|x|}{\alpha}} \, dx = \frac{1}{2} e^{-3ad}
\]

by symmetry \( p_Y(3.5d) = p_Y(-3.5d) = \frac{1}{2} e^{-3ad} \)

\[
p_Y(-2.5d) = p_Y(2.5d) = P(-3d \leq X \leq -2d) = \int_{-3d}^{-2d} \frac{\alpha}{2} e^{-\frac{|x|}{\alpha}} \, dx = \frac{1}{2} (e^{-2ad} - e^{-3ad})
\]

\[
p_Y(-1.5d) = p_Y(1.5d) = P(-2d \leq X \leq -d) = \int_{-2d}^{-d} \frac{\alpha}{2} e^{-\frac{|x|}{\alpha}} \, dx = \frac{1}{2} (e^{-ad} - e^{-2ad})
\]

\[
p_Y(-0.5d) = p_Y(0.5d) = P(-d \leq X \leq 0) = \int_{-d}^{0} \frac{\alpha}{2} e^{-\frac{|x|}{\alpha}} \, dx = \frac{1}{2} (1 - e^{-ad})
\]

and finally \( p_Y(y) = 0 \) for all other values of \( y \).

(b) \( P(|X|>4d) = 2 \int_{4d}^{\infty} f_X(x) \, dx = \int_{4d}^{\infty} \frac{\alpha}{2} e^{-\frac{|x|}{\alpha}} \, dx = e^{-4ad} \)

2. 3.53, p. 180

X, the random variable representing the grades, is Gaussian with mean \( m \) and standard deviation \( \sigma' \).

Its pdf is \( f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-m)^2}{2\sigma^2}} \).

Let \( Y = aX + b \). We wish to choose \( a \) and \( b \) so that \( Y \) is Gaussian with mean \( m' \) and standard deviation \( \sigma' \).

Solution 1: \( F_Y(y) = Pr(Y \leq y) = Pr(aX+b \leq y) = Pr(X \leq \frac{y-b}{a}) = F_X\left(\frac{y-b}{a}\right) \).

\[
f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{d}{dx} F_X(x) \bigg|_{x=(y-b)/a} \frac{d}{dy} \frac{y-b}{a} = f_X\left(\frac{y-b}{a}\right) \frac{1}{a}
\]

\[
= \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(y-b)/(a-m)^2}{2\sigma^2}} \frac{1}{a} = \frac{1}{\sqrt{2\pi a\sigma}} e^{-\frac{(y-am-b)^2}{2a^2\sigma^2}}
\]

We recognize the above as a Gaussian density with mean \( am+b \) and standard deviation \( a\sigma \).

Equation \( m' = am + b \) and \( \sigma' = a \sigma \) yields \( a = \frac{\sigma'}{\sigma} \), \( b = m' - am = m' - \frac{\sigma'}{\sigma} m \).
Solution 2: \[ m' = E[Y] = E[aX + b] = aE[X] + b \] by linearity of expectation
\[ = am + b \]
\[ (\sigma')^2 = \text{var}(Y) = E(Y - E[Y])^2 = E(aX + b - (aE[X] + b))^2 = E(a(X - E[X]))^2 \]
\[ = a^2 \text{var}(X) \] by linearity of expectation
\[ = a^2 \sigma^2 \]

Solving for \(a\) and \(b\) yields,
\[ a = \frac{\sigma'}{\sigma}, \quad b = m' - \frac{\sigma'}{\sigma}m \]

3. 3.56, p. 180

\[ P = RX^2 \] where \(X\) is Gaussian with \[ f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \]

First we find the cdf:
for \(y \geq 0\),
\[ F_P(y) = P(P \leq y) = P(RX^2 \leq y) = P(-\sqrt{y}/R \leq X \leq \sqrt{y}/R) \]
\[ = F_X(\sqrt{y}/R) - F_X(-\sqrt{y}/R) \]
and for \(y < 0\), \(F_P(y) = 0\)

Then we take the derivative of the cdf to obtain the pdf:
for \(y \geq 0\),
\[ f_P(y) = \frac{d}{dy} F_P(y) = f_X(\sqrt{y}/R) (1/2) (y/R)^{-1/2} (1/R) - f_X(-\sqrt{y}/R) (-1/2) (y/R)^{-1/2} (1/R) \]
\[ = \frac{1}{2\sqrt{yR}} (f_X(\sqrt{y}/R) + f_X(-\sqrt{y}/R)) = \frac{1}{2\sqrt{yR}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{y^2}{2\sigma^2} \right\} \]
and for \(y < 0\), \(f_P(y) = 0\)

4. 3.57 a, p. 180 (Assume \(X\) is a continuous random variable. Express your answers in terms of the cdf and/or pdf of \(X\).)

\[ g(x) \]

(a) cdf:
for \(y < -a\), \(F_Y(y) = P(Y \leq y) = 0\),
for \(-a \leq y < a\), \(F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq y) = F_X(y)\),
for \(y \geq a\), \(F_Y(y) = P(Y \leq y) = 1\)

pdf:
for \(y < -a\) or \(y > a\), \(f_Y(y) = 0\)
\[
\text{for } -a \leq y \leq a, f_Y(y) = \frac{d}{dy} F_Y(y) = P(X \leq -a) \delta(y+a) + f_X(y) + P(X \geq a) \delta(y-a)
\]
\[
= F_X(-a) \delta(y+a) + f_X(y) + (1-F_X(a)+P(X=a)) \delta(y-a)
\]
\[\text{(c) cdf: for } y < -a, F_Y(y) = 0\]
\[
\text{for } -a \leq y < a, F_Y(y) = F_X(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(z-m)^2}{2\sigma^2}\right\} \, dz
\]
\[
\text{for } y \geq a, F_Y(y) = 1
\]
\[\text{pdf: for } y < -a \text{ or } y > a, f_Y(y) = 0\]
\[
\text{for } -a \leq y \leq a, f_Y(y) = \left( \int_{-\infty}^{-a} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(z-m)^2}{2\sigma^2}\right\} \, dz \right) \delta(y+a)
\]
\[
+ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-m)^2}{2\sigma^2}\right\} + \left( \int_{a}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(z-m)^2}{2\sigma^2}\right\} \, dz \right) \delta(y-a)
\]

5. 3.65, p. 181

This problem is about a discrete random variable with outcomes \(\{1,2,...,n\}\) and pmf

\[p_X(i) = \begin{cases} \frac{1}{n}, & i \text{ in } \{1,2,...,n\} \\ 0, & \text{otherwise} \end{cases}\]

The mean

\[E[X] = \sum_{i=1}^{n} i p_X(i) = \sum_{i=1}^{n} i \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}\]

Note that an easy way to derive the fact that \(\sum_{i=1}^{n} i = (n+1)/2\) is that

\[2 \sum_{i=1}^{n} i = (1 + 2 + ... + n) + (n + n-1 + ... + 1) = (n+1) + (n+1) + ... + (n+1) = n(n+1)\]

To compute the variance we use: \(\text{var}(X) = E[X^2] - (E[X])^2\)

So we need to find

\[E[X^2] = \sum_{i=1}^{n} i^2 p_X(i) = \sum_{i=1}^{n} i^2 \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}\]

Then

\[\text{var}(X) = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}\]
6. 3.75 (a), p. 182

The net reward is \( aX^2 + bX - nd \), where \( X \) is a binomial random variable with pmf
\[
p_X(x) = \begin{cases} \binom{n}{x} 2^{-n}, & x = 0, 1, \ldots, n \\ 0, & \text{else} \end{cases}
\]
The expected net reward is
\[
E[aX^2 + bX - nd] = aE[X^2] + bE[X] - nd
\]
From Table 3.1, \( E[X] = n/2 \) and \( \text{var}(X) = n/4 \).

Therefore, \( E[X^2] = \text{var}(X) + (E[X])^2 = n^2/4 + n^2/4 = n^2/2 \). Finally,
\[
E[aX^2 + bX - nd] = a \frac{n^2}{2} + b \frac{n}{2} - nd
\]

7. 3.79, p. 182

\( X \) is a discrete random variable with pmf \( p_X(x) = \begin{cases} \frac{1}{n}, & x = 1, \ldots, n \\ 0, & \text{else} \end{cases} \)

\[
E[Y] = E[K+LX] = K + L E[X] = K + L \frac{n+1}{2} \quad \text{from Problem 3}
\]
\[
\text{var}(Y) = L^2 \text{var}(X) \quad \text{from Problem 1}
\]
\[
= L^2 \frac{n^2-1}{12} \quad \text{from Problem 3}
\]

8. The "game show problem" is due with the next assignment.