1. 3.51, p. 180 (Hint: use symmetry where possible.)

The possible values of $Y$ are $-3.5d, -2.5d, -1.5d, -.5d, .5d, 1.5d, 2.5d, 3.5d$. So $Y$ is a discrete random variable. To find the pmf of $Y$, we need to find the probability of every possible value of $Y$

(a) The pmf of $Y$ is:

$$p_Y(-3.5d) = P(-\infty \leq X \leq -3d) = \int_{-\infty}^{-3d} f_X(x) \, dx = \frac{3d}{2} e^{-\frac{a|x|}{2}} = \frac{1}{2} e^{-3ad}$$

by symmetry $p_Y(3.5d) = p_Y(-3.5d) = \frac{1}{2} e^{-3ad}$

$$p_Y(-2.5d) = p_Y(2.5d) = P(-3d \leq X \leq -2d) = \int_{-3d}^{-2d} f_X(x) \, dx = \frac{-2d}{2} e^{-\frac{a|x|}{2}} = \frac{1}{2} (e^{-2ad} - e^{-3ad})$$

$$p_Y(-1.5d) = p_Y(1.5d) = P(-2d \leq X \leq -d) = \int_{-2d}^{-d} f_X(x) \, dx = \frac{-d}{2} e^{-\frac{a|x|}{2}} = \frac{1}{2} (e^{-ad} - e^{-2ad})$$

$$p_Y(-.5d) = p_Y(.5d) = P(-d \leq X \leq 0) = \int_{-d}^{0} f_X(x) \, dx = \frac{0}{2} e^{-\frac{a|x|}{2}} = \frac{1}{2} (1 - e^{-ad})$$

and finally $p_Y(y) = 0$ for all other values of $y$

(b) $P(|X|>4d) = 2 \int_{4d}^{\infty} f_X(x) \, dx = \int_{4d}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx = e^{-4ad}$

2. 3.53, p. 180

$X$, the random variable representing the grades, is Gaussian with mean $m$ and standard deviation $\sigma'$.

Its pdf is $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$.

Let $Y = aX + b$. We wish to choose $a$ and $b$ so that $Y$ is Gaussian with mean $m'$ and standard deviation $\sigma'$.

Solution 1: $F_Y(y) = Pr(Y \leq y) = Pr(aX+b \leq y) = Pr(X \leq \frac{y-b}{a}) = F_X\left(\frac{y-b}{a}\right)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{d}{dx} F_X(x)\bigg|_{x=(y-b)/a} \frac{d}{dy} \frac{y-b}{a} = f_X\left(\frac{y-b}{a}\right) \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-b)/a - m)^2}{2\sigma^2}} \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi\sigma\sigma^2}} e^{-\frac{(y-am-b)^2}{2a^2\sigma^2}}$$

We recognize the above as a Gaussian density with mean $am+b$ and standard deviation $a\sigma$.

Equation $m' = am + b$ and $\sigma' = a\sigma$ yields $a = \frac{\sigma'}{\sigma}$, $b = m' - am = m' - \frac{\sigma'}{\sigma} m$
Solution 2: \[ m' = E[Y] = E[aX+b] = aE[X] + b \] by linearity of expectation
\[ (\sigma')^2 = \text{var}(Y) = E(Y-E[Y])^2 = E(aX+b - (a E[X] + b))^2 = E(a(X-E[X]))^2 \]
\[ = a^2 E(X-E[X])^2 \] by linearity of expectation
\[ = a^2 \sigma^2 \]
Solving for \( a \) and \( b \) yields, \[ a = \frac{\sigma'}{\sigma}, \quad b = m' - \frac{\sigma'}{\sigma} m \]

3. 3.56, p. 180

\[ P = R X^2 \] where \( X \) is Gaussian with \( f_X(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-x^2/2\sigma^2} \).

First we find the cdf:

for \( y \geq 0, \quad F_P(y) = P(P \leq y) = P(RX^2 \leq y) = P(-\sqrt{y/R} \leq X \leq \sqrt{y/R}) = F_X(\sqrt{y/R}) - F_X(-\sqrt{y/R}) \]

and for \( y < 0, \quad F_P(y) = 0 \)

Then we take the derivative of the cdf to obtain the pdf:

\[ f_P(y) \quad \text{for} \quad y \geq 0, \quad \frac{df}{dy} F_P(y) = f_X(\sqrt{y/R}) (1/2) (y/R)^{-1/2} (1/R) - f_X(-\sqrt{y/R}) (-1/2) (y/R)^{-1/2} (1/R) \]
\[ = \frac{1}{2\sqrt{yR}} (f_X(\sqrt{y/R}) + f_X(-\sqrt{y/R})) = \frac{1}{2\sqrt{2\pi\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{y^2}{2\sigma^2} \right\} \]
\[ = \frac{1}{\sqrt{2\pi\sigma^2} yR} \exp\left\{ - \frac{y^2}{2\sigma^2} \right\} \]

and \( \text{for} \quad y < 0, \quad f_P(y) = 0 \)

4. 3.57 a, p. 180 (Assume \( X \) is a continuous random variable. Express your answers in terms of the cdf and/or pdf of \( X \).)

\[ g(x) \]

(a) cdf: \( \text{for} \quad y < -a, \quad F_Y(y) = P(Y \leq y) = 0, \)
\( \text{for} \quad -a \leq y < a, \quad F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq y) = F_X(y) \)
\( \text{for} \quad y \geq a, \quad F_Y(y) = P(Y \leq y) = 1 \)
pdf: for \( y < -a \) or \( y > a \), \( f_Y(y) = 0 \)

for \(-a \leq y \leq a\), \( f_Y(y) = \frac{d}{dy} F_Y(y) = P(X \leq -a) \delta(y+a) + f_X(y) + P(X \geq a) \delta(y-a) \)

\[ = F_X(-a) \delta(y+a) + f_X(y) + (1-F_X(a)+P(X=a)) \delta(y-a) \]

*** removed solution to part (c) since it was not assigned.

5. 3.65, p. 181

This problem is about a discrete random variable with outcomes \{1,2, \ldots, n\} and pmf

\[ p_X(i) = \begin{cases} \frac{1}{n}, & \text{if } i \in \{1,2,\ldots,n\} \\ 0, & \text{otherwise} \end{cases} \]

The mean

\[ E[X] = \sum_{i=1}^{n} i p_X(i) = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2} \]

Note that an easy way to derive the fact that \( \sum_{i=1}^{n} i = (n+1)/2 \) is that

\[ 2 \sum_{i=1}^{n} i = (1 + 2 + \ldots + n) + (n + n-1 + \ldots + 1) = (n+1) + (n+1) + \ldots + (n+1) = n(n+1) \]

To compute the variance we use: \( \text{var}(X) = E[X^2] - (E[X])^2 \)

So we need to find

\[ E[X^2] = \sum_{i=1}^{n} i^2 p_X(i) = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6} \]

Then

\[ \text{var}(X) = \frac{(n+1)(2n+1)}{6} - \left( \frac{n+1}{2} \right)^2 = \frac{n^2-1}{12} \]

6. 3.75 (a), p. 182

The net reward is \( aX^2 + bX - nd \), where \( X \) is a binomial random variable with pmf

\[ p_X(x) = \begin{cases} \binom{n}{x} 2^{-n}, & x = 0, 1, \ldots, n \\ 0, & \text{else} \end{cases} \]

The expected net reward is

\[ E[aX^2 + bX - nd] = aE[X^2] + bE[X] - nd \]

From Table 3.1, \( E[X] = n/2 \) and \( \text{var}(X) = n/4 \).

*** Therefore, \( E[X^2] = \text{var}(X) + (E[X])^2 = n/4 + n^2/4 \). Finally,

*** \[ E[aX^2 + bX - nd] = a \frac{n}{4} + a \frac{n^2}{4} + b \frac{n}{2} - nd \]
7. 3.79, p. 182

\( X \) is a discrete random variable with pmf

\[
p_X(x) = \begin{cases} 
\frac{1}{n}, & x = 1, \ldots, n \\
0, & \text{else} 
\end{cases}
\]

\[
E[Y] = E[K + LX] = K + L \frac{n+1}{2} \quad \text{from Problem 3}
\]

\[
\text{var}(Y) = L^2 \text{var}(X) \quad \text{from Problem 1}
\]

\[
= L^2 \frac{n^2 - 1}{12} \quad \text{from Problem 3}
\]

*** the following solution was omitted.

8. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is
between 1 and 3 inches from the target, and 3 points if it is between 3 and 5 inches from the target. Find
the expected number of points scored if the distance from shot to the target is uniformly distributed
between 0 and 10.

Let \( X \) be a random variable representing the distance of the shot from the target. Let \( Y = \text{the number of}
points received. \) Then \( X \) is a continuous random variable with pdf

\[
f_X(x) = \begin{cases} 
\frac{1}{10}, & 0 \leq x \leq 10 \\
0, & \text{elsewhere} 
\end{cases}
\]

and \( Y = g(X) \), where

\[
g(x) = \begin{cases} 
10, & 0 < x < 1 \\
5, & 1 < x < 3 \\
3, & 3 < x < 5 \\
0, & x > 5 
\end{cases}
\]

\( Y \) is a discrete random variable so we need to find its pmf:

\[
p_Y(10) = P(0 < X < 1) = \frac{1}{10} \quad p_Y(5) = P(1 < X < 3) = \frac{2}{10}
\]

\[
p_Y(3) = P(3 < X < 5) = \frac{2}{10} \quad p_Y(0) = 1 - p_Y(10) - p_Y(5) - p_Y(3) = \frac{1}{2}
\]

\[
p_Y(y) = 0, \quad \text{for other } y \text{'s}
\]

Then the expected number of points is the expected value of \( Y \):

\[
E[Y] = \sum_y y p_Y(y) = 10 \times \frac{1}{10} + 5 \times \frac{1}{5} + 3 \times \frac{1}{5} + 0 \times \frac{1}{2} = \frac{13}{5}
\]

9. The "game show problem" is due with the next assignment.