Homework Set 6 Solutions **EECS 401** **revised 3/6/00**

changes marked with ***

1. 3.51, p. 180 (Hint: use symmetry where possible.)

The possible values of Y are -3.5d, - 2.5d, -1.5d, -.5d, .5d, 1.5d, 2.5d, 3.5d. So Y is a discrete random variable. To find the pmf of Y, we need to find the probability of every possible value of Y

(a) The pmf of Y is:

$$\mathbf{p_Y(-3.5d)} = P(-\infty \le X \le -3d) = \int_{-\infty}^{-3d} f_X(x)\,dx = \int_{-\infty}^{-3d} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} e^{-3ad}}$$

by symmetry $\mathbf{p_Y(3.5d)} = p_Y(-3.5d) = \mathbf{\frac{1}{2} e^{-3ad}}$

$$\mathbf{p_Y(-2.5d) = p_Y(2.5d)} = P(-3d \le X \le -2d) = \int_{-3d}^{-2d} f_X(x)\,dx = \int_{-3d}^{-2d} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} (e^{-2ad} - e^{-3ad})}$$

$$\mathbf{p_Y(-1.5d) = p_Y(1.5d)} = P(-2d \le X \le -d) = \int_{-2d}^{-d} f_X(x)\,dx = \int_{-2d}^{-d} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} (e^{-ad} - e^{-2ad})}$$

$$\mathbf{p_Y(-.5d) = p_Y(.5d)} = P(-d \le X \le 0) = \int_{-d}^{0} f_X(x)\,dx = \int_{-d}^{0} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{\frac{1}{2} (1 - e^{-ad})}$$

and finally $\mathbf{p_Y(y) = 0}$ **for all other values of y**

(b) $\mathbf{P(|X|>4d)} = 2\int_{4d}^{\infty} f_X(x)\,dx = \int_{4d}^{\infty} \frac{\alpha}{2} e^{-\alpha|x|}\,dx = \mathbf{e^{-4ad}}$

2. 3.53, p. 180

X, the random variable representing the grades, is Gaussian with mean m and standard deviation σ'.

Its pdf is $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$.

Let Y = aX + b. We wish to choose a and b so that Y is Gaussian with mean m' and standard deviation σ'.

Solution 1: $F_Y(y) = Pr(Y \le y) = Pr(aX+b \le y) = Pr(X \le \frac{y-b}{a}) = F_X(\frac{y-b}{a})$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\frac{y-b}{a}) = \frac{d}{dx} F_X(x)\Big|_{x=(y-b)/a} \frac{d}{dy} \frac{y-b}{a} = f_X(\frac{y-b}{a}) \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{((y-b)/a - m)^2}{2\sigma^2}} \frac{1}{a}$$

$$= \frac{1}{\sqrt{2\pi} a\sigma} e^{-\frac{(y-am-b)^2}{2a^2\sigma^2}} .$$

We recognize the above as a Gaussian density with mean am+b and standard deviation $a\sigma$.

Equation m' = am + b and $\sigma' = a\,\sigma$ yields $\mathbf{a} = \frac{\sigma'}{\sigma}$, $\mathbf{b} = m' - am = \mathbf{m' - \frac{\sigma'}{\sigma} m}$

Solution 2: $m' = E[Y] = E[aX+b] = a\,E[X] + b$ by linearity of expectation

$$= am + b$$

$$(\sigma')^2 = \text{var}(Y) = E\,(Y-E[Y])^2 = E\,(aX+b - (a\,E[X]+b))^2 = E(a(X-E[X]))^2$$

$$= a^2\,E\,(X-E[X])^2 \quad \text{by linearity of expectation}$$

$$= a^2\,\sigma^2$$

Solving for a and b yields, $\mathbf{a = \frac{\sigma'}{\sigma}}$, $\mathbf{b = m' - \frac{\sigma'}{\sigma}\,m}$

3. 3.56, p. 180

$P = R\,X^2$ where X is Gaussian with $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$.

First we find the cdf:

for $y \geq 0$, $F_P(y) = P(P\leq y) = P(RX^2\leq y) = P(-\sqrt{y/R} \leq X \leq \sqrt{y/R}\,)$

$$= F_X(\sqrt{y/R}) - F_X(-\sqrt{y/R})$$

and for $y < 0$, $F_P(y) = 0$

Then we take the derivative of the cdf to obtain the pdf:

for $y \geq 0$,

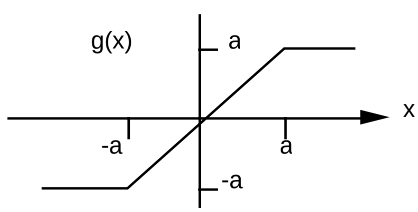
$$\mathbf{f_P(y)} = \frac{d}{df} F_P(y) = f_X(\sqrt{y/R})\,(1/2)\,(y/R)^{-1/2}\,(1/R) - f_X(-\sqrt{y/R})\,(-1/2)\,(y/R)^{-1/2}\,(1/R)$$

$$= \frac{1}{2\sqrt{yR}}\left(f_X(\sqrt{y/R}) + f_X(-\sqrt{y/R})\right) = \frac{1}{2\sqrt{yR}}\,2\,\frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{y}{R}\frac{1}{2\sigma^2}\}$$

$$= \mathbf{\frac{1}{\sqrt{2\pi\sigma^2 y R}}\ \exp\{-\frac{y}{2\sigma^2 R}\}}$$

and **for $y < 0$, $f_P(y) = 0$**

4. 3.57 a, p. 180 (Assume X is a continuous random variable. Express your answers in terms of the cdf and/or pdf of X.)



(a) cdf: **for $y < -a$, $F_Y(y)$** $= P(Y\leq y)$ **= 0**,

for $-a \leq y < a$, $F_Y(y)$ $= P(Y\leq y) = P(g(X)\leq y) = P(X\leq y)$ **$= F_X(y)$**

for $y \geq a$, $F_Y(y)$ $= P(Y\leq y)$ **= 1**

pdf: **for $y < -a$ or $y > a$, $f_Y(y) = 0$**

for $-a \le y \le a$, $f_Y(y)$ $= \frac{d}{dy} F_Y(y)$ **$= P(X \le -a)\ \delta(y+a) + f_X(y) + P(X \ge a)\ \delta(y-a)$**

$= F_X(-a)\ \delta(y+a) + f_X(y) + (1 - F_X(a) + P(X=a))\ \delta(y-a)$

*** removed solution to part (c) since it was not assigned.

5. 3.65, p. 181

This problem is about a discrete random variable with outcomes {1,2, ..., n} and pmf

$$p_X(i) = \begin{cases} \frac{1}{n}, & i \text{ in } \{1,2,\ldots,n\} \\ 0, & \text{otherwise} \end{cases}$$

The mean

$$E[X] = \sum_{i=1}^{n} i\, p_X(i) = \sum_{i=1}^{n} i\, \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$$

Note that an easy way to derive the fact that $\sum_{i=1}^{n} i = (n+1)/2$ is that

$$2 \sum_{i=1}^{n} i = (1 + 2 + \ldots + n) + (n + n-1 + \ldots + 1) = (n+1) + (n+1) + \ldots + (n+1) = n(n+1)$$

To compute the variance we use: $\text{var}(X) = E[X^2] - (E[X])^2$

So we need to find

$$\mathbf{E[X^2]} = \sum_{i=1}^{n} i^2\, p_X(i) = \sum_{i=1}^{n} i^2\, \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} i^2 = \frac{1}{n} \frac{n(n+1)(2n+1)}{6} = \mathbf{\frac{(n+1)(2n+1)}{6}}$$

Then

$$\mathbf{var(X)} = \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \mathbf{\frac{n^2-1}{12}}$$

6. 3.75 (a), p. 182

The net reward is $aX^2 + bX - nd$, where X is a binomial random variable with pmf

$$p_X(x) = \begin{cases} \binom{n}{x} 2^{-n}, & x = 0, 1, \ldots, n \\ 0, & \text{else} \end{cases}$$

The expected net reward is

$$E[\, aX^2 + bX - nd \,] = a\, E[X^2] + b\, E[X] - nd$$

From Table 3.1, $E[X] = n/2$ and $\text{var}(X) = n/4$.

*** Therefore, $E[X^2] = \text{var}(X) + (E[X])^2 = n/4 + n^2/4$. Finally,

*** $$\mathbf{E[\, aX^2 + bX - nd \,] = a \frac{n}{4} + a \frac{n^2}{4} + b \frac{n}{2} - nd}$$

7. 3.79, p. 182

X is a discrete random variable with pmf $p_X(x) = \begin{cases} \frac{1}{n}, & x = 1, \ldots, n \\ 0, & \text{else} \end{cases}$

$$\mathbf{E[Y]} = E[K+LX] = K + L\,E[X] = \mathbf{K + L\frac{n+1}{2}} \quad \text{from Problem 3}$$

$$\mathbf{var(Y)} = L^2\,\text{var}(X) \quad \text{from Problem 1}$$

$$= \mathbf{L^2 \frac{n^2-1}{12}} \quad \text{from Problem 3}$$

******* the following solution was omitted.

8. *A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches from the target, and 3 points if it is between 3 and 5 inches from the target. Find the expected number of points scored if the distance from shot to the target is uniformly distributed between 0 and 10.*

Let X be a random variable representing the distance of the shot from the target. Let Y = the number of points received. Then X is a continuous random variable with pdf

$$f_X(x) = \begin{cases} 1/10, & 0 \le x \le 10 \\ 0, & \text{elsewhere} \end{cases}$$

and Y = g(X), where

$$g(x) = \begin{cases} 10, & 0 < x < 1 \\ 5, & 1 < x < 3 \\ 3, & 3 < x < 5 \\ 0, & x > 5 \end{cases}$$

Y is a discrete random variable so we need to find its pmf:

$$p_Y(10) = P(0<X<1) = \frac{1}{10}, \quad p_Y(5) = P(1<X<3) = \frac{2}{10}$$

$$p_Y(3) = P(3<X<5) = \frac{2}{10}, \quad p_Y(0) = 1-p_Y(10)-p_Y(5)-p_Y(3) = \frac{1}{2}$$

$$p_Y(y) = 0, \text{ for other y's}$$

Then the expected number of points is the expected value of Y:

$$\mathbf{E[Y]} = \sum_y y\,p_Y(y) = 10 \times \frac{1}{10} + 5 \times \frac{1}{5} + 3 \times \frac{1}{5} + 0 \times \frac{1}{2} = \mathbf{\frac{13}{5}}$$

9. The "game show problem" is due with the next assignment.