Homework Set 7 Solutions *DRAFT* **EECS 401** **March 14, 2000**

1. 3.82, p. 182

We are given that $Y = X/n$ where the pmf of X is $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, ..., n$, and from Table 3.1, $E[X] = np$ and $var(X) = np(1-p)$.

Notice that $E[y] = E[X/n] = E[X]/n = p$ and $var(Y) = var(X/n) = var(X)/n^2 = p(1-p)/n$

Now $\mathbf{P(|Y-p| > a)} = P(|Y-E[Y]| > a) \leq \frac{var(Y)}{a^2}$ by the Chebychev inequality

$$= \mathbf{\frac{p(1-p)/n}{a^2}}$$

We see that as $n \to \infty$, $\mathbf{P(|Y-p|>a) \to 0}$.

2. 3.113, p. 185

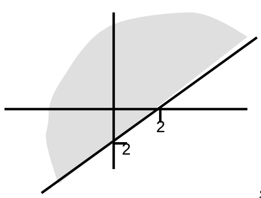
For the Laplacian random variable, the pdf is $f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$ and the cdf is

$$F_X(x) = \begin{cases} \frac{1}{2} e^{\alpha x}, & x \leq 0 \\ 1-\frac{1}{2} e^{-\alpha x}, & x \geq 0 \end{cases}$$

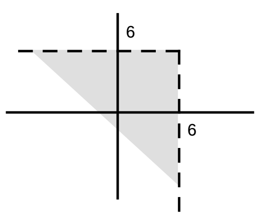
The transformation we use is:

$$\mathbf{T(u) = F_X^{-1}(u) = \begin{cases} \frac{1}{\alpha} \ln 2u, & 0 < u < \frac{1}{2} \\ -\frac{1}{\alpha} \ln 2(1-u), & \frac{1}{2} \leq u \leq 1 \end{cases}}$$

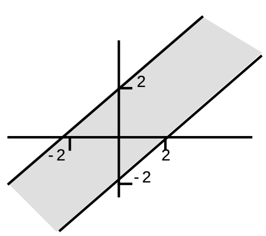
3. 4.1, a,c,d,e p. 256

a. $\{X-Y \leq 2\}$ 

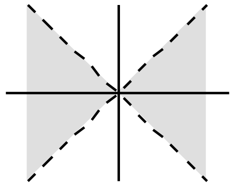
, **not product-form** (i.e. not a cartesian product set)

c. $\{\max(X,Y) < 6\}$ 

product-form

d. $\{|X-Y| \leq 2\}$ 

not product-form

e. $\{|X| > |Y|\}$  **not product-form**

4. 4.5, p. 257

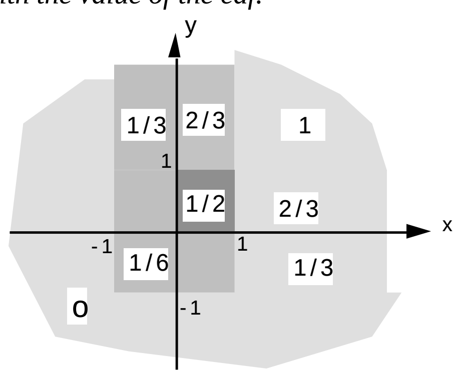
   a. All three cases have the same marginal pmf's for X and Y, namely,

   $p_X(-1) = p_X(0) = p_X(1) = \frac{1}{3}$ and $p_Y(-1) = p_Y(0) = p_Y(1) = \frac{1}{3}$

   b.

	i	ii	iii
$P(A) = P(X\le 0) = p_X(-1) + p_X(0) =$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{2}{3}}$
$P(B) = P(X\le Y) = p_{XY}(-1,-1)+p_{XY}(-1,0)+p_{XY}(-1,1)$ $+p_{XY}(0,0)+p_{XY}(0,1)++p_{XY}(1,1) =$	$\mathbf{\frac{5}{6}}$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{2}{3}}$
$P(C) = P(X=-Y) = p_{XY}(-1,1)+p_{XY}(1,-1)+p_{XY}(0,0) =$	$\mathbf{\frac{2}{3}}$	$\mathbf{\frac{1}{3}}$	**1**

5. 4.6, p. 257, *Just for pmf i. To make the required sketch, draw x-y axes, divide the x-y plane into regions, and label each region with the value of the cdf.*



6. *X and Y are independent random variables. X is discrete with pmf $p_X(0) = p_X(1) = \frac{1}{2}$ and Y is uniformly distributed on the interval [0,1]. Find their joint cdf.*

$$F_{XY}(x,y) = \Pr(X\le x, Y\le y) = \Pr(X\le x)\Pr(Y\le y) \quad \text{since X and Y are independent}$$
$$= F_X(x)\,F_Y(y)$$

$$F_X(x) = \begin{cases} 0, & x<0 \\ \frac{1}{2}, & 0\le x<1 \\ 1, & x\ge 1 \end{cases} \quad , \quad F_Y(y) = \begin{cases} 1, & 0\le x\le 1 \\ 0, & \text{else} \end{cases}$$

Therefore,

$$\mathbf{F_{XY}(x,y) = \begin{cases} 0, & x<0, y<0 \\ \frac{1}{2}y, & 0\le x<1, 0\le y\le 1 \\ \frac{1}{2}, & 0\le x<1, y>1 \\ y, & x\ge 1, 0\le y\le 1 \\ 1, & x\ge 1, y>1 \end{cases}}$$

7. a. 4.10, p. 257

a. $f(x,y) = k(x+y),\ 0 < x < 1,\ 0 < y < 1$

$$1 = k\int_0^1\int_0^1 (x+y)\,dx\,dy = k\int_0^1\left(\frac{x^2}{2} + xy\right)\Big|_0^1 dy = k\int_0^1\left(\frac{1}{2} + y\right)dy = k\left[\frac{y}{2} + \frac{y^2}{2}\right]_0^1 = k$$

so $\mathbf{k = 1}$.

b.

if $x<0$ or $y<0$, then $F(x,y) = 0$,

if $0 \le x \le 1,\ 0 \le y \le 1$, then

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x',y')\,dx'\,dy' = \int_0^x\int_0^y (x+y)\,dy'\,dx' = \int_0^x\int_0^y x'\,dy'\,dx' + \int_0^x\int_0^y y'\,dy'\,dx'$$

$$= \int_{-\infty}^{x} x'\,y\,dx' + \int_{-\infty}^{x}\frac{1}{2}y^2\,dx' = \frac{1}{2}x^2y + \frac{1}{2}y^2x = \frac{1}{2}xy(x+y)$$

if $0 \le x \le 1,\ y>1$, then

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x',y')\,dx'\,dy' = \int_0^x\int_0^1 (x+y)\,dy'\,dx' = \int_0^x\int_0^1 x'\,dy'\,dx' + \int_0^x\int_0^1 y'\,dy'\,dx'$$

$$= \int_{-\infty}^{x} x'\,dx' + \int_{-\infty}^{x}\frac{1}{2}\,dx' = \frac{1}{2}x^2 + \frac{1}{2}x = \frac{1}{2}x(x+1)$$

by symmetry, if $0 \le y \le 1,\ x>1$, then

$$F(x,y) = \frac{1}{2}y(y+1)$$

if $x>1,\ y>1$, $F(x,y) = 1$

In summary,

$$\mathbf{F(x,y)} = \begin{cases} \mathbf{0,\ x<0\ or\ y<0} \\ \mathbf{\frac{1}{2}xy(x+y),\ 0 \le x \le 1\ ,\ 0 \le y \le 1} \\ \mathbf{\frac{1}{2}x(x+1),\ 0 \le x \le 1\ ,\ y>1} \\ \mathbf{\frac{1}{2}y(y+1),\ 0 \le y \le 1\ ,\ x>1} \\ \mathbf{1,\ x>1,\ y>1} \end{cases}$$

c. $\mathbf{f_X(x)} = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 (x+y)\,dy = x + \frac{y^2}{2}\Big|_0^1 = \mathbf{x + \frac{1}{2}}$

$\mathbf{f_Y(x)} = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^1 (x+y)\,dx = y + \frac{x^2}{2}\Big|_0^1 = \mathbf{y + \frac{1}{2}}$

b. *Are X and Y independent? Justify your answer.*

No, x and y are not independent because $f_{XY}(x,y) \neq f_X(x)\,f_Y(y)$, as can be seen by computing $f_X(x)$ and $f_Y(y)$.

8. i. 4.12, p. 257

$f_{XY}(x,y) = 2e^{-x}e^{-2y}$, $x > 0$, $y > 0$

(a) $\mathbf{P(X+Y \le 8)} = \int_0^8 \int_0^{8-x} 2e^{-x} e^{-2y}\,dy\,dx = \int_0^8 e^{-x}(1-e^{-2(8-x)})\,dx = \mathbf{1 - 2e^{-8} + e^{-16}}$

(b) $\mathbf{P(X<Y)} = \int_0^\infty \int_x^\infty 2e^{-x}e^{-2y}\,dy\,dx = \int_0^\infty e^{-x} e^{-2x}\,dx = \mathbf{\frac{1}{3}}$

(c) $\mathbf{P(X-Y\le 10)} = \int_0^\infty \int_0^{y+10} 2e^{-x} e^{-2y}\,dx\,dy = \int_0^\infty (1-e^{-y-10})e^{-2y}\,dy = \mathbf{1 - \frac{2}{3} e^{-10}}$

(d) $\mathbf{P(X^2<Y)} = \int_0^\infty \int_{x^2}^\infty 2e^{-x} e^{-2y}\,dy\,dx = \int_0^\infty e^{-x} e^{-2x^2}\,dx = e^{1/8} \int_0^\infty \exp\left\{-\frac{x^2+\frac{1}{2}x+\frac{1}{16}}{1/2}\right\} dx$

$$= e^{1/8}\sqrt{2\pi/4} \int_0^\infty \frac{1}{\sqrt{2\pi/4}} \exp\left\{-\frac{(x+1/4)^2}{2(1/4)}\right\} dx$$

$$= e^{1/8}\sqrt{\pi/2} \int_{1/2}^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dx \quad \text{with } z = \frac{x+1/4}{1/2}$$

$$= \mathbf{e^{1/8}\sqrt{\pi/2}\; Q(1/2) = 0.439}$$

ii. *Find the marginal pdf's*

$\mathbf{f_X(x)} = \int_{-\infty}^\infty f_{XY}(x,y)\,dy = \int_0^\infty 2e^{-x}e^{-2y}\,dy = \mathbf{e^{-x}}$

$\mathbf{f_Y(y)} = \int_{-\infty}^\infty f_{XY}(x,y)\,dx = \int_0^\infty 2e^{-x}e^{-2y}\,dx = \mathbf{2e^{-2y}}$

iii. *Are X and Y independent?*

Yes, since $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

9. 4.23 a, p. 259

a. $\mathbf{P(a < X \le b,\ Y \le d)} = P(a < X \le b)\, P(Y \le d)$ since X and Y are independent

$$= \mathbf{(F_X(b) - F_X(a))\, F_Y(d)}$$

10. 4.24 b,c, p. 259

b. $\mathbf{P(X/2 < 1,\ Y > 0)} = P(X<2)\, P(Y>0) = \mathbf{1}$

c. Notice that $f_{XY}(x,y) = f_X(x)\, f_Y(y) = 1$, $0 < x < 1$, $0 < y < 1$. Therefore,

$\mathbf{P(XY < 1/2)} = \int_0^{1/2} \int_0^1 1\,dy\,dx + \int_{1/2}^1 \int_0^{1/2x} 1\,dy\,dx = \frac{1}{2} + \int_{1/2}^1 \frac{1}{2x}\,dx = \frac{1}{2} + \frac{1}{2}\ln|x|\Big|_{1/2}^1 = \mathbf{.85}$

11. *It is found that from the time a husband and wife marry, the number of years that the husband lives can be modelled as random with an average of 50 and an exponential distribution. Similarly, from the time of marriage, the number of years that the wife lives can be modelled as random with an average of 60 and an exponential distribution. (The numbers of years need not be integers.) In addition, in this model, the time of death of one spouse has no influence on the time of death of the other. Find the probability that the husband outlives the wife.*

$$f_{HW}(h,w) = f_H(h)\, f_W(w), \quad f_H(h) = \frac{1}{50} e^{-h/50}, \; h\geq 0, \quad f_W(w) = \frac{1}{60} e^{-w/60}, \; w\geq 0$$

$$\mathbf{Pr(H\geq W)} = \int_0^{\infty}\int_0^{h} f_{HW}(h,w)\, dw\, dh = \int_0^{\infty} f_H(h) \int_0^{h} f_W(w)\, dw\, dh$$

$$= \int_0^{\infty} \frac{1}{50} e^{-h/50} \int_0^{h} \frac{1}{60} e^{-w/60}\, dw\, dh = \int_0^{\infty} \frac{1}{50} e^{-h/50} \left(1-e^{-h/60}\right) dh$$

$$= 1 - \frac{1}{50}\int_0^{\infty} e^{-11h/300}\, dh = 1 - \frac{1}{50}\frac{300}{11} e^{-11h/300}\Big|_0^{\infty} = 1 - \frac{6}{11} = \mathbf{\frac{5}{11}}$$