Homework Set 8 Solutions **EECS 401** **revised March 28, 2000**

revisions marked with ***

1. 4.30, p. 259

*** The pmf of Y given X = -1 is $\mathbf{p_{Y|X}(y|-1)} = \frac{p_{XY}(x,y)}{p_X(-1)}$ as given in the following table

	Y = -1	**0**	**1**
i	$\frac{1}{2}$	**0**	$\frac{1}{2}$
ii	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
iii	**0**	**0**	**1**

2. 4.32, case (iii), p. 259

*** $f_{XY}(x,y) = 2,\ 0 \le x \le 1,\ 0 \le y \le 1\text{-}x.$ $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \int_0^{1-x} 2\, dy = 2\text{-}2x$

$$\mathbf{f_{Y|X}(y|x)} = \frac{f_{XY}(x,y)}{f_X(x)} = \begin{cases} \frac{2}{2-2x} = \frac{1}{1-x}, & 0 \le x \le 1,\ 0 \le y \le 1-x \\ 0, & \text{else} \end{cases}$$

3. 4.35, p. 260

The problem statement gives us the conditional pdf of T given that the clerk is i. This is a kind of nonumerical conditioning. Specifically, it tells us that

$$f_T(t|i) = \alpha_i\, e^{-\alpha_i t},\ t \ge 0,$$

The conditional cdf is $F_T(t|i) = 1\text{-}e^{-\alpha_i t}$

(a) To find the pdf of T we first find the cdf and then differentiate. Using the law of total probability

$$F_T(t) = \Pr(T \le t) = \sum_{i=1}^{n} p_i \Pr(T \le t|i) = \sum_{i=1}^{n} p_i F_T(t|i)$$

Taking the derivative gives

$$\mathbf{f_T(t)} = \frac{d}{dt} F_T(t) = \frac{d}{dt} \sum_{i=1}^{n} p_i F_T(t|i) = \sum_{i=1}^{n} p_i \frac{d}{dt} F_T(t|i) = \sum_{i=1}^{n} p_i f_T(t|i)$$

$$= \sum_{i=1}^{n} p_i\, \alpha_i\, e^{-\alpha_i t},\quad t \ge 0$$

Notice that the above is a kind of law of total probability for densities.

(b) $$\mathbf{E[T]} = \int_{-\infty}^{\infty} t\, f_T(t)\, dt = \int_0^{\infty} t \sum_{i=1}^{n} p_i\, \alpha_i\, e^{-\alpha_i t}\, dt = \sum_{i=1}^{n} p_i \int_0^{\infty} t\, \alpha_i\, e^{-\alpha_i t}\, dt = \sum_{i=1}^{n} p_i \frac{1}{\alpha_i}$$

$$E[T^2] = \int_{-\infty}^{\infty} t^2 f_T(t)\, dt = \int_0^{\infty} t^2 \sum_{i=1}^{n} p_i\, \alpha_i\, e^{-\alpha_i t}\, dt = \sum_{i=1}^{n} p_i \int_0^{\infty} t^2\, \alpha_i\, e^{-\alpha_i t}\, dt = \sum_{i=1}^{n} p_i \frac{2}{\alpha_i^2}$$

$$\mathbf{var(T)} = E[T^2] - (E[T])^2 = \sum_{i=1}^{n} p_i \frac{2}{\alpha_i^2} - \left(\sum_{i=1}^{n} p_i \frac{1}{\alpha_i} \right)^2$$

4. *A random variable X is exponentially distributed with expected value 10. Let Y be a random variable whose conditional pdf given X = x is*

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{10}\, e^{(x-y)/10}, & y \ge x \\ 0, & \text{else} \end{cases}$$

Find the conditional pdf of X given Y = y.

By Bayes rule for conditional densities: $f_{X|Y}(x|y) = \dfrac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$

Since X is exponentially distributed with expected value 10: $f_X(x) = \frac{1}{10} e^{-x/10}$, $x \ge 0$.

To find $f_Y(y)$ we use $f_Y(y) = \int_{-\infty}^{\infty} f_X(x)\, f_{Y|X}(y|x)\, dx$

$$f_X(x)\, f_{Y|X}(y|x) = \frac{1}{10} e^{-x/10} \frac{1}{10} e^{(x-y)/10} = \frac{1}{100} e^{-y/10}, \quad 0 \le x \le y$$

So ... $f_Y(y) = \int_0^y \frac{1}{100} e^{-y/10}\, dx = \frac{1}{100}\, y\, e^{-y/10}$

Substituting into Bayes rule gives:

$$\mathbf{f_{X|Y}(x|y)} = \frac{\frac{1}{100} e^{-y/10}}{\frac{1}{100}\, y\, e^{-y/10}} = \mathbf{\frac{1}{y}}, \quad \mathbf{0 \le x \le y}$$

5. *Joe buys his gasoline at the local cut-rate gas station. The station sells two grades of gas, A and B, but the station owner won't tell which you get. Joe is concerned because his car mileage varies uniformly between 20 and 32 miles per gallon with brand A and between 16 and 26 miles per gallon with brand B. Suppose Joe knows the probabilities of the gas station having brands A and B are .3 and .7, respectively. Let X be a random variable representing Joe's gas mileage.*

(a) *Find the cumulative distribution function of X.*

From the problem statement, we deduce Pr(brand A gas) =0.3, Pr(brand B gas)=0.7; and

$$f_X(x|A) = \begin{cases} \frac{1}{12}, & 20 \le x \le 32 \\ 0, & \text{else} \end{cases} \qquad f_X(x|B) = \begin{cases} \frac{1}{10}, & 16 \le x \le 26. \\ 0, & \text{else} \end{cases}$$

The cdf is

$$\mathbf{F_X(x)} = \Pr(X \le x) = \Pr(X \le x\,|A)\Pr(A) + \Pr(X \le x|B)\Pr(B)$$

$$= \int_{-\infty}^{x} f_X(x|A)\, dx \times 0.3 + \int_{-\infty}^{x} f_X(x|B)\, dx \times 0.7$$

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$$= \begin{cases} 0, & x<16 \\ 0.7 \int_{16}^{x} \frac{1}{10}\, dx, & 16 \le x < 20 \\ 0.3 \int_{20}^{x} \frac{1}{12}\, dx + 0.7 \int_{16}^{x} \frac{1}{10}\, dx, & 20 \le x < 26 \\ 0.7 + 0.3 \int_{-20}^{x} \frac{1}{12}\, dx, & 26 \le x < 32 \\ 1, & x \ge 32 \end{cases} = \begin{cases} \mathbf{0,} & \mathbf{x<16} \\ \mathbf{0.7\, \frac{x-16}{10},} & \mathbf{16 \le x < 20} \\ \mathbf{0.3\, \frac{x-20}{12} + 0.7\, \frac{x-16}{10},} & \mathbf{20 \le x < 26} \\ \mathbf{0.7 + 0.3\, \frac{x-20}{12},} & \mathbf{26 \le x < 32} \\ \mathbf{1,} & \mathbf{x \ge 32} \end{cases}$$

(b) *Joe has found that his gas mileage is at least 24 . Given this, determine the probability that he is using brand B.*

$$\mathbf{Pr(B|X\geq 24)} = \frac{Pr(X\geq 24|B)\ Pr(B)}{Pr(X\geq 24)} = \frac{\int_{24}^{\infty} f_X(x|B)\ dx \times 0.7}{1-F_X(24)} = \frac{\int_{24}^{26} \frac{1}{10}\ dx \times 0.7}{1-(0.3\times 4/12+0.7\times 8/10)}$$

$$= \frac{\frac{1}{5}\times 0.7}{\frac{17}{50}} = \mathbf{\frac{7}{17}}$$

6. *A gambler brings X dollars to a casino where X is a random variable with density*

$$f_X(x) = \begin{cases} \frac{x}{80{,}000}, & 0 \leq x \leq 400 \\ 0, & otherwise \end{cases}$$

After a night of gambling the gambler leaves the casino with Y dollars, where Y is uniformly distributed between 0 and X .

(a) *Given the gambler leaves the casino with less than \$200 dollars, find the probability that he brought less than \$200.*

We deduce from the problem statement, that

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0\leq y\leq x \\ 0, & else \end{cases}$$

and from this, we deduce that

$$f_{XY}(x,y) = f_X(x)\ f_{Y|X}(y|x) = \begin{cases} \frac{1}{80{,}000}, & 0\leq x\leq 400, 0\leq y\leq x \\ 0, & else \end{cases}$$

Now, we must find

$$Pr(X\leq 200|Y\leq 200) = \frac{Pr(X\leq 200,Y\leq 200)}{Pr(Y\leq 200)}$$

The numerator:

$$Pr(X\leq 200,Y\leq 200) = \int_{-\infty}^{200}\int_{-\infty}^{200} f_{XY}(x,y)\ dy\ dx = \int_{0}^{200}\int_{0}^{x} \frac{1}{80{,}000}\ dy\ dx = \int_{0}^{200} \frac{x}{80{,}000}\ dx$$

$$= \frac{1}{80{,}000}\ \frac{1}{2}\ x^2\Big|_0^{200} = \frac{1}{4}$$

The denominator:

$$Pr(Y\leq 200) = \int_{-\infty}^{200} f_Y(y)\ dy = \int_{-\infty}^{200}\int_{-\infty}^{\infty} f_{XY}(x,y)\ dx\ dx = \int_{0}^{200}\int_{y}^{400} \frac{1}{80{,}000}\ dx\ dy$$

$$= \int_{0}^{200} \frac{400-y}{80{,}000}\ dy = 1 - \frac{1}{80{,}000}\frac{1}{2}\ y^2\Big|_0^{200} = \frac{3}{4}$$

Substituting for the numerator and denominator gives

$$\mathbf{Pr(X\leq 200|Y\leq 200)} = \mathbf{\frac{1}{3}}$$

(b) *Find the probability that his loss was less than $100.*

His loss is X-Y, so we need to find

$$\mathbf{Pr(X\text{-}Y<100)} \;\; = \int_{-\infty}^{\infty} \int_{x-100}^{x} f_{XY}(x,y)\, dy\, dx \;=\; \int_{0}^{400} \int_{\max(0,x-100)}^{x} \frac{1}{80{,}000}\, dy\, dx$$

$$= \int_{0}^{100}\int_{0}^{x} \frac{1}{80{,}000}\, dy\, dx + \int_{100}^{400}\int_{x-100}^{x} \frac{1}{80{,}000}\, dy\, dx$$

$$= \int_{0}^{100} \frac{x}{80{,}000}\, dy + \int_{100}^{400} \frac{1}{800}\, dx \;=\; \frac{1}{80{,}000}\,\frac{1}{2}\,x^2\Big|_{0}^{100} + \frac{300}{800} = \frac{1}{16} + \frac{3}{8} \;\; = \;\; \mathbf{\frac{7}{16}}$$

(c) *Find the probability that his loss was exactly $75.*

$$\mathbf{Pr(X\text{-}Y=75)} \;\; = \int_{-\infty}^{\infty} \int_{x-75}^{x-75} f_{XY}(x,y)\, dy\, dx \; = \; \mathbf{0}$$ because the area of integration is zero

(d) *Find the density of Y.*

$$\mathbf{f_Y(y)} \;\; = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx \;=\; \begin{cases} \displaystyle\int_{y}^{400} \frac{1}{80{,}000}\, dx, & 0 \le y \le 400 \\ 0, & \text{else} \end{cases} \;=\; \begin{cases} \mathbf{\dfrac{400-y}{80{,}000}}, & \mathbf{0 \le y \le 400} \\ \mathbf{0}, & \textbf{else} \end{cases}$$

7. *Suppose X and Y are jointly continuous, independent random variables. Show that*

$$P(Y \le X) \;=\; \int_{-\infty}^{\infty} F_Y(y)\, f_X(y)\, dy$$

$$\Pr(Y \le X) \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{XY}(x,y)\, dy\, dx \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_X(x)\, f_Y(y)\, dy\, dx$$ since X and Y are independent

$$= \int_{-\infty}^{\infty} f_X(x) \int_{-\infty}^{x} f_Y(y)\, dy\, dx \;=\; \int_{-\infty}^{\infty} f_X(x)\, F_Y(x)\, dx \;=\; \int_{-\infty}^{\infty} f_X(y)\, F_Y(y)\, dy$$