Study Problem Solutions **EECS 401** **April 19, 2000**

1. *The autocorrelation function of a widesense stationary random process X(.) is*

$$R_X(\tau) = a\, e^{-b\tau^2}$$

(a) *Find its power spectral density.*

The power spectral density is

$$\mathbf{S_X(f)} = \text{Fourier transform of } R_X(\tau) = \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} a\, e^{-b\tau^2} e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{\infty} a \exp\left\{- b(\tau+j2\pi f/2b)^2\right\} \exp\left\{-4\pi^2 f^2/4b\right\} d\tau$$

$$= a\, e^{-\pi^2 f^2/b} \int_{-\infty}^{\infty} \exp\left\{- bu^2 \right\} du \;, \text{ letting } u = \tau+j2\pi f/2b$$

$$= a\, e^{-\pi^2 f^2/b} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi/2b}} \exp\left\{- \frac{u^2}{2/(1/2b)} \right\} du\ \sqrt{2\pi/2b}$$

$$= \boldsymbol{\sqrt{\pi/b}\; a\, e^{-\pi^2 f^2/b}} \text{, since the integral of a Gaussian density is one}$$

(b) *Find the mean of this random process.*

EX(t) = 0, because if not zero, the power spectral density would contain a delta function.

(c) *Find the (average) power of this random process.*

power = E X²(t) = R_X(0) = a .

$$\text{Alternatively,}\quad \text{power} = \int_{-\infty}^{\infty} S_X(f)\, df = \int_{\infty}^{\infty} \sqrt{\pi/b}\; a\, e^{-\pi^2 f^2/b}\, df$$

$$= a \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-f^2/2\sigma^2} df\,, \quad \text{where } \sigma^2 = \frac{b}{2\pi^2}$$

$= a$ since the integral is one

(d) *How much power does this random process have in the frequency band [2,3]? (Hint: Q-function tables are useful.)*

$$\text{power in band [2,3]} = 2\int_2^3 S_X(f)\, df = 2\int_2^3 \frac{a}{\sqrt{2\pi\sigma^2}} e^{-f^2/2\sigma^2} df, \quad \text{where } \sigma^2 = \frac{b}{2\pi^2}$$

$$= 2\, a\, Q\left(\frac{2}{\sigma}\right) - 2\, a\, Q\left(\frac{3}{\sigma}\right)$$

$$= 2a\, Q(2\pi\sqrt{2/b}) - 2a\, Q(3\pi\sqrt{2/b})$$

We can't proceed farther because the values of a and b are not given.

2. *A widesense stationary random process X(.) has power spectral density*

$$S_X(f) = \frac{6f^2}{1+f^4} \quad \text{because } \omega = 2\pi f$$

Find the average power of the random process.

The power is $\mathbf{P} = \int_{-\infty}^{\infty} S_X(f)\,df = \int_{-\infty}^{\infty} \frac{f^2}{1+f^4}\,df = 2\int_{0}^{\infty} \frac{f^2}{1+f^4}\,df$

$$= \mathbf{12}\ \frac{\pi}{\mathbf{4\sin(3\pi/4)}} \quad \text{(from tables of definite integrals)} \ = \ \mathbf{13.3}$$

3. *Let $X(t)$ and $Y(t)$ be independent widesense stationary random processes with power spectral densities $S_X(f)$ and $S_Y(f)$, respectively. Find the power spectral density of the random processes $U(t)$ and $V(t)$ defined by*

   (a) *$U(t) = a + b\,X(t)$*

   The autocorrelation function of $U(t)$ is

$$R_U(t,s) = E[U(t)U(s)] = E[(a + bX(t))(a+bX(s))] = a^2+abE[X(s)]+abE[X(t)] + b^2 E[X(t)X(s)]$$

$$= a^2 + 2ab\,m_X + b^2 R_X(t\text{-}s)$$

   Notice that this is just a function of t-s. This fact coupled with the fact that the mean function is a constant means that U(t) is widesense stationary.

   The power spectral density of U(t) is

$$\mathbf{S_U(f)} = \text{Fourier transform of } R_U(\tau) = \int_{-\infty}^{\infty} R_U(\tau)\,e^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} (a^2+2ab\,m_X+b^2 R_X(\tau))\,e^{-j2\pi f\tau}\,d\tau$$

$$= \mathbf{(a^2+2abm_X)\ \delta(f) + b^2\,S_X(f)} \quad \text{since Fourier transform of a constant } c \text{ is } c\delta(f)$$

   (b) *$V(t) = X(t) + Y(t)$*

   As above or from Problem 8 of Homework 12, $R_V(\tau) = a^2 R_X(\tau) + 2m_Xm_Y + R_Y(\tau)$.

   So $\mathbf{S_V(f)} = F\{R_X(\tau) + 2m_Xm_Y + R_Y(\tau)\} = \mathbf{S_X(f) + 2m_Xm_Y\,\delta(f) + S_Y(f)}$

4. *White noise with power spectral density η is applied to an ideal low pass filter with frequency response*

$$H(f) = \begin{cases} 1, & |f| \le W \\ 0, & |f| > W \end{cases}$$

   (a) *Find and sketch the autocorrelation function of the output of the filter.*

   We first find the power spectral density of the output random process. Then we take its inverse transform to find the autocorrelation function.

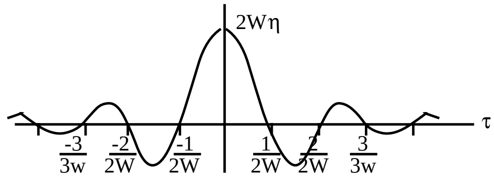
   Let X(t) and Y(t) denote the input and output random processes, respectively. Then, as shown in class $S_Y(f) = S_X(f)\,|H(f)|^2$, where $S_X(f) = \eta$ for all f. Substiuting for H(f) gives

$$S_Y(f) = \begin{cases} \eta, & |f| \le W \\ 0, & |f| > W \end{cases}$$

   The autocorrelation function is then

$$\mathbf{R_Y(\tau)} = F^{-1}\{S_Y(f)\} = \int_{-\infty}^{\infty} S_Y(f)\,e^{j2\pi f\tau}\,df = \int_{-W}^{W} \eta\,e^{j2\pi f\tau}\,df = \frac{\eta}{2\pi}\frac{1}{j\tau}\,e^{j2\pi f\tau}\Big|_{-W}^{W}$$

$$= \frac{\eta}{\mathbf{j2\pi\tau}}\,\mathbf{(e^{j2\pi W\tau} - e^{-j2\pi W\tau})} = \eta\,\frac{\mathbf{\sin 2\pi W\tau}}{\pi\tau}, \quad \text{since } \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$



(b) Let $Z_n = Y(n/2W)$ be the sample of the ouput taken at times $n/2W$. Find the autocorrelation function of the discrete-time random process $\{Z_n\}$ and comment on what you find.

$$\mathbf{R_Z(n,n+k)} = E\, Z_n Z_{n+k} = E\, Y(n/2W)\, Y((n+k)/2W) = R_Y(k/2W) = \begin{cases} \mathbf{2W\eta}, & \mathbf{k=0} \\ \mathbf{0}, & \mathbf{k \neq 0} \end{cases}$$

In addition, we know that $EZ_n = E[Y(n/2W)] = E[X(t)] \int_{-\infty}^{\infty} h(t)\, dt = 0$ since white noise has $E[X(t)] = 0$. Therefore, we see that the Z_n's are uncorrelated with each other.

5. *A wide-sense stationary Gaussian random process $X(t)$ with mean 0 and autocorrelation function $R_X(\tau) = 3e^{-|\tau|}$ is the input to a filter (linear time-invariant system) with impulse response $h(t) = e^{-t}$, $t \geq 0$ and $h(t) = 0$ for $t < 0$.*

(a) *Find the probability that the output of the filter is less than 3 at time 5.*

Let $Y(t)$ denote the output random process. Then $Y(t)$ is Gaussian, because the output of a filter with a Gaussian input is Gaussian. Its mean function is

$$E[Y(t)] = m_X \int_{-\infty}^{\infty} h(t)\, dt = 0 \quad \text{because} \quad m_X = 0$$

It's power at time t is $E[Y(t)^2] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\, df$ and $S_Y(f) = S_X(f)\, |H(f)|^2$

$$S_X(f) = \text{fourier transform of } 3e^{-|\tau|} = \int_{-\infty}^{\infty} 3\, e^{-|\tau|} e^{-j2\pi f\tau}\, d\tau$$

$$= \int_0^{\infty} 3\, e^{-\tau-j2\pi f\tau}\, d\tau + \int_{-\infty}^{0} 3\, e^{\tau-j2\pi f\tau}\, d\tau = 3\, \frac{e^{-\tau-j2\pi f\tau}}{-1-j2\pi f}\Big|_0^{\infty} + 3\, \frac{e^{\tau-j2\pi f\tau}}{1-j2\pi f}\Big|_{-\infty}^{0}$$

$$= \frac{3}{1+j2\pi f} + \frac{3}{1-j2\pi f} = \frac{6}{1+4\pi^2 f^2}$$

$$H(f) = \text{fourier transform of } h(t) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}\, dt = \int_0^{\infty} e^{-t}\, e^{-j2\pi ft}\, dt = \frac{e^{-t-j2\pi ft}}{-1-j2\pi f}\Big|_0^{\infty} = \frac{1}{1+j2\pi f}.$$

$$|H(\omega)|^2 = \frac{1}{(1+j2\pi f)(1-j2\pi f)} = \frac{1}{1+4\pi^2 f^2}$$

Substituting gives: $S_Y(f) = \dfrac{6}{(1+4\pi^2 f^2)^2}$ and so

$$\sigma^2_{Y(t)} = E[Y(t)^2] = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\, df = \int_{-\infty}^{\infty} \frac{6}{(1+4\pi^2 f^2)^2}\, df = 12 \int_0^{\infty} \frac{1}{(1+4\pi^2 f^2)^2}\, df$$

$$= 12 \int_0^{\infty} \frac{1}{(1+\omega^2)^2}\, d\omega\, \frac{1}{2\pi}, \quad \text{with } \omega = 2\pi f$$

$$= \frac{6}{\pi}\, \frac{3\pi}{4} = \frac{9}{2} \text{ from tables of definite integrals}$$

Finally,

$$\mathbf{Pr(Y(5) < 3)} \;=\; \int_{-\infty}^{3} f_{Y(5)}(y)\,dy \;=\; 1 - \int_{3}^{\infty} f_{Y(5)}(y)\,dy \;=\; 1 - \int_{3}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2_{Y(5)}}} \exp\left\{-\frac{y^2}{2\sigma^2_{Y(5)}}\right\} dy$$

$$= 1 - \int_{3/\sigma_{Y(5)}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} dy \;=\; 1 - Q\left(\frac{3}{\sigma_{Y(5)}}\right)$$

$$= 1 - Q\left(\frac{3}{\sqrt{9/2}}\right) \cong 1 - Q(1.4) = 1 - .08 = \mathbf{0.92}$$

(b) *Find an expression for the power of the output of the filter in the frequency band [1,2] . You may leave your answer as an integral.*

$$\textbf{Power in frequency band [1,2]} \;=\; \int_{-2}^{-1} S_Y(f)\,df + \int_{1}^{2} S_Y(f)\,df \;=\; 2\int_{1}^{2} S_Y(f)\,df \;=\; \mathbf{2\int_{1}^{2} \frac{6}{(1+4\pi^2 f^2)^2}\,df}$$

6. 7.8, p. 451, $Z(t) = X(t)\,Y(t)$ where $\{X(t)\}$ and $\{Y(t)\}$ are independent and WSS.

(a) $m_Z(t) = E[Z(t)] = E[X(t)Y(t)] = E[X(t)]\,E[Y(t)]$ since $X(t)$ & $Y(t)$ are independent

$= m_X\; m_Y$, since $\{X(t)\}$ and $\{Y(t)\}$ are WSS

$R_Z(t,s) = E[Z(t)Z(s)] = E[X(t)Y(t)\,X(s)Y(s)]$

$= E[X(t)X(s)]\,E[Y(t)Y(s)]$, since $\{X(t)\}$ and $\{Y(t)\}$ are independent

$= R_X(t-s)\,R_Y(t-s)$

Since $m_Z(t)$ is a constant and $R_Z(t,s)$ depends only on t-s, we conclude that $\{Z(t)\}$ is WSS.

(b) As found above, $\mathbf{R_Z}(\tau) = R_X(\tau)\,R_Y(\tau)$

$$S_Z(f) = \mathcal{F}\{R_Z(\tau)\} = \mathcal{F}\{R_X(\tau)\,R_Y(\tau)\} = \mathbf{S_X(f)*S_Y(f)}$$

7. 7.19, p. 452, except to simplify the problem a bit, let $Y(t) = \frac{1}{T}\int_{-T/2}^{T/2} X(t')\,dt'$

(a) $Y(t) = \frac{1}{T}\int_{t-T}^{t} X(t')\,dt' = X(t) * h(t)$ where $h(t) = \begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, & \text{else} \end{cases}$

Therefore $S_Y(f) = S_X(f)\,|H(f)|^2$ where

$$H(f) = \mathcal{F}\{h(t)\} = \int_0^T \frac{1}{T} e^{-j2\pi ft}\,dt = \frac{1}{T}\,\frac{1}{-j2\pi f}\left(e^{-j2\pi ft}\right)_0^T = \frac{(1-e^{-j2\pi fT})}{j2\pi fT}$$

$$= \frac{(e^{j2\pi fT/2} - e^{-j2\pi fT/2})}{j2\pi fT}\, e^{-j2\pi fT/2} = \frac{\sin \pi fT}{\pi fT}\, e^{-j2\pi fT/2}$$

Therefore, $\mathbf{S_Y(f)} = S_X(f)\,|H(f)|^2 = \mathbf{S_X(f)\left(\frac{\sin \pi fT}{\pi fT}\right)^2}$

(b) $\mathbf{E[Y^2(t)]} = R_Y(0) = \int_{-\infty}^{\infty} S_Y(f)\,df = \mathbf{\int_{-\infty}^{\infty} S_X(f)\left(\frac{\sin \pi fT}{\pi fT}\right)^2 df}$