Summary of Random Variable Concepts **March 17, 2000**

This is a list of important concepts we have covered, rather than a review that devives or explains them.

Types of random variables

- discrete

  A random variable X is discrete if there is a discrete set A (i.e. finite our countably infinite) such that $\Pr(X\in A) = 1$.

- continuous

  A random variable X is continuous if $\Pr(X=x) = 0$ for all values x.

- mixed (other)

  A random variable X is continuous if it is neither discrete nor continuous. In other words there is a least one value x such that $\Pr(X=x)>0$ and the sum of the probabilities of all values x with positive probability is not one.

The Probability Distribution of a Random Variable

The probability distribution of a random variable X can be described by the following three types of functions

- probability mass function (pmf) $p_X(x)$: for discrete random variables only

  Defn: $p_X(x) = \Pr(X=x)$

  Key property: $\Pr(X\in B) = \sum_{x\in B} p_X(x)$,

- probability density function (pdf) $f_X(x)$: basically for continuous random variables, but with delta functions a density can also be used for discrete and mixed random variables. $f_X(x)$ is defined to be a function such that

  $\Pr(X\in B) = \int_B f_X(x)\, dx$, this is also how it is principally used.

- cumulative distribution function (cdf) $F_X(x)$

  Defn: $F_X(x) = \Pr(X\leq x)$

  $\Pr(a<X\leq b) = F_X(b)-F_X(a)$,

Relationships between pdf and cdf:

- $F_X(x) = \int_{-\infty}^{x} f_X(x')\, dx'$
- $f_X(x) = \frac{d}{dx} F_X(x)$

Some common probability distributions:

- Discrete: binary, binomial, Poisson, geometric
- Continuous: uniform, Gaussian, exponential, Laplacian

Functions of Random Variables

Suppose X is a random variable and $Y = g(X)$

- Key fact: $\Pr(Y \in B) = \Pr(X \in \{x : g(x) \in B\})$
- Common question: Given knowledge of the function g and the distribution of X (i.e. of its pdf, pmf or cdf), find the probability distribution of Y (i.e. its pdf, pmf or cdf)
- Important preliminary steps:

  1. Identify the possible values of Y.

  2. Determine the <u>type</u> of Y (discrete, continuous or mixed).

- Special case, if X is continuous r.v. with density $f_X(x)$ and $g(x)$ is a function with no flat segments except those on which $f_X(x) = 0$, then Y is a continuous r.v. with pdf

  $$f_Y(y) = \sum_i f_X(x_i) \left|\frac{1}{g'(x_i)}\right|$$

  where $x_1, x_2, \ldots$ are the values of x such that $g(x) = y$.

  If $g(x)$ has flat segments, then it is possible for $f_Y(y)$ to contain delta functions.

Expected Values

- For a random variable X with density $f_X(x)$ (X could be continuous, discrete or mixed), the expected value or mean value of X is

  $$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

- For a discrete random variable with pmf $p_X(x)$, an alternative and usually more useful formula is

  $$E[X] = \sum_x x\, p_X(x)\,, \qquad \text{where } \sum_x \text{ means to sum over all the possiblex values of } x$$

- Fundamental theorem of expectation: When $Y = g(X)$,

  $$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

- Linearity of expectation: $E[aX+b] = a\,E[X] + b$
  (Generally speaking, for nonlinear functions, $E[g(X)] \neq g(E[X])$)
- Moments: The nth moment of X is $E[X^n]$
- Central moments: The nth central moment of X is $E[(X-E[X])^n]$
- Variance of X: $\mathrm{var}(X) = E[(X-E[X])^2] = E[X^2] - (E[X])^2$
- Chebychev's Inequality

  $$\Pr(|X-E[X]| \geq \varepsilon) \leq \frac{\sigma_X^2}{\varepsilon^2}$$

Pairs of Random Variables

Let X and Y be random variables. Each could be discrete, continuous or mixed. They need not be of the same type.

The Joint Distribution of a Pair of Random Variables:

Can be described by the following three types of functions

- Joint pmf: applies only when both X and Y are discrete

  Defn: $p_{XY}(x,y) = \Pr(X=x, Y=x)$

  Key property: $\Pr((X,Y) \in A) = \sum_{(x,y)\in A} p_{XY}(x,y)$

- Joint pdf: applies only when both X and Y are continuous AND $\Pr((X,Y)\in A) = 0$ for every set A having zero area.

  Defn: the joint pdf is a function $f_{XY}(x,y)$ such that

  $$\Pr((X,Y) \in A) = \int\int_A f_{XY}(x,y)\, dx\, dy$$

  An example of a pair of continuous random variables that do not have joint density:
  X is continuous and Y = g(X).

  Note: We have not introduced delta functions for use in joint densities. For pairs of random variables, they are too complicated to be of use.

- Joint cdf:

  Defn: $F_{XY}(x,y) = \Pr(X\le x, Y\le y)$

  Property: $\Pr(a\le X\le b \text{ and } c \le Y \le d) = F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,c)$

Interrelationships

- $F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x',y')\, dy'\, dx'$ if X and Y have joint density

- $F_{XY}(x,y) = \sum_{x'\le x, y'\le y} p_{XY}(x,y)$ if X and Y are discrete

- $f_{XY}(x,y) = \frac{d^2}{dx\,dy} F_{XY}(x,y)$ if X and Y have joint density

- marginal distribution of X (similar relationships for Y)

  $F_X(x) = F_{XY}(x,\infty)$

  $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

  $p_X(x) = \sum_y p_{XY}(x,y$

- A pair of Gaussian random variables has joint density of the form

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left\{ - \frac{\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2r(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{2\sigma_Y^2}}{2(1-r^2)} \right\}$$

where $-1 \le r \le 1$. The marginal density of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{ - \frac{(x-\bar{X})^2}{2\sigma_X^2} \right\}$$

Independence

- Defn: Random variables X and Y are independent if the events $\{X\in A\}$ and $\{Y\in B\}$ are indepenendent for all A and B. In other words X and Y are independent if

$$\Pr(X\in A, Y\in B) = \Pr(X\in A) \Pr(X\in B) \text{ for all } A \text{ and } B$$

Each of the following is an equivalent conditions for independence:

- $p_{XY}(x,y) = p_X(x)\, p_Y(y)$ all x,y (applies only if X and Y are discrete)

- $f_{XY}(x,y) = f_X(x)\, f_Y(y)$ all x,y (applies only if X and Y are continuous with a joint density)

- $F_{XY}(x,y) = F_X(x)\, F_Y(y)$ all x,y (applies to any pair of random variables)

Equal vs. Identical (not covered on Exam 2)

- Defn: Random variables X and Y are equal, i.e. $X = Y$, if $\Pr(X=Y)=1$, i.e. the produce the same value with probability one.

- Defn: Random variables X and Y are identical if they have the same probability distribution, i.e. they have the same cdf and pdf and if discrete they have the same pmf.

- equal $\Rightarrow$ identical. identical $\not\Rightarrow$ equal.

- independent $\Rightarrow$ not equal; equivalently equal $\Rightarrow$ not independent

Conditional Probability Distributions

	type of functtion describing conditional probability distribution		
type of conditioning	pmf	pdf	cdf
$X\in B$	$p_X(x\|B)$	$f_X(x\|B)$	$F_X(x\|B)$
$Y\in B$	$p_X(x\|Y\in B)$	$f_X(x\|Y\in B)$	$F_X(x\|Y\in B)$
$Y=y$	$p_{X\|Y}(x\|y)$	$f_{X\|Y}(x\|y)$	$F_{X\|Y}(x\|y)$
nonumerical event	$p_X(x\|\text{event})$	$p_X(x\|\text{event})$	$F_X(x\|\text{event})$

Conditional pmf's: These are defined by

$p_X(x|B) = \Pr(X=x|X\in B)$

$p_X(x|Y\in B) = \Pr(X=x|X\in B)$

$p_{X|Y}(x|y) = \Pr(X=x|Y=y)$

$p_X(x|\text{event}) = \Pr(X=x|\text{event})$

As functions of x, the conditional pmf's have all the usual properties of pmf's. Most importantly they are summed to compute conditional probabilities, e.g.

$$\Pr(X\in A|Y=y) = \sum_{x\in A} p_{X|Y}(x|y)$$

Conditional pdf's: The conditional pdf's are defined as functions that one integrates to compute a conditional probability. Specifically,

$$\Pr(X\in A|X\in B) = \int_A f_X(x|B)\, dx$$

$$\Pr(X\in A|Y\in B) = \int_A f_X(x|Y\in B)\, dx$$

$$\Pr(X\in A|Y=y) = \int_A f_{X|Y}(x|y)\, dx$$

$$\Pr(X\in A|\text{event}) = \int_A f_X(x|\text{event})\, dx$$

As functions of x, the conditional pmf's have all the usual properties of pdf's.

Conditional cdf's: These are defined by

$F_X(x|B) = \Pr(X\le x|X\in B)$

$F_X(x|Y\in B) = \Pr(X\le x|X\in B)$

$F_{X|Y}(x|y) = \Pr(X\le x|Y=y)$

$F_X(x|\text{event}) = \Pr(X\le x|\text{event})$

As functions of x, the conditional pmf's have all the usual properties of cdf's.

Notes:

- Of all the above functions, the following are generally the most useful and nicest to work with:

  $p_X(x)$, $p_Y(y)$, $p_{XY}(x,y)$, $p_{X|Y}(x|y)$, $p_{Y|X}(y|x)$, $f_X(x)$, $f_Y(y)$, $f_{XY}(x,y)$, $f_{X|Y}(x|y)$, $f_{Y|X}(y|x)$

- Conditioning can change the type of a random variable X. For example, X could be continuous, but conditioned on $Y = y$, X could be discrete.

  We use a conditional pmf when X is conditionally discrete. We use a conditional pdf when X is conditionally continuous. We can also use a conditional pmf when X is conditionally discrete or mixed, but it will contain delta functions.

The Big Four Rrelationships

	X&Y discrete	X&Y continuous with a joint density
1.	$p_{XY}(x,y) = p_Y(y)\, p_{X\|Y}(x\|y)$	$f_{XY}(x,y) = f_Y(y)\, f_{X\|Y}(x\|y)$
2.	$p_{Y\|X}(y\|x) = \frac{p_{XY}(x,y)}{p_X(x)}$	$f_{Y\|X}(y\|x) = \frac{f_{XY}(x,y)}{f_X(x)}$
3. Bayes rule	$p_{X\|Y}(x\|y) = \frac{p_{Y\|X}(y\|x)p_X(x)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{Y\|X}(y\|x)f_X(x)}{f_Y(y)}$
4. Total prob.	$p_X(x) = \sum_y p_{XY}(x,y) = \sum_y p_Y(y)\, p_{X\|Y}(x\|y)$	$f_X(x) = \int f_{XY}(x,y)\, dy = \int f_Y(y) f_{X\|Y}(x\|y)\, dy$

Other relationships:

- $f_X(x|B) = \begin{cases} f_X(x), & x \in B \\ \Pr(X\in B, & \text{else}) \end{cases}$

- $F_X(x|B) = \int_{-\infty}^{x} f_X(x'|B)\, dx'$ when X is conditionally continuous

  $= \sum_{x'\le x} p_X(x'|B)$ when X is conditionally discrete

- $f_X(x|B) = \frac{d}{dx} F_X(x|B)$

- Same as the above but for conditioning on $Y\in B$ or $Y=y$ or nonnumerical conditioning.

- $f_X(x|Y\in B) = \dfrac{\int_B f_{XY}(x,y)\, dy}{\Pr(Y\in B)}$

- Other total probability laws: These can be straightforwardly derived, though we haven't derived them in class.

  If $B_1,\ldots,B_n$ form a partition, then

  $$f_X(x) = \sum_{i=1}^{n} f_X(x|B_i)\, \Pr(X\in B_i)$$

  $$p_X(x) = \sum_{i=1}^{n} p_X(x|B_i)\, \Pr(X\in B_i)$$

  $$\Pr(X\in A) = \int_{-\infty}^{\infty} \Pr(X\in A|Y=y)\, f_Y(y)\, dy$$

  $$\Pr(X\in A) = \sum_y \Pr(X\in A|Y=y)\, p_Y(y)$$