2.3   SAMPLING BANDPASS SIGNALS

Although satisfying the majority of sampling requirements, the sampling of low-pass signals, as in Figure 2–6, is not the only sampling scheme used in practice. We can use a technique known as *bandpass sampling* to sample a continuous bandpass signal that is centered about some frequency other than zero Hz. When a continuous input signal's bandwidth and center frequency permit us to do so, bandpass sampling not only reduces the speed requirement of A/D converters below that necessary with traditional low-pass sampling; it also reduces the amount of digital memory necessary to capture a given time interval of a continuous signal.

By way of example, consider sampling the band-limited signal shown in Figure 2–7(a) centered at f_c = 20 MHz, with a bandwidth B = 5 MHz. We use the term bandpass sampling for the process of sampling continuous signals whose center frequencies have been translated up from zero Hz. What we're calling bandpass sampling goes by various other names in the literature, such as IF sampling, harmonic sampling[2], sub-Nyquist sampling, and under-sampling[3]. In bandpass sampling, we're more concerned with a signal's bandwidth than its highest frequency component. Note that the negative frequency portion of the signal, centered at $-f_c$, is the mirror image of the positive frequency portion—as it must be for real signals. Our bandpass signal's highest frequency component is 22.5 MHz. Conforming to the Nyquist criterion (sampling at twice the highest frequency content of the signal) implies that the sampling frequency must be a minimum of 45 MHz. Consider the effect if the sample rate is 17.5 MHz shown in Figure 2–7(b). Note that the original spectral components remain located at $\pm f_c$, and spectral replications are located exactly
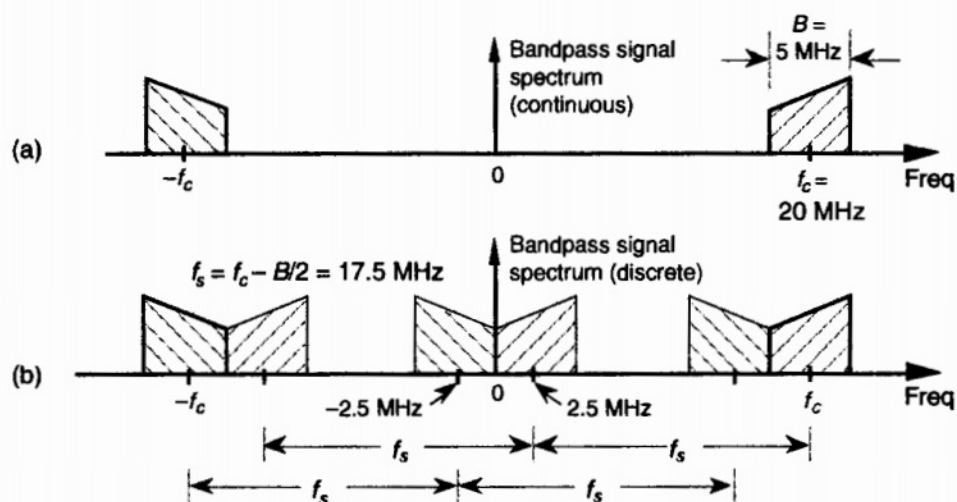

Figure 2-7   Bandpass signal sampling: (a) original continuous signal spectrum; (b) sampled signal spectrum replications when sample rate is 17.5 MHz.

at baseband, i.e., butting up against each other at zero Hz. Figure 2–7(b) shows that sampling at 45 MHz was unnecessary to avoid aliasing—instead we've used the spectral replicating effects of Eq. (2–5) to our advantage.

Bandpass sampling performs digitization and frequency translation in a single process, often called *sampling translation*. The processes of sampling and frequency translation are intimately bound together in the world of digital signal processing, and every sampling operation inherently results in spectral replications. The inquisitive reader may ask, "Can we sample at some still lower rate and avoid aliasing?" The answer is yes, but, to find out how, we have to grind through the derivation of an important bandpass sampling relationship. Our reward, however, will be worth the trouble because here's where bandpass sampling really gets interesting.

Let's assume we have a continuous input bandpass signal of bandwidth B. Its *carrier frequency* is f_c Hz, i.e., the bandpass signal is centered at f_c Hz, and its sampled value spectrum is that shown in Figure 2–8(a). We can sample that continuous signal at a rate, say $f_{s'}$ Hz, so the spectral replications of the positive and negative bands, Q and P, just butt up against each other exactly at zero Hz. This situation, depicted in Figure 2–8(a), is reminiscent of Figure 2–7(b). With an arbitrary number of replications, say m, in the range of $2f_c - B$, we see that

$$mf_{s'} = 2f_c - B \quad \text{or} \quad f_{s'} = \frac{2f_c - B}{m} \; . \tag{2-6}$$

In Figure 2–8(a), $m = 6$ for illustrative purposes only. Of course m can be any positive integer so long as $f_{s'}$ is never less than $2B$. If the sample rate $f_{s'}$ is in-
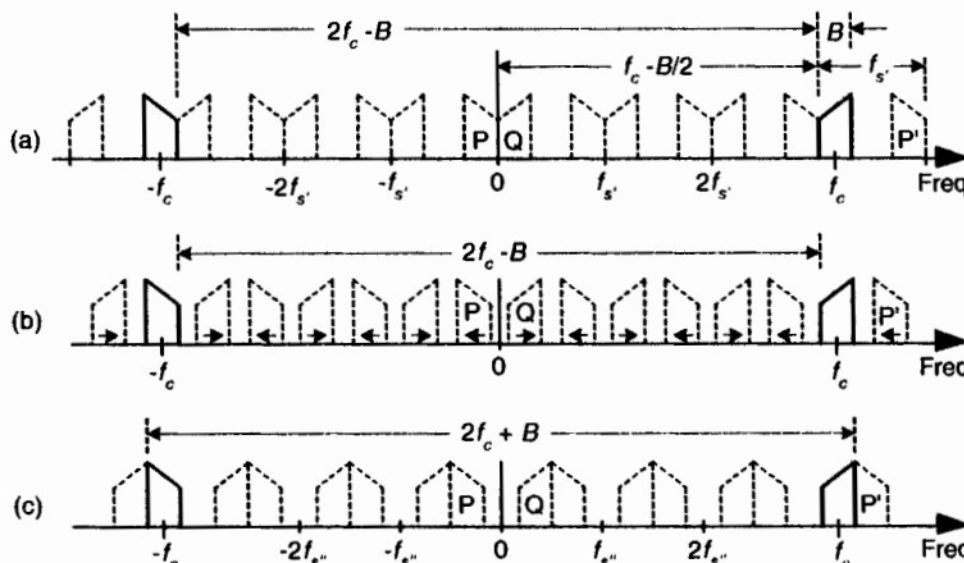

Figure 2-8   Bandpass sampling frequency limits: (a) sample rate $f_{s'} = (2f_c - B)/6$; (b) sample rate is less than $f_{s'}$; (c) minimum sample rate $f_{s''} < f_{s'}$.

creased, the original spectra (bold) do not shift, but all the replications will shift. At zero Hz, the P band will shift to the right, and the Q band will shift to the left. These replications will overlap and aliasing occurs. Thus, from Eq. (2–6), for an arbitrary m, there is a frequency that the sample rate must not exceed, or

$$f_{s'} \leq \frac{2f_c - B}{m} \quad \text{or} \quad \frac{2f_c - B}{m} \geq f_{s'} . \tag{2–7}$$

If we reduce the sample rate below the $f_{s'}$ value shown in Figure 2–8(a), the spacing between replications will decrease in the direction of the arrows in Figure 2–8(b). Again, the original spectra do not shift when the sample rate is changed. At some new sample rate $f_{s''}$, where $f_{s''} < f_{s'}$, the replication P′ will just butt up against the positive original spectrum centered at f_c as shown in Figure 2–8(c). In this condition, we know that

$$(m+1)f_{s''} = 2f_c + B \quad \text{or} \quad f_{s''} = \frac{2f_c + B}{m+1} . \tag{2–8}$$

Should $f_{s''}$ be decreased in value, P′ will shift further down in frequency and start to overlap with the positive original spectrum at f_c and aliasing occurs. Therefore, from Eq. (2-8) and for $m+1$, there is a frequency that the sample rate must always exceed, or

$$f_{s''} \geq \frac{2f_c + B}{m+1} . \tag{2–9}$$

We can now combine Eqs. (2–7) and (2–9) to say that f_s may be chosen anywhere in the range between $f_{s''}$ and $f_{s'}$ to avoid aliasing, or

$$\frac{2f_c - B}{m} \geq f_s \geq \frac{2f_c + B}{m+1} , \tag{2–10}$$

where m is an arbitrary, positive integer ensuring that $f_s \geq 2B$. (For this type of periodic sampling of real signals, known as real or first-order sampling, the Nyquist criterion $f_s \geq 2B$ must still be satisfied.)

    To appreciate the important relationships in Eq. (2–10), let's return to our bandpass signal example, where Eq. (2–10) enables the generation of Table 2–1. This table tells us that our sample rate can be anywhere in the range of 22.5 to 35 MHz, anywhere in the range of 15 to 17.5 MHz, or anywhere in the range of 11.25 to 11.66 MHz. Any sample rate below 11.25 MHz is unacceptable because it will not satisfy Eq. (2–10) as well as $f_s \geq 2B$. The spectra resulting from several of the sampling rates from Table 2–1 are shown in Figure 2–9 for our bandpass signal example. Notice in Figure 2–9(f) that when f_s equals 7.5 MHz (m = 5), we have aliasing problems because neither

Table 2-1 Equation (2-10) Applied to the Bandpass Signal Example

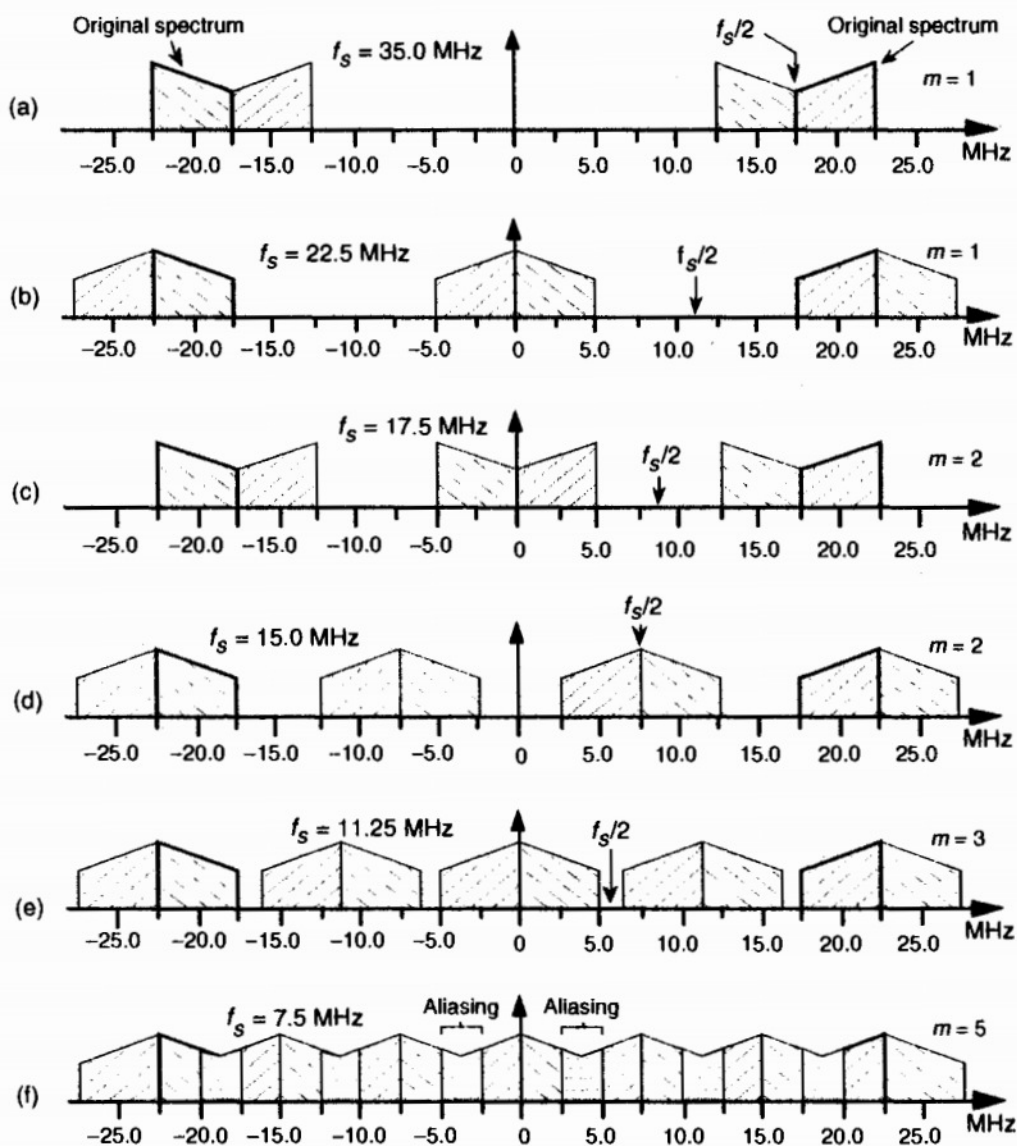| m	$(2f_c-B)/m$	$(2f_c+B)/(m+1)$	Optimum sampling rate
1	35.0 MHz	22.5 MHz	22.5 MHz
2	17.5 MHz	15.0 MHz	17.5 MHz
3	11.66 MHz	11.25 MHz	11.25 MHz
4	8.75 MHz	9.0 MHz	—
5	7.0 MHz	7.5 MHz	—

Figure 2-9 Various spectral replications from Table 2-1: (a) f_s = 35 MHz; (b) f_s = 22.5 MHz; (c) f_s = 17.5 MHz; (d) f_s = 15 MHz; (e) f_s = 11.25 MHz; (f) f_s = 7.5 MHz.

the greater than relationships in Eq. (2–10) nor $f_s \geq 2B$ have been satisfied. The $m = 4$ condition is also unacceptable because $f_s \geq 2B$ is not satisfied. The last column in Table 2–1 gives the *optimum* sampling frequency for each acceptable m value. Optimum sampling frequency is defined here as that frequency where spectral replications do not butt up against each other except at zero Hz. For example, in the $m = 1$ range of permissible sampling frequencies, it is much easier to perform subsequent digital filtering or other processing on the signal samples whose spectrum is that of Figure 2–9(b), as opposed to the spectrum in Figure 2–9(a).

The reader may wonder, "Is the optimum sample rate always equal to the minimum permissible value for f_s using Eq. (2–10)?" The answer depends on the specific application—perhaps there are certain system constraints that must be considered. For example, in digital telephony, to simplify the follow-on processing, sample frequencies are chosen to be integer multiples of 8 kHz[4]. Another application-specific factor in choosing the optimum f_s is the shape of analog anti-aliasing filters[5]. Often, in practice, high-performance A/D converters have their hardware components *fine-tuned* during manufacture to ensure maximum linearity at high frequencies (>5 MHz). Their use at lower frequencies is not recommended.

An interesting way of illustrating the nature of Eq. (2–10) is to plot the minimum sampling rate, $(2f_c+B)/(m+1)$, for various values of m, as a function of R defined as

$$R = \frac{\text{highest signal frequency component}}{\text{bandwidth}} = \frac{f_c + B/2}{B} . \qquad (2\text{–}11)$$

If we normalize the minimum sample rate from Eq. (2–10) by dividing it by the bandwidth B, we get a bold-line plot whose axes are normalized to the bandwidth shown as the solid curve in Figure 2–10. This figure shows us the minimum normalized sample rate as a function of the normalized highest frequency component in the bandpass signal. Notice that, regardless of the value of R, the minimum sampling rate need never exceed $4B$ and approaches $2B$ as the carrier frequency increases. Surprisingly, the minimum acceptable sampling frequency actually decreases as the bandpass signal's carrier frequency increases. We can interpret Figure 2–10 by reconsidering our bandpass signal example from Figure 2–7 where $R = 22.5/5 = 4.5$. This R value is indicated by the dashed line in Figure 2–10 showing that $m = 3$ and f_s/B is 2.25. With $B = 5$ MHz, then, the minimum $f_s = 11.25$ MHz in agreement with Table 2–1. The leftmost line in Figure 2–10 shows the low-pass sampling case, where the sample rate f_s must be twice the signal's highest frequency component. So the normalized sample rate f_s/B is twice the highest frequency component over B or $2R$.
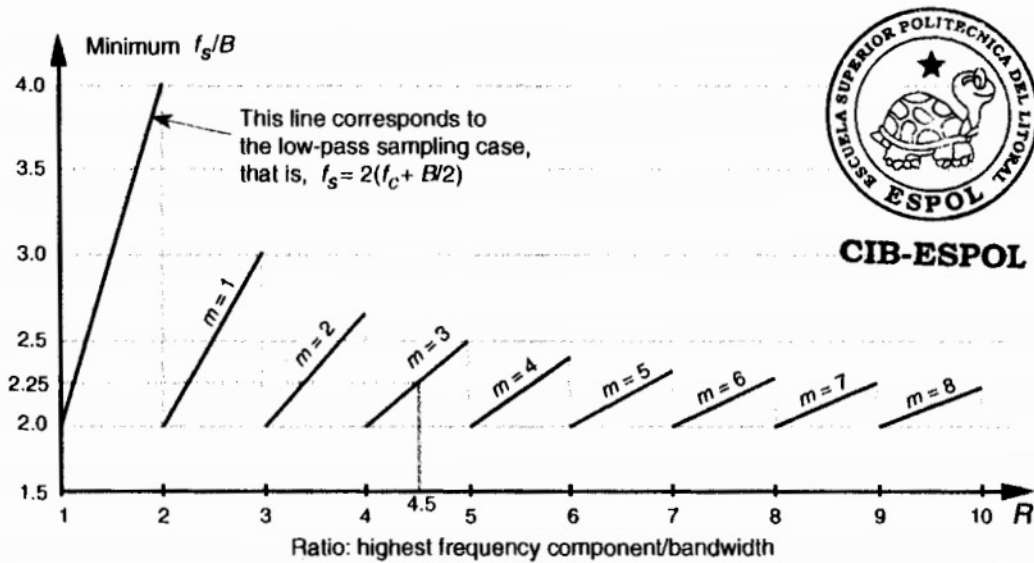
Figure 2-10   Minimum bandpass sampling rate from Eq. (2-10).

Figure 2–10 has been prominent in the literature, but its normal presentation enables the reader to jump to the false conclusion that any sample rate above the minimum shown in the figure will be an acceptable sample rate[6–12]. There's a clever way to avoid any misunderstanding[13]. If we plot the acceptable ranges of bandpass sample frequencies from Eq. (2–10) as a function of R we get the depiction shown in Figure 2–11. As we saw from Eq. (2–10), Table 2–1, and Figure 2–9, acceptable bandpass sample rates are a series of frequency ranges separated by unacceptable ranges of sample rate frequencies, that is, an acceptable bandpass sample frequency must be above the minimum shown in Figure 2–10, but cannot be just any frequency above that minimum. The shaded region in Figure 2–11 shows those normalized bandpass sample rates that will lead to spectral aliasing. Sample rates within the white regions of Figure 2–11 are acceptable. So, for bandpass sampling, we want our sample rate to be in the white wedged areas associated with some value of m from Eq. (2–10). Let's understand the significance of Figure 2–11 by again using our previous bandpass signal example from Figure 2–7.

Figure 2–12 shows our bandpass signal example R value (highest frequency component/bandwidth) of 4.5 as the dashed vertical line. Because that line intersects just three white wedged areas, we see that there are only three frequency regions of acceptable sample rates, and this agrees with our results from Table 2–1. The intersection of the $R = 4.5$ line and the borders of the white wedged areas are those sample rate frequencies listed in Table 2–1. So Figure 2–11 gives a depiction of bandpass sampling restrictions much more realistic than that given in Figure 2–10.

Although Figures 2–11 and 2–12 indicate that we can use a sample rate that lies on the boundary between a white and shaded area, these sample
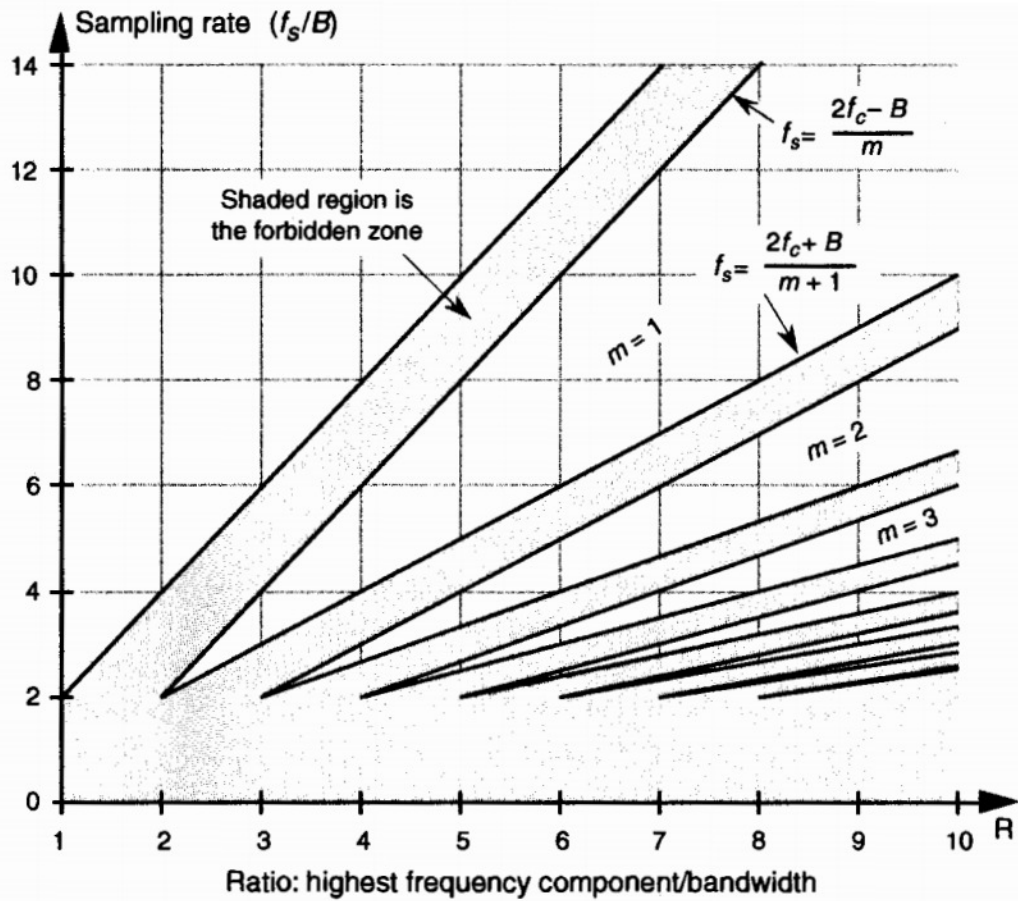
Figure 2-11 Regions of acceptable bandpass sampling rates from Eq. (2-10), normalized to the sample rate over the signal bandwidth (f_s/B).
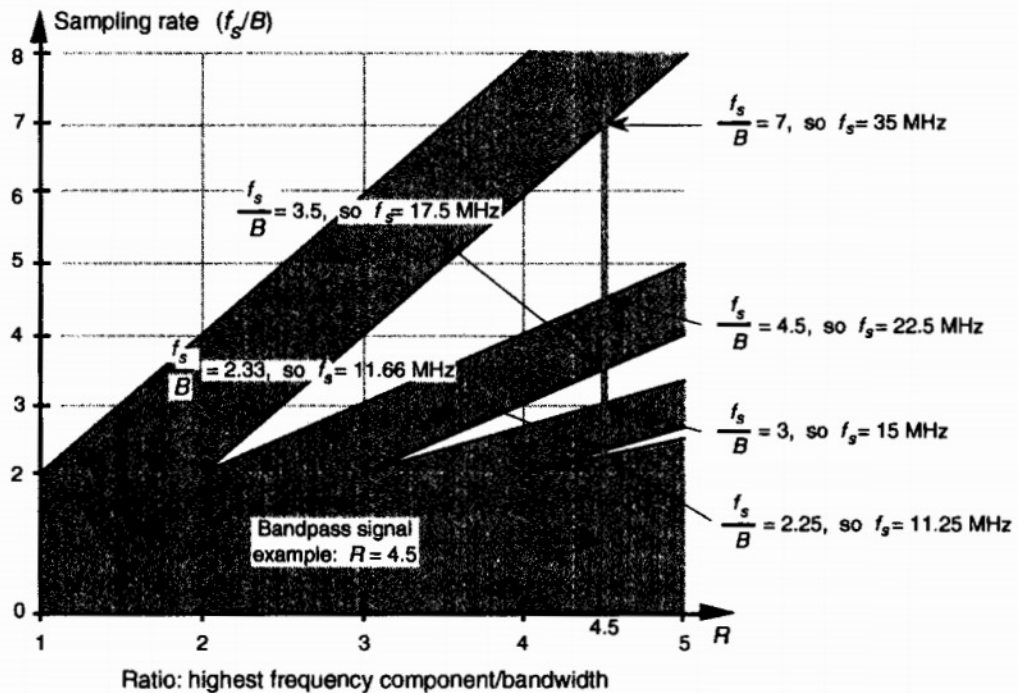

Figure 2-12 Acceptable sample rates for the bandpass signal example ($B = 5$ MHz) with a value of $R = 4.5$.

rates should be avoided in practice. Nonideal analog bandpass filters, sample rate clock generator instabilities, and slight imperfections in available A/D converters make this *ideal* case impossible to achieve exactly. It's prudent to keep f_s somewhat separated from the boundaries. Consider the bandpass sampling scenario shown in Figure 2–13. With a typical (nonideal) analog bandpass filter, whose frequency response is indicated by the dashed line, it's prudent to consider the filter's bandwidth not as B, but as B_{gb} in our equations. That is, we create a guard band on either side of our filter so that there can be a small amount of aliasing in the discrete spectrum without distorting our desired signal, as shown at the bottom of Figure 2–13.

We can relate this idea of using guard bands to Figure 2–11 by looking more closely at one of the white wedges. As shown in Figure 2–14, we'd like to set our sample rate as far down toward the vertex of the white area as we can—lower in the wedge means a lower sampling rate. However, the closer we operate to the boundary of a shaded area, the more narrow the guard band must be, requiring a sharper analog bandpass filter, as well as the tighter the tolerance we must impose on the stability and accuracy of our A/D clock generator. (Remember, operating on the boundary between a white and shaded area in Figure 2–11 causes spectral replications to butt up against each other.) So, to be safe, we operate at some intermediate point
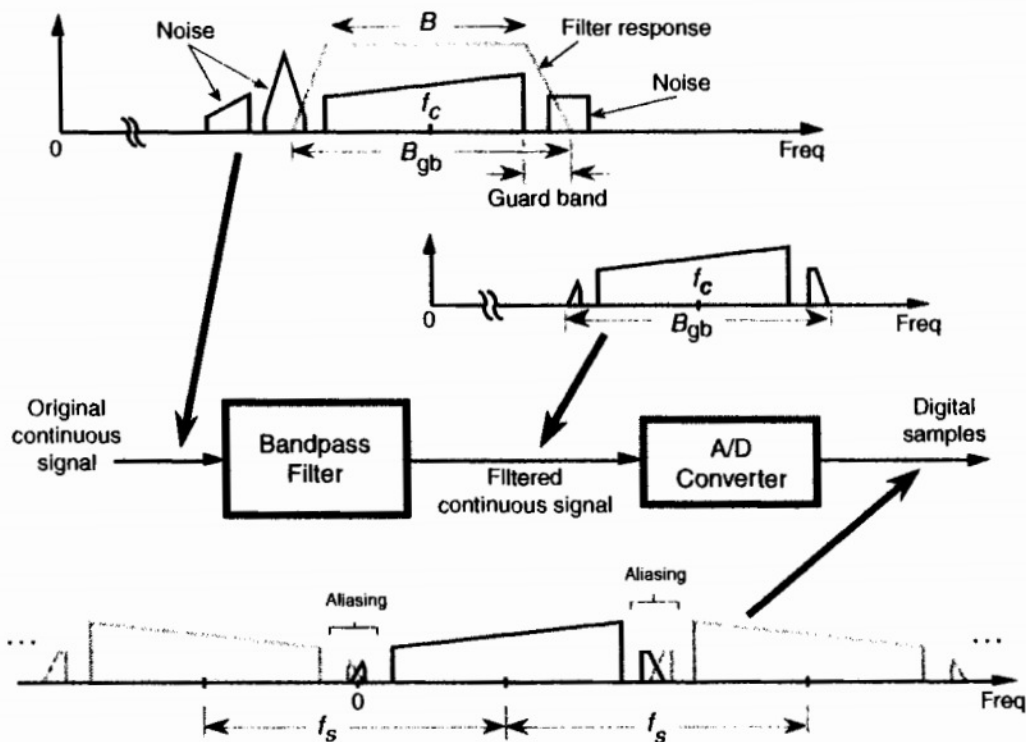

Figure 2-13   Bandpass sampling with aliasing occurring only in the filter guard bands.
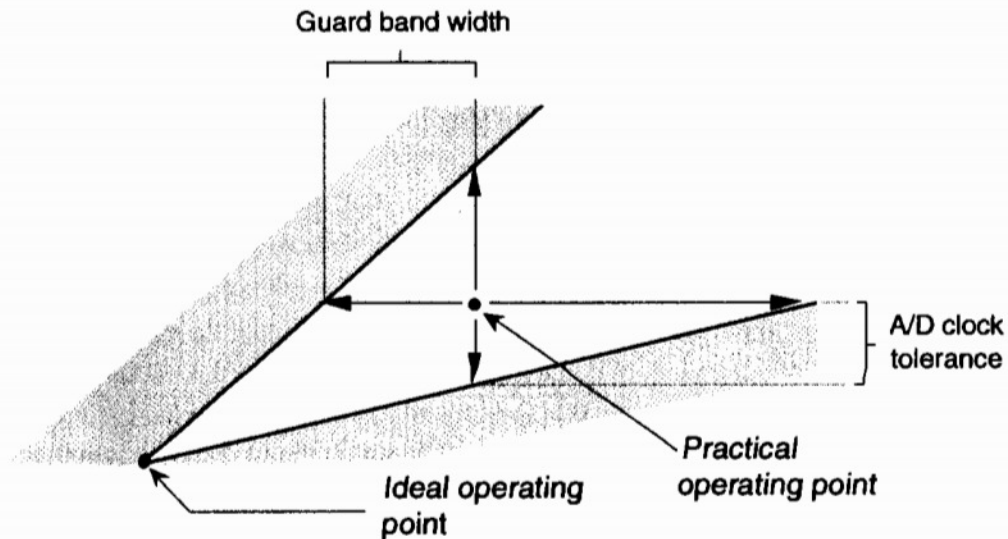
Figure 2-14 Typical operating point for f_s to compensate for nonideal hardware.

away from any shaded boundaries as shown in Figure 2–14. Further analysis of how guard bandwidths and A/D clock parameters relate to the geometry of Figure 2–14 is available in reference [13]. For this discussion, we'll just state that it's a good idea to ensure that our selected sample rate does not lie too close to the boundary between a white and shaded area in Figure 2–11.

There are a couple of ways to make sure we're not operating near a boundary. One way is to set the sample rate in the middle of a white wedge for a given value of R. We do this by taking the average between the maximum and minimum sample rate terms in Eq. (2–10) for a particular value of m, that is, to center the sample rate operating point within a wedge we use a sample rate of

$$f_{s_{cntr}} = \frac{1}{2} \cdot \left[\frac{2f_c - B}{m} + \frac{2f_c + B}{m+1} \right] = \frac{f_c - B/2}{m} + \frac{f_c + B/2}{m+1} . \qquad (2\text{–}12)$$

Another way to avoid the boundaries of Figure 2–14 is to use the following expression to determine an intermediate f_{s_i} operating point:

$$f_{s_i} = \frac{4f_c}{m_{odd}} , \qquad (2\text{–}13)$$

where m_{odd} is an odd integer[14]. Using Eq. (2–13) yields the useful property that the sampled signal of interest will be centered at one fourth the sample rate $(f_{s_i}/4)$. This situation is attractive because it greatly simplifies follow-on complex downconversion (frequency translation) used in many digital communications applications. Of course the choice of m_{odd} must ensure that the
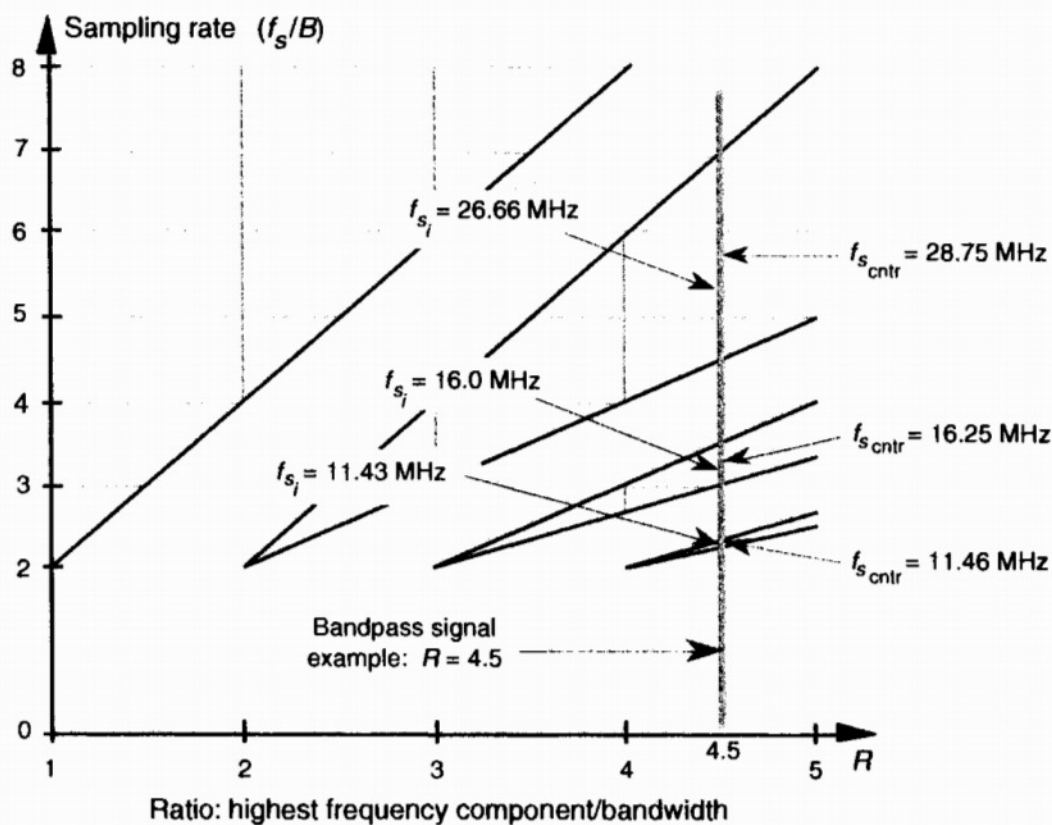
Figure 2-15 Intermediate f_{s_i} and $f_{s_{cntr}}$ operating points, from Eqs. (2–12) and (2–13), to avoid operating at the shaded boundaries for the bandpass signal example. $B = 5$ MHz and $R = 4.5$.

Nyquist restriction of $f_{s_i} > 2B$ be satisfied. We show the results of Eqs. (2–12) and (2–13) for our bandpass signal example in Figure 2–15.

2.4 SPECTRAL INVERSION IN BANDPASS SAMPLING

Some of the permissible f_s values from Eq. (2–10) will, although avoiding aliasing problems, provide a sampled baseband spectrum (located near zero Hz) that is inverted from the original positive and negative spectral shapes, that is, the positive baseband will have the inverted shape of the negative half from the original spectrum. This spectral inversion happens whenever m, in Eq. (2–10), is an odd integer, as illustrated in Figures 2–9(b) and 2–9(e). When the original positive spectral bandpass components are symmetrical about the f_c frequency, spectral inversion presents no problem and any nonaliasing value for f_s from Eq. (2–10) may be chosen. However, if spectral inversion is something to be avoided, for example, when single sideband signals are being processed, the minimum applicable sample rate to avoid spectral inversion is defined by Eq. (2–10) with the restriction that m is the largest even inte-