Today: Spectrum analysis, windowing  
STFT, DFT filterbanks

Announcements: HW5 due today.  
Parts order due on Friday.  
Midterm next Wednesday.

References: See last slide.

Please keep the lab clean and organized.  
Last one out should close door!!!!

“Of course the first novel idea was to do the factorization, which you do on pencil and paper, put together a program. To get an efficient program you have to have some way of indexing.”  
— Jim Cooley talking about how he and Tukey discovered the FFT
The $k$-th coefficient of the N-point DFT of $x[n]$ is a sample of the DTFT of $x[n]$ at digital frequency $f = k/N$.

If $x[n]$ are time samples $x(nT_s)$ of a continuous time signal $x(t)$, then DTFT is an approximation to the finite time FT of $x(t)$ over the time window $t \in [0, (N - 1)T_s)$.

There are several issues that need to be addressed

- Spectral leakage
- Spectral resolution
- Time varying spectra

To build intuition we start by considering an example: DFT of sinusoidal signal.
DFT of a sinusoid at frequency $f_c = m/N$

DFT of sinusoid $x[n] = \cos(2\pi f_c n + \phi)$?

Assume sinusoidal frequency satisfies $f_c = m/N$ for integer $m \in \{0, \ldots, N/2\}$

Use Euler formula: $\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$

$$X_{DFT}(k) = \frac{e^{j\phi}}{2} \sum_{n=0}^{N-1} e^{-j2\pi \frac{k-m}{N} n} + \frac{e^{-j\phi}}{2} \sum_{n=0}^{N-1} e^{-j2\pi \frac{k+m}{N} n}$$

$$= \begin{cases} 
\frac{Ne^{j\phi}}{2}, & k = m \\
\frac{Ne^{-j\phi}}{2}, & k = N - m 
\end{cases}$$

($\Delta[n]$ is kronecker delta function)
DFT of single sinusoid at arbitrary frequency

DFT of sinusoid \( x[n] = \cos(2\pi f_c n + \phi) \)?

Assume sinusoidal frequency does not satisfy \( f_c = m / N \) for integer \( m \in \{0, \ldots, N/2\} \)

\[
X_{DFT}(k) = \frac{e^{j\phi}}{2} \sum_{n=0}^{N-1} e^{-j2\pi \frac{k-Nf_c}{N} n} + \frac{e^{-j\phi}}{2} \sum_{n=0}^{N-1} e^{-j2\pi \frac{k+Nf_c}{N} n}
\]

\[
\neq N\Delta[k-m] \quad \neq N\Delta[N-k-m]
\]

This is the leakage phenomenon and it occurs when \( f_c \neq m / N \).
DFT: a sinusoid $f_c = m/N, \ m = 4, \ N = 64$
DFT: single sinusoid $f_c = 4.5/N$, not an integer/N
DFT: two sinusoids $f_{ci} = m_i/N$, $m_1 = 4$, $m_2 = 5$, $N = 64$
DFT: two sinusoids \( f_{ci} = \frac{m_i}{N} \), \( m_1 = 4 \), \( m_2 = 4.5 \), \( N = 64 \)
Another view on leakage

A sinusoid at frequency $f_c = F_c / F_s$:

$x[n] = \cos(2\pi f_c n), \quad n = 0, \ldots, N - 1.$

$$X_{DTFT}(f) = \sum_{n=0}^{N-1} \cos(2\pi f_c n) e^{-2\pi f n} = \frac{1}{2} \sum_{n=0}^{N-1} (e^{j2\pi f_c n} + e^{-j2\pi f_c n}) e^{-j2\pi f n}$$

$$= \frac{1}{2} g_N(f - f_c) + \frac{1}{2} g_N(f + f_c)$$

$$g_N(\nu) = \sum_{n=0}^{N-1} e^{-j2\pi \nu n}$$

Use geometric series formula $\sum_{n=0}^{M} a^n = (1 - a^{M+1}) / (1 - a)$ to obtain

$$g_N(\nu) = Ne^{-j\pi \nu (N-1)} \frac{\sin(\pi \nu N)}{N \sin(\pi \nu)}$$

Dirichlet kernel
DTFT of 16 samples of sinusoid: $f_c = F_c/F_s = k/N$

Peaks are at $f_c$, $1 - f_c$. Zeros occur at $F_c/F_s \pm k/N$, $k$ an integer.
DFT of 16 samples of sinusoid: \( f_c = \frac{F_c}{F_s} = \frac{k}{N} \)

There is no leakage since \( f_c \) is equal to \( \frac{k}{N} \) for the integer \( k = 4 \).
DTFT of 16 samples of sinusoid: $f_c = F_c / F_s = 0.27$

Peaks occur near $f_c$, $1 - f_c$. There are no zeros in DTFT
DFT of 16 samples of sinusoid: $f_c = \frac{F_c}{F_s} = 0.27$

There is leakage since $f_c$ is not equal to $k/N$ for any integer $k$. 
DFT spectrum of sum of sinusoids (1/2)

Time domain waveforms in spectra shown on previous slide ($N = 64$)

$$x[n] = \cos(2\pi f_1 n) + \frac{1}{2} \cos(2\pi f_2 n), \quad n = 0, \ldots, N - 1$$

- Left panel: $f_1 = 4/N$, $f_2 = 5/N$. Both analysis frequencies of N-point DFT.

- Right panel: $f_1 = 4/N$, $f_2 = 4.5/N$. $f_2$ not an analysis frequency of N-point DFT. $|f_1 - f_2|$ is under DFT’s spectral resolution $1/N$. 
Q. What can we conclude about the time domain signal $x[n]$ by observing peaks in $|X(k)|$ at frequencies $f = k_1/N, \ldots, k_p/N$?

A. Not much unless $|X(k)|$ at all other frequencies is zero.

The reason for the ambiguity on right panel is spectral leakage.
Sometimes increasing the resolution is sufficient

Two frequencies $f_1 = 0.1641 (= 10.5/N)$ and $f_2 = 0.3203 (= 20.5/N)$. N=64-point FFT.

$$x[n] = \cos(2\pi f_1 n) + \frac{1}{2} \cos(2\pi f_2 n), \quad n = 0, \ldots, N - 1$$
Leakage disappears if double the $N$: $f_1 = 0.1641$ ($= 21/N$) and $f_2 = 0.3203$ ($= 41/N$). $N = 128$-point FFT.

$$x[n] = \cos(2\pi f_1 n) + \frac{1}{2} \cos(2\pi f_2 n), \quad n = 0, \ldots, N - 1$$
Resolution vs sensitivity of DFT spectrum

Resolution and sensitivity are the primary "quality" measures of a spectral analysis method.

**Frequency resolution**: the minimum detectable frequency separation of two sinusoids in the absence of noise.

Frequency resolution is \( F_s / N = 1 / (N T_s) = 1 / T \) Hz.

**Spectral sensitivity**: the minimum amplitude of a sinusoid required for detection against noise background.

Spectral sensitivity depends on several factors

- Nature of background noise
- Number of bits of amplitude resolution (Q(15), Q(31))
- The length of the analysis window \( T \)
- Signal-to-noise power ratio (SNR)
DFT spectrum: 10k vs 100k pts

Top: 16384-pt ($2^{14}$) FFT, Bottom 131072-pt ($2^{17}$) FFT
DFT spectrum: 10k vs 100k pts zoomed in

Top: 16384-pt ($2^{14}$) FFT, Bottom 131072-pt ($2^{17}$) FFT
DFT w/0dB noise: 10k vs 100k pts

Top: 16384-pt ($2^{14}$) FFT, Bottom 131072-pt ($2^{17}$) FFT
DFT w/0dB noise: 10k vs 100k pts zoomed in

Top: 16384-pt (2^14) FFT, Bottom 131072-pt (2^17) FFT
Summarize: DFT spectrum

$|X_{DFT}[k]|, \; k = 0, \ldots, N - 1$

- DFT index $k$ corresponds to digital frequency $f_c = k/N$ and Hz frequency $F_c = F_s k/N$.
- Leakage occurs for any frequency component not at one of DFT analysis frequencies $F_s k/N$, $k = 0, \ldots, N/2$.
- Frequency resolution of DFT spectrum is $F_s/N$. This is the minimum frequency separation that can be detected.
- If $x(t)$ is a sum of $p$ sinusoids

$$x(t) = A_1 \sin(2\pi F_1 t + \phi_1) + \cdots + A_p \sin(2\pi F_p t + \phi_p)$$

Then sinusoids can be detected from DFT spectrum if:

- There are no more than $p = N/2 - 1$ sinusoids
- The sinusoidal frequencies $F_i$ are all less than $F_s/2$ Hz.
- The frequency $F_c$ of each sinusoid is distinct and satisfies

$$F_c/F_s = k/N, \; k \in \{0, \ldots, N/2\}$$
How to combat leakage and ambiguity?

Method that is effective: use longer analysis window (increase $T$) → this always reduces leakage for "long duration" (stationary) signals.

Methods that are not effective for leakage mitigation
▶ Zero padding, decimating or interpolating the DFT
▶ Computing the full DTFT

Methods that can be effective
▶ If frequency estimation is the objective, use a different "high resolution" spectrum estimator (signal subspace, MUSIC)
▶ Apply a non-rectangular time window to data prior to DFT ("windowing the data")
Compensate for leakage

The IDFT of $X_{DFT}[k]$ is periodic with period $N$:

$$x_{IDFT}[n] = \sum_{k=0}^{N-1} X_{DFT}[k] e^{j2\pi kn/N}$$

Therefore a cyclic shift of the input does not change magnitude spectrum.

The following have identical magnitude DFT’s:

$\{x[0], \ldots, x[N-1]\}$ and $\{x[N/2], \ldots, x[N-1], x[0], \ldots, x[N/2-1]\}$

(DFT enforces periodicity...).

Spectral leakage can be attributed to the "discontinuity" at the endpoints of the analysis window.
Illustration of cyclic discontinuity effect

Plot of samples and the rect. window function.

Weighted samples shown re-centered at end point splice.

dB plot of the spectrum of the windowed samples.
Windowing data to compensate for leakage

Can mitigate leakage by downweighting the input near endpoints by multiplying data $x[n]$ with a window function.

There is a cost to doing this. Multiplication in the time domain results in a convolution in the frequency domain. The response will be smeared a bit and the values will be attenuated some.
Windowing

Select portion of waveform to analyze.
Weight or shade the data to minimize end effects.

\[
X(k) = \sum_{n=0}^{N-1} w[n]x[n]e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N - 1.
\]

Multiplication in the time domain corresponds to convolution (filtering) in the frequency domain.
Many window functions to choose from

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
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<tbody>
<tr>
<td>bartlett</td>
<td>Bartlett window</td>
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<tr>
<td>blackman</td>
<td>Blackman window</td>
</tr>
<tr>
<td>blackmanharris</td>
<td>Minimum 4-term Blackman-Harris window</td>
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<tr>
<td>bohmanwin</td>
<td>Bohman window</td>
</tr>
<tr>
<td>chebwin</td>
<td>Chebyshev window</td>
</tr>
<tr>
<td>dpss</td>
<td>Discrete prolate spheroidal (Slepian) sequences</td>
</tr>
<tr>
<td>dpssclear</td>
<td>Remove discrete prolate spheroidal sequences from database</td>
</tr>
<tr>
<td>dpssdir</td>
<td>Discrete prolate spheroidal sequences database directory</td>
</tr>
<tr>
<td>dpssload</td>
<td>Load discrete prolate spheroidal sequences from database</td>
</tr>
<tr>
<td>flattopwin</td>
<td>Flat Top weighted window</td>
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<td>gausswin</td>
<td>Gaussian window</td>
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<td>Hamming window</td>
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<td>hann</td>
<td>Hann (Hanning) window</td>
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<td>kaiser</td>
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<td>nuttallwin</td>
<td>Nuttall-defined minimum 4-term Blackman-Harris window</td>
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<td>parzenwin</td>
<td>Parzen (de la Valle-Poussin) window</td>
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<td>Rectangular window</td>
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<td>sigwin</td>
<td>Signal processing window object</td>
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<td>triang</td>
<td>Triangular window</td>
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<tr>
<td>tukeywin</td>
<td>Tukey (tapered cosine) window</td>
</tr>
</tbody>
</table>
Window functions used in lab

![Graph showing window functions]

- Rectangle
- Hamming
- Chebyshev
Rectangular window (no window)

Plot of samples and the window function.

Weighted samples shown re-centered at end point splice.

dB plot of the spectrum of the windowed samples.
Hamming window

Plot of samples and the window function.

Weighted samples shown re-centered at end point splice.

dB plot of the spectrum of the windowed samples.
Chebyshev 72 dB window

Plot of samples and the window function.

Weighted samples shown re-centered at end point splice.

dB plot of the spectrum of the windowed samples.
Unmasking a low level sinewave

- Raw DFT magnitude (N=256)
- Hamming windowed DFT magnitude (N=256)
- Chebyshev windowed DFT magnitude (N=256)

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What are the downsides of windowing?

Main lobe width spreads energy of a frequency component (line) in DTFT. This causes loss of nearby resolution.

Frequency component line amplitudes are reduced.

Scalloping loss causes masking of frequency line that falls midway between adjacent lines.

May need increased numeric precision to implement a window accurately.
The short time Fourier transform (STFT)

A method for performing time varying spectral analysis with the DFT.

The STFT of a discrete time signal $x[n]$ is defined as

$$X_m(f) = \sum_{n=-\infty}^{\infty} x[n]w[n - mL]e^{-j2\pi fm}$$

Where:

- $w[n]$ is a length $N$ window function, e.g., rectangular, hanning, hamming, etc
- $L$ controls the overlap of successive windows for successive output times $m$ ($L = N$ no overlap, $L = 1$ overlap by $N - 1$ samples).
- $f$ is analysis frequency of interest

Note: $X_0(f)$ is ordinary windowed DTFT
Short time Fourier transform example

also see
http://www.ceremade.dauphine.fr/~peyre/numerical-tour/tours/audio_1_processing/.
Using DFT as a filter

Define the *sliding DFT* (identical to STFT for rectangular window and \( L = 1 \))

\[
X_n[k] = \sum_{m=0}^{N-1} x[n - m]e^{-j2\pi f_k m}, \quad f_k = k/N
\]

This produces a time varying DFT that changes over sequential samples. For a fixed value of \( k \) we can think of the sliding DFT as a filter with input \( x[n] \) and output \( y[n] \).

\[
y[n] = \sum_{m=\infty}^{\infty} h_k[m]x[n - m]
\]

where \( h_k[m] = w_N(m)e^{-j2\pi f_k m} \), \( w_N(m) \) is rectangular window

\[
\{h_k[0], \ldots, h_k[N - 1]\} = \{1, e^{-j2\pi f_k}, \ldots, e^{-j2\pi f_k(N-1)}\}
\]
Magnitude transfer function $|H_k(f)|$ of DFT bandpass filter with $k = 21$ (magnitude of DFT at bin 21).

Sliding DFT as a bank of $N$ bandpass filters with passbands at $f_k = k/N$. 
Summary of what we covered today

- Leakage
- Windowing
- Spectral estimation
References

