EECS 455 Solutions to Problem Set 1

1. Let $X(t)$ be a zero mean wide sense stationary Gaussian random process with power spectral density

$$S_X(\omega) = \begin{cases} 1\,, & |\omega| \le 5 \\ 0\,, & |\omega| > 5\,. \end{cases}$$

Let $H(\omega) = 5$ for all ω be the transfer function of a linear system with input $X(t)$ and output $Y(t)$. Find the mean and variance of the output $Y(t)$.

The mean is zero and the variance is

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} S_x(f)|H(f)|^2 df \\ &= \int_{-5/(2\pi)}^{5/(2\pi)} 25 df \\ &= 25 \times 10/(2\pi) \\ &= 125/\pi \end{aligned}$$

2. A zero mean white Gaussian random process $X(t)$ with power spectral density $N_0/2$ is the input to a linear time invariant system.

(a) The inpulse response of the linear system is

$$h(t) = A\cos(\omega_c t) p_T(t)$$

where $p_T(t)$ is 1 for $0 \le t \le T$ and is zero elsewhere. Also $\omega_c T = 2\pi n$ (an integer number of cycles in T seconds). Find the mena and variance of the output of the filter.

The mean is zero. The variance is calculated as

$$\begin{aligned} \sigma^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t) dt \\ &= \frac{N_0}{2}\int_0^T A^2\cos^2(\omega_c t) dt \\ &= \frac{N_0}{2}\int_0^T \frac{A^2}{2}[1+\cos(2\omega_c t) dt \\ &= \frac{N_0 A^2 T}{4} \end{aligned}$$

(b) If the transfer function of (a different) linear system is triangular,

$$H(f) = \begin{cases} T(fT+1), & -\frac{1}{T} < f <\le 0 \\ T(1-fT), & 0 \le f < \frac{1}{T} \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of the output of the filter.

The mean is zero. The variance is calculated as

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df \\
&= \frac{N_0}{2}\int_{-\frac{1}{T}} T^2(fT+1)^2 dt + \frac{N_0}{2}\int_0^{\frac{1}{T}} T^2(1-fT)^2 dt \\
&= \frac{N_0 T}{3}
\end{aligned}$$

3. Problem 4.9 of Text 1)

$$\begin{aligned}
x < -1 &\Rightarrow F_X(x) = 0 \\
-1 \le x \le 0 &\Rightarrow F_X(x) = \int_{-1}^{x}(v+1)dv = (\frac{1}{2}v^2+v)\Big|_{-1}^{x} = \frac{1}{2}x^2+x+\frac{1}{2} \\
0 \le x \le 1 &\Rightarrow F_X(x) = \int_{-1}^{0}(v+1)dv + \int_0^x(-v+1)dv = -\frac{1}{2}x^2+x+\frac{1}{2} \\
1 \le x &\Rightarrow F_X(x) = 1
\end{aligned}$$

2)

$$p(X > \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = 1 - \frac{7}{8} = \frac{1}{8}$$

3)

$$p(X > 0 \big| X < \frac{1}{2}) = \frac{p(X > 0,\, X < \frac{1}{2})}{p(X < \frac{1}{2})} = \frac{F_X(\frac{1}{2}) - F_X(0)}{1 - p(X > \frac{1}{2})} = \frac{3}{7}$$

4) We find first the CDF

$$F_X(x \big| X > \frac{1}{2}) = p(X \le x \big| X > \frac{1}{2}) = \frac{p(X \le x,\, X > \frac{1}{2})}{p(X > \frac{1}{2})}$$

If $x \le \frac{1}{2}$ then $p(X \le x \big| X > \frac{1}{2}) = 0$ since the events $E_1 = \{X \le \frac{1}{2}\}$ and $E_1 = \{X > \frac{1}{2}\}$ are disjoint. If $x > \frac{1}{2}$ then $p(X \le x \big| X > \frac{1}{2}) = F_X(x) - F_X(\frac{1}{2})$ so that

$$F_X(x \big| X > \frac{1}{2}) = \frac{F_X(x) - F_X(\frac{1}{2})}{1 - F_X(\frac{1}{2})}$$

Differentiating this equation with respect to x we obtain

$$f_X(x \big| X > \frac{1}{2}) = \begin{cases} \frac{f_X(x)}{1-F_X(\frac{1}{2})} & x > \frac{1}{2} \\ 0 & x \le \frac{1}{2} \end{cases}$$

5)

$$\begin{aligned}
E[X\big|X>1/2] &= \int_{-\infty}^{\infty} x f_X(x|X>1/2)dx \\
&= \frac{1}{1-F_X(1/2)}\int_{\frac{1}{2}}^{\infty} x f_X(x)dx \\
&= 8\int_{\frac{1}{2}}^{\infty} x(-x+1)dx = 8(-\frac{1}{3}x^3+\frac{1}{2}x^2)\bigg|_{\frac{1}{2}}^{1} \\
&= \frac{2}{3}
\end{aligned}$$

4. Problem 4.11 of Text 1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned}
f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|}+\frac{f_X(x_2)}{|g'(x_2)|} \\
&= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}}+\frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\
&= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}}e^{-\frac{y}{2a\sigma^2}}
\end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \le y \le b$, then for a fixed y, $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \le b < y$ for every x; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned}
f_Y(y) &= F_X(-b)\delta(y+b)+(1-F_X(b))\delta(y-b)+f_X(y)[u_{-1}(y+b)-u_{-1}(y-b)] \\
&= Q\left(\frac{b}{\sigma}\right)(\delta(y+b)+\delta(y-b))+\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{y^2}{2\sigma^2}}[u_{-1}(y+b)-u_{-1}(y-b)]
\end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned}
p(Y=b) &= p(X<0) = F_X(0) = \frac{1}{2} \\
p(Y=a) &= p(X>0) = 1-F_X(0) = \frac{1}{2}
\end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y-b) + (1 - F_X(0))\delta(y-a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$p(Y = y_n) = p(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

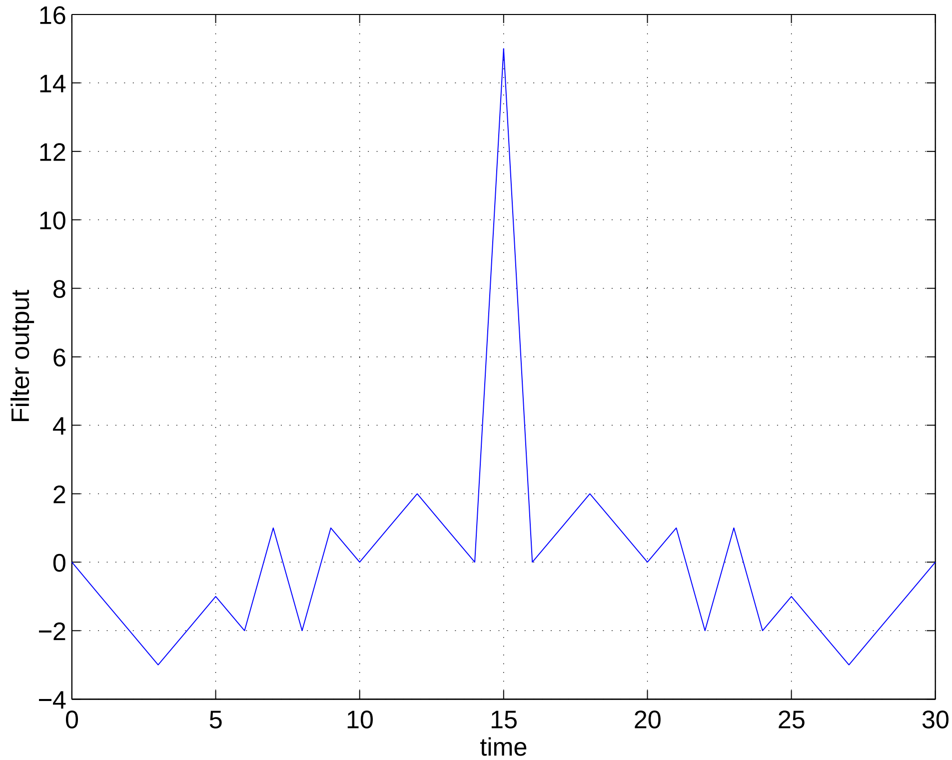
Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{N} [F_X(a_{i+1}) - F_X(a_i)]\,\delta(y - x_i) \\ &= \sum_{i=1}^{N} \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right)\right] \delta(y - x_i) \end{aligned}$$

5. A signal $s(t)$ of duration T consists of 15 consecutive pulses (of duration $T/15$) of amplitude ± 1. The sequence of amplitudes is (-1 -1 -1 -1 +1 -1 +1 -1 -1 +1 +1 -1 +1 +1 +1). Assume that this signal is input to a linear time-invariant system (filter) with impulse response $h(t) = s(T-t)$. Find (plot) the output of the filter.

The correlations for different offsets are given in the table below

l	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	15	0	1	2	1	0	1	-2	1	-2	-1	-2	-3	-2	-1	0



6. Consider the UWB channel which goes from 3.1GHz to 10.6 GHz. Suppose the noise power spectral density is $N_0 = kT = 1.38 \times 10^{-23} 290 = 4 \times 10^{-21}$ Watts/Hz. The allowed (by the FCC) transmitted power density is -41.3dBm/MHz. Suppose the received power is related to the transmitted power by

$$P_r = P_t/d^4$$

where the d is the distance in meters (independent of frequency). Compute the largest possible data rate that can be communicated reliably at a distance of 100 m and 1000 m.

Solution:

The total transmitted power is determined as follows. First determine power in a 1 MHz bandwidth. Then multiply by 7500 to get power in 7.5 GHz.

$$\begin{aligned} P_t|_{1MHz} &= 10^{(-71.3/10)} \\ &= 7.41 \times 10^{-8} W/MHz \end{aligned}$$

$$\begin{aligned} P_t &= 7.41 \times 10^{-8}(7500) \\ &= .556 \times 10^{-3} W \\ &= 556\mu\text{Watts} \end{aligned}$$

Now the power received is (for distance d) is

$$\begin{aligned} P_r &= .556 \times 10^{-3}/d^4 \\ &= \begin{cases} 5.556 \times 10^{-12} & d = 100 \\ 5.556 \times 10^{-16} & d = 1000 \end{cases} \end{aligned}$$

The capacity is then

$$\begin{aligned} C &= W \log_2(1 + \frac{P_r}{N_0 W}) \\ &= \begin{cases} 1.8\text{Gbps} & d = 100 \\ 200\text{kbps} & d = 1000 \end{cases} \end{aligned}$$