EECS 455 Solutions to Problem Set 2

1. Problem 2.1 of Text: 1)

$$\begin{aligned}
\varepsilon^2 &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} \left(x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right) \left(x^*(t) - \sum_{j=1}^{N} \alpha_j^* \phi_j^*(t) \right) dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt \\
&\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t) \phi_j^*(t) dt \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^{N} |\alpha_i|^2 - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt
\end{aligned}$$

Completing the square in terms of α_i we obtain

$$\varepsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 + \sum_{i=1}^{N} \left| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2$$

The first two terms are independent of α's and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt$$

which causes the last term to vanish.

2) With this choice of α_i's

$$\begin{aligned}
\varepsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 \\
&= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2
\end{aligned}$$

2. Problem 2-9 of Text 1) Since $(a-b)^2 \geq 0$ we have that

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

with equality if $a = b$. Let

$$A = \left[\sum_{i=1}^{n} \alpha_i^2 \right]^{\frac{1}{2}}, \qquad B = \left[\sum_{i=1}^{n} \beta_i^2 \right]^{\frac{1}{2}}$$

Then substituting α_i/A for a and β_i/B for b in the previous inequality we obtain

$$\frac{\alpha_i}{A}\frac{\beta_i}{B} \leq \frac{1}{2}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\frac{\beta_i^2}{B^2}$$

with equality if $\frac{\alpha_i}{\beta_i} = \frac{A}{B} = k$ or $\alpha_i = k\beta_i$ for all i. Summing both sides from $i = 1$ to n we obtain

$$\begin{aligned}\sum_{i=1}^{n} \frac{\alpha_i\beta_i}{AB} &\leq \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha_i^2}{A^2} + \frac{1}{2}\sum_{i=1}^{n}\frac{\beta_i^2}{B^2} \\ &= \frac{1}{2A^2}\sum_{i=1}^{n}\alpha_i^2 + \frac{1}{2B^2}\sum_{i=1}^{n}\beta_i^2 = \frac{1}{2A^2}A^2 + \frac{1}{2B^2}B^2 = 1\end{aligned}$$

Thus,

$$\frac{1}{AB}\sum_{i=1}^{n}\alpha_i\beta_i \leq 1 \Rightarrow \sum_{i=1}^{n}\alpha_i\beta_i \leq \left[\sum_{i=1}^{n}\alpha_i^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{n}\beta_i^2\right]^{\frac{1}{2}}$$

Equality holds if $\alpha_i = k\beta_i$, for $i = 1, \ldots, n$.

2) The second equation is trivial since $|x_i y_i^*| = |x_i||y_i^*|$. To see this write x_i and y_i in polar coordinates as $x_i = \rho_{x_i}e^{j\theta_{x_i}}$ and $y_i = \rho_{y_i}e^{j\theta_{y_i}}$. Then, $|x_i y_i^*| = |\rho_{x_i}\rho_{y_i}e^{j(\theta_{x_i}-\theta_{y_i})}| = \rho_{x_i}\rho_{y_i} = |x_i||y_i| = |x_i||y_i^*|$. We turn now to prove the first inequality. Let z_i be any complex with real and imaginary components $z_{i,R}$ and $z_{i,I}$ respectively. Then,

$$\begin{aligned}\left|\sum_{i=1}^{n} z_i\right|^2 &= \left|\sum_{i=1}^{n} z_{i,R} + j\sum_{i=1}^{n} z_{i,I}\right|^2 = \left(\sum_{i=1}^{n} z_{i,R}\right)^2 + \left(\sum_{i=1}^{n} z_{i,I}\right)^2 \\ &= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})\end{aligned}$$

Since $(z_{i,R}z_{m,I} - z_{m,R}z_{i,I})^2 \geq 0$ we obtain

$$(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})^2 \leq (z_{i,R}^2 + z_{i,I}^2)(z_{m,R}^2 + z_{m,I}^2)$$

Using this inequality in the previous equation we get

$$\begin{aligned}\left|\sum_{i=1}^{n} z_i\right|^2 &= \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}z_{m,R} + z_{i,I}z_{m,I}) \\ &\leq \sum_{i=1}^{n}\sum_{m=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)\left(\sum_{m=1}^{n}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\right) = \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2\end{aligned}$$

Thus

$$\left|\sum_{i=1}^{n} z_i\right|^2 \leq \left(\sum_{i=1}^{n}(z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2 \quad \text{or} \quad \left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n}|z_i|$$

The inequality now follows if we substitute $z_i = x_i y_i^*$. Equality is obtained if $\frac{z_{i,R}}{z_{i,I}} = \frac{z_{m,R}}{z_{m,I}} = k_1$ or $\angle z_i = \angle z_m = \theta$.

3) From 2) we obtain

$$\left| \sum_{i=1}^{n} x_i y_i^* \right|^2 \leq \sum_{i=1}^{n} |x_i||y_i|$$

But $|x_i|$, $|y_i|$ are real positive numbers so from 1)

$$\sum_{i=1}^{n} |x_i||y_i| \leq \left[\sum_{i=1}^{n} |x_i|^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2 \right]^{\frac{1}{2}}$$

Combining the two inequalities we get

$$\left| \sum_{i=1}^{n} x_i y_i^* \right|^2 \leq \left[\sum_{i=1}^{n} |x_i|^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2 \right]^{\frac{1}{2}}$$

From part 1) equality holds if $\alpha_i = k\beta_i$ or $|x_i| = k|y_i|$ and from part 2) $x_i y_i^* = |x_i y_i^*| e^{j\theta}$. Therefore, the two conditions are

$$\begin{cases} |x_i| = k|y_i| \\ \angle x_i - \angle y_i = \theta \end{cases}$$

which imply that for all i, $x_i = K y_i$ for some complex constant K.

3) The same procedure can be used to prove the Cauchy-Schwartz inequality for integrals. An easier approach is obtained if one considers the inequality

$$|x(t) + \alpha y(t)| \geq 0, \qquad \text{for all } \alpha$$

Then

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} |x(t) + \alpha y(t)|^2 dt = \int_{-\infty}^{\infty} (x(t) + \alpha y(t))(x^*(t) + \alpha^* y^*(t)) dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt + \alpha \int_{-\infty}^{\infty} x^*(t) y(t) dt + \alpha^* \int_{-\infty}^{\infty} x(t) y^*(t) dt + |\alpha|^2 \int_{-\infty}^{\infty} |y(t)|^2 dt \end{aligned}$$

The inequality is true for $\int_{-\infty}^{\infty} x^*(t) y(t) dt = 0$. Suppose that $\int_{-\infty}^{\infty} x^*(t) y(t) dt \neq 0$ and set

$$\alpha = -\frac{\int_{-\infty}^{\infty} |x(t)|^2 dt}{\int_{-\infty}^{\infty} x^*(t) y(t) dt}$$

Then,

$$0 \leq -\int_{-\infty}^{\infty} |x(t)|^2 dt + \frac{[\int_{-\infty}^{\infty} |x(t)|^2 dt]^2 \int_{-\infty}^{\infty} |y(t)|^2 dt}{|\int_{-\infty}^{\infty} x(t) y^*(t) dt|^2}$$

and

$$\left| \int_{-\infty}^{\infty} x(t) y^*(t) dt \right| \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |y(t)|^2 dt \right]^{\frac{1}{2}}$$

Equality holds if $x(t) = -\alpha y(t)$ a.e. for some complex α.

3. A communication system transmits one of three signals:

$$s_0(t) = A\cos\omega_c t\, p_T(t)$$

$$s_1(t) = 0$$

$$s_2(t) = -A\cos\omega_c t\, p_T(t)$$

over an additive white Gaussian noise channel with spectral density $N_0/2$. Let $r(t)$ denote the received signal ($r(t) = s_i(t) + n(t)$). The receiver computes the quantity

$$Z = \int_0^T r(t)\cos\omega_c t dt.$$

Assume $\omega_c T = 2\pi n$ for some integer n. Z is compared with a threshold γ and a threshold $-\gamma$. If $Z > \gamma$, the decision is made that $s_0(t)$ was sent. If $Z < -\gamma$, the decision is made that $s_2(t)$ was sent. If $-\gamma < Z < \gamma$ the the decision is made in favor of $s_1(t)$

(a) Determine the three conditional probabilities of error: $P_{e,0}$ = probability of error given s_0 sent, $P_{e,1}$ =probability of error given s_1 sent, and $P_{e,2}$

Solution: Assume signal 0 is transmitted. The decision variable is

$$\begin{aligned}
Z &= \int_0^T r(t)\cos\omega_c t dt. \\
Z &= \int_0^T (s_0(t) + n(t))\cos\omega_c t dt. \\
&= \int_0^T A\cos(\omega_c t)\cos(\omega_c t) dt + \eta. \\
&= \int_0^T A[1/2 + 1/2\cos(2\omega_c t)] dt + \eta. \\
&= \int_0^T A[1/2 + 1/2\cos(2\omega_c t)] dt + \eta. \\
&= AT/2 + \eta.
\end{aligned}$$

where η is a Gaussian random variable. The mean of η is zero and the variance of η is calculated as

$$\begin{aligned}
\sigma^2 = \mathrm{Var}\{\eta\} &= E[\int_0^T n(t)\cos\omega_c t dt \int_0^T n(s)\cos\omega_c s ds] \\
&= \int_0^T \int_0^T E[n(t)n(s)]\cos(\omega_c t)\cos(\omega_c s) dt ds \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\cos(\omega_c t)\cos(\omega_c s) dt ds \\
&= \int_0^T \frac{N_0}{2}\cos^2(\omega_c t) dt \\
&= \frac{N_0}{2}\int_0^T [1/2 + 1/2\cos(2\omega_c t)] dt
\end{aligned}$$

$$
\begin{aligned}
&= \frac{N_0}{2}T/2 \\
&= \frac{N_0 T}{4}
\end{aligned}
$$

The probability of error given signal 0 transmitted is then

$$
\begin{aligned}
P_{e,0} &= P\{AT/2+\eta<\gamma\} \\
&= P\{\eta<\gamma-AT/2\} \\
&= \int_{-\infty}^{\gamma-AT/2} \frac{1}{\sqrt{2\pi}\sigma} e^{-u^2/(2\sigma^2)} du \\
&= Q(\frac{AT/2-\gamma}{\sigma})
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P_{e,2} &= P\{-AT/2+\eta>\gamma\} \\
&= Q(\frac{AT/2-\gamma}{\sigma}).
\end{aligned}
$$

Finally

$$
\begin{aligned}
P_{e,1} &= 1-P\{-\gamma<\eta<\gamma\} \\
&= 1-[\Phi(\gamma/\sigma)-\Phi(-\gamma/\sigma)] \\
&= 1-Q(-\gamma/\sigma)+Q(\gamma/\sigma) \\
&= 2Q(\gamma/\sigma)
\end{aligned}
$$

(b) Determine the average error probability assuming that all three signals are equally probable of being transmitted. **Solution:** The average error probability is

$$
\bar{P_e} = \frac{1}{3}P_{e,0} + \frac{1}{3}P_{e,1} + \frac{1}{3}P_{e,2}
$$

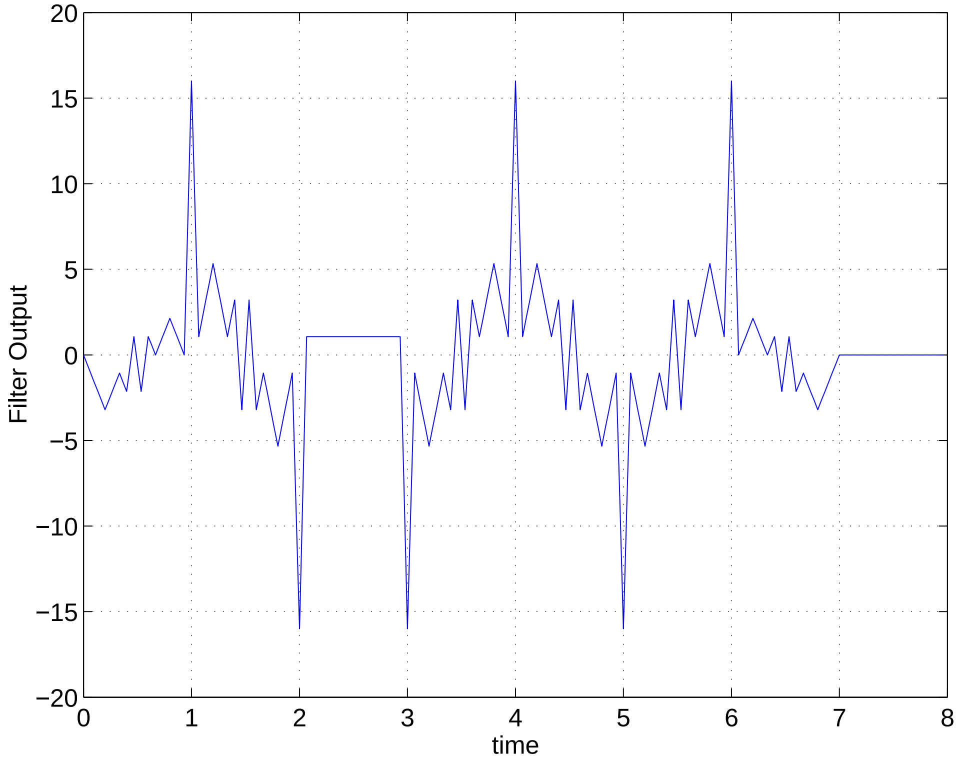
4. A signal $s(t)$ of duration T consists of 15 consecutive pulses (of duration $T/15$) of amplitude ± 1. The sequence of amplitudes is (-1 -1 -1 -1 +1 -1 +1 -1 -1 +1 +1 -1 +1 +1 +1). Data is transmitted using this signal. The data sequence 011010 is transmitted in 6T seconds by transmitting the signal $s(t)$ for the first T seconds, $-s(t-T)$ during the interval $[T,2T]$, $-s(t-2T)$ during the interval $[2T,3T]$ and so on. The total signal transmitted is thus

$$
s(t)-s(t-T)-s(t-2T)+s(t-3T)-s(t-4T)+s(t-5T)
$$

Assume that this signal is input to a linear time-invariant system (filter) with impulse response $h(t)=s(T-t)$. Find (plot) the output of the filter.

Solution: This is most easily calculated by using the output to a single T second waveform and adding delayed and multiplied version together.

The output is shown below.



5. A data signal consists of an infinite sequence of rectangular pulses of duration T. That is

$$s(t) = \sum_{l=-\infty}^{\infty} b_l p_T(t - lT)$$

where $p_T(t)$ is 1 for $0 \leq t \leq T$ and zero elsewhere. The data is represented by b_l and is either +1 or -1. The signal is filtered by a low pass RC filter with impulse response

$$h(t) = e^{-\alpha t} u(t)$$

where $u(t)$ is one for $t > 0$ and is 0 otherwise. The filter output is sampled every T seconds. Find the largest possible value (over all possible data sequences) of the sampled output and the smallest possible positive value for the sampled output.

Solution: The largest possible value is obtained when the data sequence is all ones. The filter output corresponding to that data sequence is

$$\begin{aligned} \hat{s}(T) &= \int_{-\infty}^{T} h(T-\tau) d\tau \\ &= 1 \end{aligned}$$

The smallest possible positive value is obtained when the data sequence is all negative except the last bit. The output at time 0 due to a constant negative pulse starting at time $-\infty$ and ending at time 0 is -1. At time T the output due to this pulse is $-e^{-\alpha T}$. The output at time T due to a positive pulse starting at time 0 and ending at time T is $1-e^{-\alpha T}$. The total output is then the sum which is $1-2e^{-\alpha T}$.

6. White Gaussian noise with power spectral density $N_0/2$ is the input to an RC filter with impulse response

$$h(t) = e^{-\alpha t}u(t)$$

where $u(t)$ is one for $t > 0$ and is 0 otherwise. Find the variance of the noise at the output of the filter.

Solution: The variance of the noise at the output is

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt \\
&= \frac{N_0}{2}\int_{0}^{\infty} e^{-2\alpha t}dt \\
&= \frac{N_0}{2}\left[-\frac{1}{2\alpha}e^{-2\alpha t}\right]_0^{\infty} \\
&= \frac{N_0}{4\alpha}
\end{aligned}$$