EECS 455 Solution to Problem Set 4

1. A binary communications system operates over an AWGN channel with spectral density $N_0/2$. The transmitted signals are given by

$$s_0(t) = Ap_{T/2}(t) - Ap_{T/2}(t - T/2)$$
$$s_1(t) = 0$$

   (a) Give an expression (in terms of A, T, N_0, and $Q(x)$) for the minimum average error probability when the two signals are transmitted with equal probabilities (i.e. $\pi_0 = \pi_1$).

   Solution: We are interested in the minimum error probability. The signals are given but we need to optimize over the filter and threshold. The result is obtained from Step 2 of the notes and is

$$P_e = Q(\alpha)$$

   where

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}$$

   In this case $E_0 = A^2T$, $E_1 = 0$, so $\bar{E} = A^2T/2$. Also $r = (s_0, s_1)/\bar{E} = 0$. So

$$P_e = Q(\sqrt{\frac{A^2T}{2N_0}})$$

   (b) Assume that the optimum filter $h(t)$ is used for the signals above but the signal $s_0(t)$ is actually given by

$$s_0(t) = cAp_{T/2}(t) - cAp_{T/2}(t - T/2)$$

   instead of the one given above while $s_1(t)$ is the same. Give an expression (in terms of c, A, T, N_0 and Φ) for the average error probability if $\pi_0 = \pi_1$. Note: The signal duration is $2T$ for this system.

   Solution: In this case the filter is given (possibly suboptimum) and the threshold is given (possibly subotimum). The filter is the matched filter from part a which is

$$h(t) = s_0(T-t) - s_1(T-t) = -Ap_{T/2}(t) + Ap_{T/2}(t - T/2)$$

   The error probability is

$$\bar{P_e} = \frac{1}{2}Q(\frac{\hat{s}_0(t) - \gamma}{\sigma}) + \frac{1}{2}Q(\frac{\gamma - \hat{s}_1(t)}{\sigma})$$

   where

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} |h(t)|^2 dt \\
&= \frac{N_0}{2}\int_0^T A^2 dt \\
&= \frac{A^2TN_0}{2}
\end{aligned}$$

Now we need to calculate $\hat{s_0}(T)$ and $\hat{s_1}(T)$.

$$\begin{aligned}
\hat{s}_0(t) &= \int_{-\infty}^{\infty} h(t-\tau)s_0(t)dt \\
&= \int_{-\infty}^{\infty} [Ap_{T/2}(t) - Ap_{T/2}(t-T/2)]s_0(t)dt \\
&= \int_{0}^{T/2} [Ap_{T/2}(t)cAp_{T/2}(t)dt \\
&\quad + \int_{T/2}^{T} [-Ap_{t-T/2}(t)][-cAp_{t-T/2}(t)]dt \\
&= cA^2T
\end{aligned}$$

$$\hat{s}_1(t) = 0$$

The threshold that is used (from part a) is

$$\gamma = \frac{A^2T}{2}$$

So the error probability is

$$\begin{aligned}
\bar{P_e} &= \frac{1}{2}Q(\frac{\hat{s}_0(T)-\gamma}{\sigma}) + \frac{1}{2}Q(\frac{\gamma - \hat{s}_1(T)}{\sigma}) \\
&= \frac{1}{2}Q(\frac{cA^2T - A^2T/2}{\sigma}) + \frac{1}{2}Q(\frac{A^2T/2 - 0}{\sigma}) \\
&= \frac{1}{2}Q(\frac{A^2T(c-1/2)}{\sqrt{A^2TN_0/2}}) + \frac{1}{2}Q(\frac{A^2T/2}{\sqrt{A^2TN_0/2}}) \\
&= \frac{1}{2}Q(\sqrt{\frac{2A^2T(c-1/2)^2}{N_0}}) + \frac{1}{2}Q(\sqrt{\frac{A^2T}{4N_0}})
\end{aligned}$$

2. For each signal set below, find the average error probability for binary communication via an AWGN channel (spectral density $N_0/2$). Assume for each signal that the receiver consists of an ideal matched filter, a sampler which samples at an optimal time, and a threshold device with the optimum threshold.

   (a)

$$\begin{aligned}
s_0(t) &= \begin{cases} A & 0 \le t < T/3 \\ -A & 2T/3 \le t < T \\ 0 & \text{elsewhere} \end{cases} \\
s_1(t) &= \begin{cases} A & 2T/3 \le t < T \\ -A & T/3 \le t < 2T/3 \\ 0 & \text{elsewhere} \end{cases}
\end{aligned}$$

Solution: The average error probability with the optimum filter and threshold (with equally likely signals) is just

$$\bar{P_e} = Q(\sqrt{\frac{E(1-r)}{N_0}})$$

So we only need to evaluate r and $\bar{E}$ in order to determine the error probability. In this case $r = -0.5$ and $\bar{E} = 2A^2T/3$. Thus

$$\begin{aligned}
\bar{P_e} &= Q(\sqrt{\frac{\bar{E}(1-r)}{N_0}}) \\
&= Q(\sqrt{\frac{A^2T}{N_0}})
\end{aligned}$$

(b)

$$\begin{aligned}
s_0(t) &= A|\cos\omega_0 t|p_T(t) \\
s_1(t) &= A|\sin\omega_0 t|p_T(t)
\end{aligned}$$

Solution: The correlation between signal 0 and 1 is

$$\begin{aligned}
r &= [\int_0^T s_0(t)s_1(t)dt]/(A^2T/2) \\
\int_0^T s_0(t)s_1(t)dt &= [\int_0^T A^2|\cos(\omega_0 t)||\sin(\omega_0 t)|dt] \\
&= A^2[\int_0^T |\cos(\omega_0 t)||\sin(\omega_0 t)|dt] \\
&= A^2(\frac{T}{1/4f_0})\int_0^{1/(4f_0)} \cos(2\pi f_0 t)\sin(2\pi f_0 t)dt \\
&= A^2(4Tf_0)\int_0^{1/(4f_0)} \frac{1}{2}\sin(4\pi f_0 t)dt \\
&= A^2(4Tf_0)[-\frac{1}{8\pi f_0}\cos(4\pi f_0 t)]|_0^{1/(4f_0)} \\
&= A^2(4Tf_0)[\frac{1}{8\pi f_0}] \\
&= \frac{A^2T}{2\pi} \\
r &= \frac{1}{\pi}
\end{aligned}$$

The error probability is then

$$\bar{P_e} = Q(\sqrt{\frac{\bar{E}(1-r)}{N_0}})$$

$$
\begin{aligned}
\bar{P_e} &= Q(\sqrt{\frac{.682\bar{E}}{N_0}}) \\
&= Q(\sqrt{\frac{A^2T(1-1/\pi)}{2N_0}}) \\
&= Q(\sqrt{\frac{0.341A^2T}{N_0}})
\end{aligned}
$$

(c)

$$
\begin{aligned}
s_0(t) &= A(1+\cos\omega_0 t)p_T(t) \\
s_1(t) &= A(1+\sin\omega_0 t)p_T(t)
\end{aligned}
$$

Solution: The correlation between signal 0 and 1 is The energy is

$$
\begin{aligned}
E_1 = E_0 &= \int_0^T s_0^2(t)dt \\
&= \int_0^T A^2(1+\cos(\omega_0 t))^2 dt \\
&= \int_0^T A^2(1+2\cos(\omega_0 t)+\cos^2(\omega_0 t))dt \\
&= A^2[T+\int_0^T 2\cos(\omega_0 t)+\cos^2(\omega_0 t))dt] \\
&= A^2[T+\int_0^T \frac{1}{2}(1+\cos(2\omega_0 t))dt] \\
&= A^2[T+\frac{T}{2}] \\
&= \frac{3A^2T}{2}
\end{aligned}
$$

$$
\begin{aligned}
r &= [\int_0^T s_0(t)s_1(t)dt]/(E) \\
\int_0^T s_0(t)s_1(t)dt &= \int_0^T A^2[1+\cos(\omega_0 t)][1+\sin(\omega_0 t)|dt \\
&= A^2\int_0^T [1+\cos(\omega_0 t)+\sin(\omega_0 t)+\cos(\omega_0 t)\sin(\omega_0 t)dt] \\
&= A^2[T+\int_0^T \cos(\omega_0 t)+\sin(\omega_0 t)+\cos(\omega_0 t)\sin(\omega_0 t)dt] \\
&= A^2T \\
r &= \frac{A^2T}{3A^2T/2} \\
r &= 2/3
\end{aligned}
$$

In parts (b) and (c), assume $\omega_0 T = 2\pi n$ for some integer n.

3. Consider a binary communication system that transmits one of two signal $s_0(t)$ and $s_1(t)$ over an additive white Gaussian noise channel (spectral density $N_0/2$) where

$$s_0(t) = A_0 p_{T/2}(t)$$

and

$$s_1(t) = A_1 p_{T/2}(t - T/2) \; ;$$

that is s_0 is a pulse of amplitude A_0 from 0 to $T/2$ and s_1 is a pulse of amplitude A_1 from $T/2$ to T. The receiver shown below consist of a filter $h(t)$ which is sampled at time T and a threshold device.

   (a) If $h(t) = -p_{T/2}(t) + p_{T/2}(t - T/2)$ find the threshold that will minimize the *average* of the error probability $\bar{P}_e = 0.5P_{e,0} + 0.5P_{e,1}$. Find the error average error probability.

   Solution: The output due to signal 0 is $\hat{s}_0(T) = A_0 T/2$. The output due to signal 1 is $\hat{s}_1(T) = -A_1 T/2$. The optimum threshold is

$$\begin{aligned} \gamma_{opt} &= \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2} \\ &= \frac{(A_0 - A_1)T}{4} \end{aligned}$$

   The corresponding error probability is

$$\bar{P}_e = Q(\alpha\lambda)$$

$$\begin{aligned} \lambda &= \frac{(h, s_T)}{||h||||s_T||} \\ &= \frac{(A_0 - A_1)T/2}{\sqrt{T}\sqrt{A_0^2 T/2 + A_1^2 T/2}} \\ &= \frac{(A_0 - A_1)\sqrt{2}}{\sqrt{A_0^2 + A_1^2}} \end{aligned}$$

$$\begin{aligned} \alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}} \\ &= \sqrt{\frac{T(A_0^2 + A_1^2)}{2N_0}} \end{aligned}$$

$$\begin{aligned}\alpha\lambda &= \sqrt{\frac{T(A_0^2+A_1^2)}{2N_0}}\frac{(A_0-A_1)\sqrt{2}}{\sqrt{A_0^2+A_1^2}} \\ &= \sqrt{\frac{(A_0-A_1)^2 T}{N_0}}\end{aligned}$$

So

$$\bar{P_e} = Q(\sqrt{\frac{(A_0-A_1)^2 T}{N_0}}).$$

(b) Find the matched filter for the same system and find the corresponding threshold that minimizes $P_{e,m}$. Also find $\bar{P_e}$.

Solution:

$$h_{opt} = s_0(T-t) - s_1(T-t) = -A_1 p_{T/2}(t) + A_0 p_{T/2}(t-t/2)$$

$$\gamma_{opt} = \frac{T}{4}(A_0^2 - A_1^2)$$

$$\begin{aligned}\bar{P_e} &= Q(\alpha) \\ &= Q(\sqrt{\frac{T(A_0^2+A_1^2)}{4N_0}})\end{aligned}$$

(c) If the s_0 is transmitted with probability $\pi_0 = 1/4$ and s_1 is transmitted with probability $\pi_1 = 3/4$ and the filter of part (a) is used find the threshold that minimizes the *average* error probability. What is the average error probability with this threshold?

Solution:

$$\begin{aligned}\gamma_{opt} &= \frac{\sigma_N^2 \ln\frac{\pi_1}{\pi_0}}{\hat{s}_0(T)-\hat{s}_1(T)} + \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2} \\ &= \frac{(N_0T/2)\ln(3)}{A_0T/2 - A_1T/2} + \frac{A_0T/2 + A_1T/2}{2}\end{aligned}$$

$$\bar{P_e} = \frac{1}{4}Q\left(\lambda\alpha - \frac{\beta}{\lambda}\right) + \frac{3}{4}Q\left(\lambda\alpha + \frac{\beta}{\lambda}\right)$$

$$\begin{aligned}\alpha\lambda &= \sqrt{\frac{(A_0-A_1)^2 T}{N_0}} \\ \frac{\beta}{\lambda} &= \sqrt{\frac{N_0}{(A_0-A_1)^2 T}}\ln(3)\end{aligned}$$

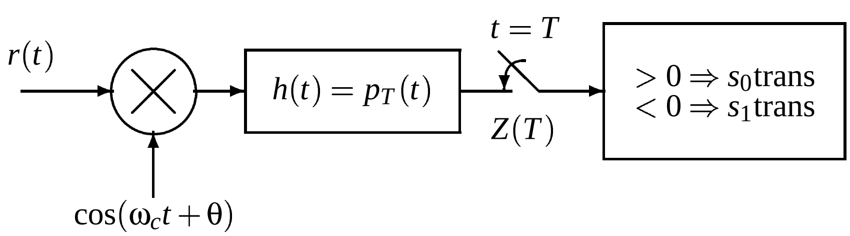
$$
\begin{aligned}
\bar{P_e} &= \frac{1}{4}Q\left(\sqrt{\frac{(A_0-A_1)^2T}{N_0}} - \sqrt{\frac{N_0}{(A_0-A_1)^2T}}\ln(3)\right) \\
&\quad + \frac{3}{4}Q\left(\sqrt{\frac{(A_0-A_1)^2T}{N_0}} + \sqrt{\frac{N_0}{(A_0-A_1)^2T}}\ln(3)\right)
\end{aligned}
$$

(d) Repeat part (c) if the matched filter is used. **Solution:**

$$
\gamma_{opt} = (A_0^2T/2 - A_1^2T/2) + \frac{N_0}{2}\ln(3)
$$

$$
\begin{aligned}
\bar{P_e} &= \frac{1}{4}Q(\alpha-\beta) + \frac{3}{4}Q(\alpha+\beta) \\
&= \frac{1}{4}Q\left(\sqrt{\frac{(A_0^2+A_1^2)T}{2N_0}} - \sqrt{\frac{2N_0}{A_0^2+A_1^2)T)}}\ln(3)\right) \\
&\quad + \frac{3}{4}Q\left(\sqrt{\frac{(A_0^2+A_1^2)T}{2N_0}} + \sqrt{\frac{2N_0}{A_0^2+A_1^2)T)}}\ln(3)\right)
\end{aligned}
$$

4. A binary communication system transmits one of two equally likely signals $s_0(t)$ and $s_1(t)$ of duration T given by
$$
s_i(t) = A(-1)^i\cos(\omega_0 t)p_T(t)
$$
The noise in the system is white Gaussian noise with power spectral density $N_0/2$. The receiver shown below is used to demodulate the signal. However, as shown, the phase of the received signal is not know completely accurately. In fact there is a discrepancy of θ radians between the received signal (in the absence of noise) and the local reference signal. Determine the error probability at the output of the demodulator as a function of θ. (Assume $\omega_c T = 2\pi n$ for some integer n).



Solution: The output of the receiver due to signal alone is

$$
\begin{aligned}
Z(T) &= \int_0^T A(-1)^i\cos(\omega t)\cos(\omega t+\theta)dt \\
&= \int_0^T A(-1)^i[1/2\cos(\theta) + 1/2\cos(2\omega t+\theta)]dt \\
&= AT(-1)^i 1/2\cos(\theta)
\end{aligned}
$$

The output due to noise η is a Gaussian random variable with mean 0 and variance

$$\begin{aligned}
\sigma^2 &= E[\eta^2] \\
&= E[\int_0^T n(t)\cos(\omega t+\theta)dt \int_0^T n(s)\cos(\omega s+\theta)ds] \\
&= \int_0^T \int_0^T E[n(t)n(s)]\cos(\omega t+\theta)\cos(\omega s+\theta)dtds] \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\cos(\omega t+\theta)\cos(\omega s+\theta)dtds] \\
&= \int_0^T \frac{N_0}{2}\cos^2(\omega t+\theta)dt \\
&= \int_0^T \frac{N_0}{2}[1/2+1/2\cos(2\omega t+2\theta)]dt \\
&= \frac{N_0 T}{4}
\end{aligned}$$

The probability of error given signal 0 transmitted is

$$\begin{aligned}
P_{e,0} &= P\{AT\cos(\theta)/2+\eta<0\} \\
&= P\{\eta < -AT\cos(\theta)/2\} \\
&= Q(\frac{AT\cos(\theta)/2}{\sigma}) \\
&= Q(\frac{AT\cos(\theta)/2}{\sqrt{N_0T/4}}) \\
&= Q(\sqrt{\frac{A^2T\cos^2(\theta)}{N_0}}) \\
&= Q(\sqrt{\frac{2E\cos^2(\theta)}{N_0}})
\end{aligned}$$

The error probability given signal 1 transmitted is identical to the error probability given signal 0 transmitted.

5. A transmitter uses one of four equally likely signals to convey two bits of information. The signals are $s_0(t), s_1(t), s_2(t)$, and $s_3(t)$. The following table indicates the mapping between information bits and signals.

Information bits	Signals
00	$s_0(t)$
01	$s_1(t)$
11	$s_2(t)$
10	$s_3(t)$

The signals are received in the presence of white Gaussian noise with power spectral density $N_0/2$. The receiver consist of a filter, a sampler and a threshold device. The sampled output is denoted by $Z(T)$. The threshold device uses the following table to make a decision.

$Z(T) > 2$	decide $s_0(t)$ transmitted
$0 < Z(T) < 2$	decide $s_1(t)$ transmitted
$-2 < Z(T) < 0$	decide $s_2(t)$ transmitted
$Z(T) < -2$	decide $s_2(t)$ transmitted

It is known that the output of the filter due to the signals alone at the sampling time is

$$\begin{aligned} \hat{s}_0(T) &= +3 \\ \hat{s}_1(T) &= +1 \\ \hat{s}_2(T) &= -1 \\ \hat{s}_3(T) &= -3 \end{aligned}$$

It is also know that the variance of the output due to noise alone is $\sigma^2 = 4$.

(a) Determine the probability of error given signal $s_i(t)$ is transmitted for $i = 0, 1, 2, 3$. (Express your answers in terms of the Q function).

Solution: Consider the case of signal 0 transmitted first. Let η be the output of the filter due to noise alone (Gaussian with mean 0 and $\sigma = 2$. Then the probability of error given signal 0 transmitted is

$$\begin{aligned} P_{e,0} &= P\{\text{error}|s_0 \text{ trans.}\} \\ &= P\{Z(T) < 2|s_0 \text{ trans.}\} \\ &= P\{3 + \eta < 2\} \\ &= P\{\eta < -1\} \\ &= \Phi(\frac{-1}{2}) \\ &= Q(\frac{1}{2}). \end{aligned}$$

For signal 1 transmitted

$$\begin{aligned} P_{e,1} &= P\{\text{error}|s_1 \text{ trans.}\} \\ &= P\{Z(T) < 0 \text{ or } Z(T) > 2|s_1 \text{ trans.}\} \\ &= P\{1 + \eta < 0\} + P\{1 + \eta > 2\} \\ &= P\{\eta < -1\} + P\{\eta > 1\} \\ &= \Phi(\frac{-1}{2}) + Q(\frac{1}{2}) \\ &= 2Q(\frac{1}{2}). \end{aligned}$$

For signal 2 transmitted

$$\begin{aligned}
P_{e,2} &= P\{\text{error}|s_2 \text{ trans.}\} \\
&= P\{Z(T) < -2 \text{ or } Z(T) > 0|s_2 \text{ trans.}\} \\
&= P\{-1+\eta < -2\} + P\{-1+\eta > 0\} \\
&= P\{\eta < -1\} + P\{\eta > 1\} \\
&= 2Q(\frac{1}{2}).
\end{aligned}$$

$$\begin{aligned}
P_{e,3} &= P\{\text{error}|s_3 \text{ trans.}\} \\
&= P\{Z(T) > -2|s_3 \text{ trans.}\} \\
&= P\{-3+\eta > -2\} \\
&= P\{\eta > 1\} \\
&= Q(\frac{1}{2}).
\end{aligned}$$

(b) Determine the probability that the receiver makes an error in the first bit given the first bit is 0. (The first bit being 0 means either signal $s_0(t)$ was transmitted or $s_1(t)$ was transmitted).

Solution: Let b_0 be the first bit. Let $\hat{b}_0$ be the decision on the first bit. Let P_e be the probabiltiy of error for the first bit. Then

$$\begin{aligned}
P_e &= P\{\hat{b}_0 = 1|b_0 = 0\} \\
&= P\{\hat{b}_0 = 1 \cap s_0 \text{ trans.}|b_0 = 0\} + P\{\hat{b}_0 = 1 \cap s_1 \text{ trans.}|b_0 = 0\} \\
&= P\{\hat{b}_0 = 1|s_0 \text{ trans.} \cap b_0 = 0\}P\{s_0 \text{ trans.}|b_0 = 0\} \\
&\quad + P\{\hat{b}_0 = 1|s_1 \text{ trans.} \cap b_0 = 0\}P\{s_1 \text{ trans.}|b_0 = 0\} \\
&= P\{Z < 0|s_0 \text{ trans.}\}P\{s_0 \text{ trans.}|b_0 = 0\} + P\{Z < 0|s_1 \text{ trans.}\}P\{s_1 \text{ trans.}|b_0 = 0\} \\
&= P\{3+\eta < 0\}\frac{1}{2} + P\{1+\eta < 0\}\frac{1}{2} \\
&= P\{\eta < -3\}\frac{1}{2} + P\{\eta < -1\}\frac{1}{2} \\
&= \frac{1}{2}Q(\frac{3}{2}) + \frac{1}{2}Q(\frac{1}{2})
\end{aligned}$$