EECS 455 Solutions to Problem Set 5

1. Consider a binary communications system with $s_i(t) = (-1)^i A p_T(t),\ \ i = 0, 1$ and an additive white Gaussian noise (AWGN) channel. Let $h(t)$ be the impulse response of the receiver filter with

$$h(t) = \frac{1}{\sqrt{2\pi}\alpha} \exp\{-t^2/2\alpha^2\}.$$

The noise process $X(t)$ has spectral density $N_0/2$.

(a) Find the optimal sampling time T_0. First recognize that the variance of the noise does not depend on the time we sample the filter but the output due to signal does depend on the time we sample the output. Notice also that the output due to $s_1(t)$ is the opposite of the output due to $s_0(t)$.

The output of the filter due to signal $s_0(t)$ is

$$\begin{aligned}
\hat{s}_0(T_0) &= \int_{-\infty}^{\infty} h(\tau) s_0(T_0 - \tau) d\tau \\
&= A \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\alpha} \exp\{-t^2/2\alpha^2\} p_T(T_0 - \tau) d\tau \\
&= A \int_{T_0 - T}^{T_0} \frac{1}{\sqrt{2\pi}\alpha} \exp\{-t^2/2\alpha^2\} \tau \\
&= A[\Phi(\frac{T_0}{\alpha}) - \Phi(\frac{T_0 - T}{\alpha})]
\end{aligned}$$

The largest possible output can be found by taking the derivative of $\hat{s}_0(T_0)$ with respect to T_0 and setting the result to 0.

$$\begin{aligned}
\frac{\partial(\hat{s}_0(T_0))}{\partial T_0} &= \frac{\partial A[\Phi(\frac{T_0}{\alpha}) - \Phi(\frac{T_0 - T}{\alpha})]}{\partial T_0} \\
&= A \frac{1}{\sqrt{2\pi}\alpha} \exp\{-T_0^2/2\alpha^2\} - A \frac{1}{\sqrt{2\pi}\alpha} \exp\{-(T_0 - T)^2/2\alpha^2\} = 0
\end{aligned}$$

Simplifying yields

$$\begin{aligned}
\exp\{-T_0^2/2\alpha^2\} &= \exp\{-(T_0 - T)^2/2\alpha^2\} \\
(T_0)^2 &= (T_0 - T)^2 \\
T_0^* &= T/2
\end{aligned}$$

The optimal sampling is thus at $T_0 = T/2$. Note that we are sampling at time $T/2$ and the signal is not over till time T. However, we have a noncausal filter.

$$\begin{aligned}
\hat{s}_0(T_0^*) &= A[\Phi(\frac{T}{2\alpha}) - \Phi(-\frac{T}{2\alpha})] \\
&= A[\Phi(\frac{T}{2\alpha}) - [1 - \Phi(\frac{T}{2\alpha})]] \\
&= A[2\Phi(\frac{T}{2\alpha}) - 1]
\end{aligned}$$

(b) Find the optimal parameter α. The variance of the the noise is calculated to be

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty} \frac{1}{2\pi\alpha^2}\exp\{-t^2/\alpha^2\}dt \\
&= \frac{N_0}{2}\frac{1}{\sqrt{2\pi}\alpha\sqrt{2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}(\alpha/\sqrt{2})}\exp\{-\frac{t^2}{2(\alpha/\sqrt{2})^2}\}dt \\
&= \frac{N_0}{4\sqrt{\pi}\alpha}
\end{aligned}$$

The probability of error is thus

$$P_e = Q(\frac{\hat{s}_0(T_0^*)}{\sigma})$$

$$\begin{aligned}
\frac{\hat{s}_0(T_0^*)}{\sigma} &= \frac{A[2\Phi(\frac{T}{2\alpha})-1]}{\sqrt{\frac{N_0}{4\sqrt{\pi}\alpha}}} \\
&= \frac{2A\sqrt{\sqrt{\pi}\alpha}[2\Phi(\frac{T}{2\alpha})-1]}{\sqrt{N_0}}
\end{aligned}$$

Now let $x = T/(2\alpha)$.

$$\frac{\hat{s}_0(T_0^*)}{\sigma} = \sqrt{\frac{2A^2T}{N_0}}(\pi)^{1/4}\frac{[2\Phi(x)-1]}{\sqrt{x}}$$

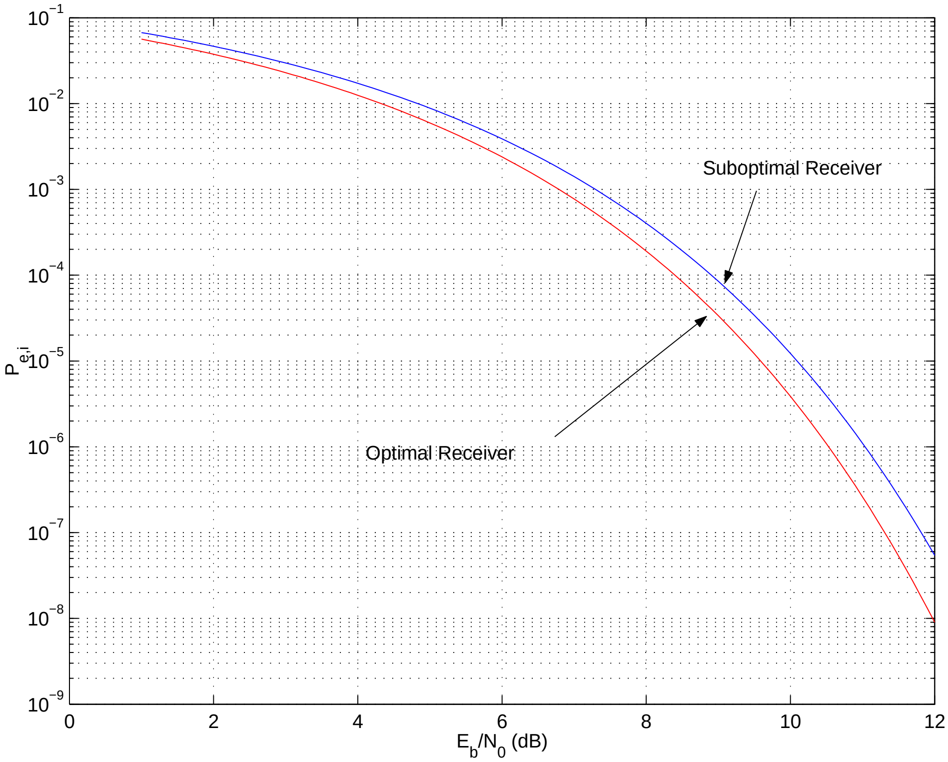
The maximum with respect to x of $(2\Phi(x)-1)/\sqrt{x}$ occurs at $x = 1.40$ and takes the value .70865. Thus the optimal α is $\alpha = T/(2.8)$. The signal-to-noise ratio for the optimal α is

$$\begin{aligned}
\max_{\alpha}\frac{\hat{s}_0(T_0^*)}{\sigma} &= \sqrt{\frac{2A^2T}{N_0}}(\pi)^{1/4}0.70865 \\
&= \sqrt{\frac{2E(0.8901)}{N_0}}
\end{aligned}$$

(c) Find $P_{e,i}$ for E/N_0= 8,9,10,11,12 dB.

E/N_0	$P_{e,i}$	$P_{e,i}$ (optimal filter)
8.0	4.02×10^{-4}	1.91×10^{-4}
9.0	8.48×10^{-5}	3.36×10^{-5}
10.0	1.23×10^{-5}	3.87×10^{-6}
11.0	1.10×10^{-6}	2.61×10^{-7}
12.0	5.43×10^{-8}	9.01×10^{-9}

(d) Compare your answer of part (c) to the error probability for the optimal receiver. The suboptimal receiver is worse by $10\log_{10}(0.89) = 0.5dB$. That is, the suboptimal receiver requires 0.5dB more energy transmitted for the same performance.



2. (a) The energy of each signal as a function of y is given below.

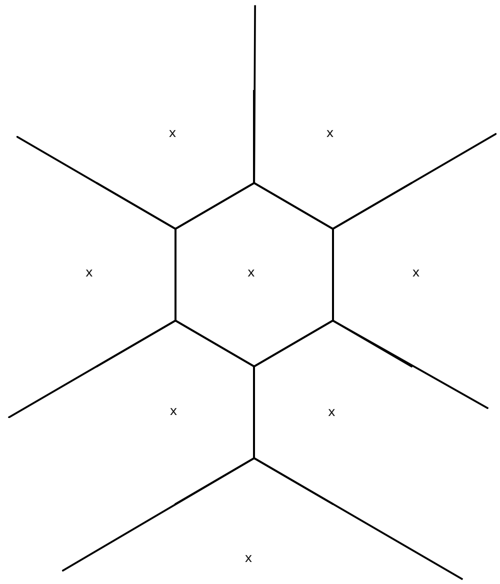
$$\begin{aligned}
E_1 &= E(1+(y+\sqrt{3})^2) \\
E_2 &= E(1+(y+\sqrt{3})^2) \\
E_3 &= E(4+y^2) \\
E_4 &= E(y^2) \\
E_5 &= E(4+y^2) \\
E_6 &= E(1+(y-\sqrt{3})^2) \\
E_6 &= E(1+(y-\sqrt{3})^2) \\
E_6 &= E(y-2\sqrt{3})^2)
\end{aligned}$$

The average energy $\bar{E}$ is

$$\bar{E} = \frac{E}{8}\left[2(1+(y+\sqrt{3})^2)+2(4+y^2)+y^2+2(1+(y-\sqrt{3})^2)+(y-2\sqrt{3})^2)\right]$$

To minimize we take the derivative with respect to y.

$$\frac{\partial \bar{E}}{\partial y} = \frac{E}{8}\left[4(y+\sqrt{3})+4y+2y+4(y-\sqrt{3})+2(y-2\sqrt{3})\right]$$



$$= \frac{E}{8}\left[16y - 4\sqrt{3}\right] = 0$$

Thus the optimal value for y is $\sqrt{3}/4$ which results in average transmitted energy of

$$\begin{aligned}
\bar{E} &= \frac{E}{8}\left[2(1+(\frac{5}{4}\sqrt{3})^2)+2(4+\frac{3}{16})+\frac{3}{16}+2(1+(\frac{3}{4}\sqrt{3})^2)+(\frac{7}{4}\sqrt{3})^2)\right] \\
&= \frac{E}{8}\left[2(1+\frac{75}{16})+2(4+\frac{3}{16})+\frac{3}{16}+2(1+\frac{27}{16})+(\frac{147}{16})\right] \\
&= \frac{E}{8}\left[\frac{182}{16}+\frac{134}{16}+\frac{3}{16}+\frac{86}{16}+\frac{147}{16}\right] \\
&= \frac{E}{8}\left[\frac{552}{16}\right] \\
&= 4.3125E.
\end{aligned}$$

The optimal decision regions are shown in the figure below.

3. The simulation for this signal set is shown below. The bound is

$$\begin{aligned}
P_e,s &\leq \sum_{l=1}^{8}\sum_{m=1,m\neq l}^{8}\frac{1}{8}P\{s_m- > s_l|s_m\} \\
&= \sum_{l=1}^{8}\sum_{m=1,m\neq l}^{8}\frac{1}{8}Q(\frac{d_E(s_m,s_l)}{2\sigma}) \\
&= \sum_{l=1}^{8}\sum_{m=1,m\neq l}^{8}\frac{1}{8}Q(\frac{d_E(s_m,s_l)}{2\sigma})
\end{aligned}$$

The squared distances are given in the table.

-	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	0	4	4	4	12	12	16	28
s_2	4	0	4	12	4	4	16	28
s_3	4	12	0	4	16	4	12	16
s_4	4	4	4	0	4	4	4	12
s_5	12	4	16	4	0	12	4	16
s_6	12	16	4	4	12	0	4	4
s_7	16	12	12	4	4	4	0	4
s_8	28	28	16	12	16	4	4	0

Using this and combining terms yields

$$P_{e,s} \leq \frac{30}{8}Q(\sqrt{\frac{4}{2N_0}}) + \frac{12}{8}Q(\sqrt{\frac{16}{2N_0}}) + \frac{4}{8}Q(\sqrt{\frac{28}{2N_0}})$$

Since the energy for the signal set is $E = 4.3125$ the energy per bit is $E_b = E/3 = 4.3125/3$.

$$P_{e,s} \leq \frac{30}{8}Q(\sqrt{\frac{6E_b}{4.3125N_0}}) + \frac{12}{8}Q(\sqrt{\frac{24E_b}{4.3125N_0}}) + \frac{4}{8}Q(\sqrt{\frac{42E_b}{4.3125N_0}})$$

Code for Simualtion and bound

```
clear all

Ns=input('Number of symbols per packet= ');
ncount=input('Number of errors = ');

y=sqrt(3)/4;
s(1)=(-1+j* (y+sqrt(3)));
s(2)=(1+j*(y+sqrt(3)));
s(3)=(-2+j*y);
s(4)=(0+j*y);
s(5)=(2+j*(y));
s(6)=(-1+j*(y-sqrt(3)));
s(7)=(1+j*(y-sqrt(3)));
s(8)=(0+j*(y-2*sqrt(3)));
s=s.';
E=4.3125

for m=1:11
EbN0dB(m)=m-1;
EbN0(m)=10.^(EbN0dB(m)/10);
Eb=E/3;
N0=Eb/EbN0(m);
sigma=sqrt(N0/2);
```

```
nerrors=0;
nbsim=0;
while(nerrors < ncount)


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                        %
%                  Generate data                         %
%                                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

b=round(rand(3,Ns));
l=[4 2 1]*b+1;
strans=s(l).';

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                        %
%                  Add Noise                             %
%                                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
n=sigma*(randn(1,Ns)+j*randn(1,Ns));
      r=strans+n;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                        %
%           Demodulate the received signal               %
%                                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        rdemod=ones(1,8)'*r;
d=abs(rdemod-s*ones(1,Ns));
[y,symbol]=min(d);
bhat0=(symbol==5)|(symbol==6)|(symbol==7)|(symbol==8);
bhat1=(symbol==3)|(symbol==4)|(symbol==7)|(symbol==8);
bhat2=(symbol==2)|(symbol==4)|(symbol==6)|(symbol==8);
bhat=[bhat0; bhat1; bhat2];


   nerrors=nerrors+sum(sum(abs(sign(bhat-b))))
   nbsim=nbsim+Ns*3;
   end

peb(m)=nerrors/nbsim
end
figure(3)
semilogy(EbN0dB,peb,'r')
grid on
axis([0 10 0.00001 1])


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%                                                        %
%           Now calculate the union                      %
%                                                        %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
sn=s*ones(1,8);
y=sn.'
d2=abs(sn-y).^2
for m=1:11
EbN0dB(m)=m-1;
EbN0(m)=10.^(EbN0dB(m)/10);
E=4.3125;
Eb=E/3;
N0=Eb/EbN0(m);
sigma=sqrt(N0/2);

Pebound(m)=0.0;
for l=1:8
for mm=1:8
if (mm~=l)
Pebound(m)=Pebound(m)+q(sqrt(d2(mm,l))/(2*sigma));
end;
   end
end
Pebound(m)=Pebound(m)/8;
end
hold on
semilogy(EbN0dB,Pebound,'g')
```