EECS 455: Solutions to Problem Set 7

1. Problem 7.37 of text **1)** If the received signal is

$$r(t) = \pm g_T(t)\cos(2\pi f_c t + \phi) + n(t)$$

then by crosscorrelating with the signal at the output of the PLL

$$\psi(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g_t(t)\cos(2\pi f_c t + \hat{\phi})$$

we obtain

$$\begin{aligned}
\int_0^T r(t)\psi(t)dt &= \pm\sqrt{\frac{2}{\mathcal{E}_g}}\int_0^T g_T^2(t)\cos(2\pi f_c t + \phi)\cos(2\pi f_c t + \hat{\phi})dt \\
&\quad + \int_0^T n(t)\sqrt{\frac{2}{\mathcal{E}_g}} g_t(t)\cos(2\pi f_c t + \hat{\phi})dt \\
&= \pm\sqrt{\frac{2}{\mathcal{E}_g}}\int_0^T \frac{g_T^2(t)}{2}\left(\cos(2\pi 2 f_c t + \phi + \hat{\phi}) + \cos(\phi - \hat{\phi})\right)dt + n \\
&= \pm\sqrt{\frac{\mathcal{E}_g}{2}}\cos(\phi - \hat{\phi}) + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. If we assume that the signal $s_1(t) = g_T(t)\cos(2\pi f_c t + \phi)$ was transmitted, then the probability of error is

$$\begin{aligned}
P(\text{error}|s_1(t)) &= P\left(\sqrt{\frac{\mathcal{E}_g}{2}}\cos(\phi - \hat{\phi}) + n < 0\right) \\
&= Q\left[\sqrt{\frac{\mathcal{E}_g\cos^2(\phi - \hat{\phi})}{N_0}}\right] = Q\left[\sqrt{\frac{2\mathcal{E}_s\cos^2(\phi - \hat{\phi})}{N_0}}\right]
\end{aligned}$$

where $\mathcal{E}_s = \mathcal{E}_g/2$ is the energy of the transmitted signal. As it is observed the phase error $\phi - \hat{\phi}$ reduces the SNR by a factor

$$\text{SNR}_{\text{loss}} = -10\log_{10}\cos^2(\phi - \hat{\phi})$$

2) When $\phi - \hat{\phi} = 45^o$, then the loss due to the phase error is

$$\text{SNR}_{\text{loss}} = -10\log_{10}\cos^2(45^o) = -10\log_{10}\frac{1}{2} = 3.01 \text{ dB}$$

2. Problem 7.51 of text **1)** If the transmitted signal is

$$u_0(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t), \qquad 0 \le t \le T$$

then the received signal is

$$r(t) = \sqrt{\frac{2\mathcal{E}_s}{T}} \cos(2\pi f_c t + \phi) + n(t)$$

In the phase-coherent demodulation of M-ary FSK signals, the received signal is correlated with each of the M-possible received signals $\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)$, where $\hat{\phi}_m$ are the carrier phase estimates. The output of the m^{th} correlator is

$$\begin{aligned}
r_m &= \int_0^T r(t)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&= \int_0^T \sqrt{\frac{2\mathcal{E}_s}{T}} \cos(2\pi f_c t + \phi)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&\quad + \int_0^T n(t)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&= \sqrt{\frac{2\mathcal{E}_s}{T}} \int_0^T \frac{1}{2}\left(\cos(2\pi 2 f_c t + 2\pi m\Delta f t + \hat{\phi}_m + \phi) + \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)\right) + n \\
&= \sqrt{\frac{2\mathcal{E}_s}{T}} \frac{1}{2} \int_0^T \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)dt + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

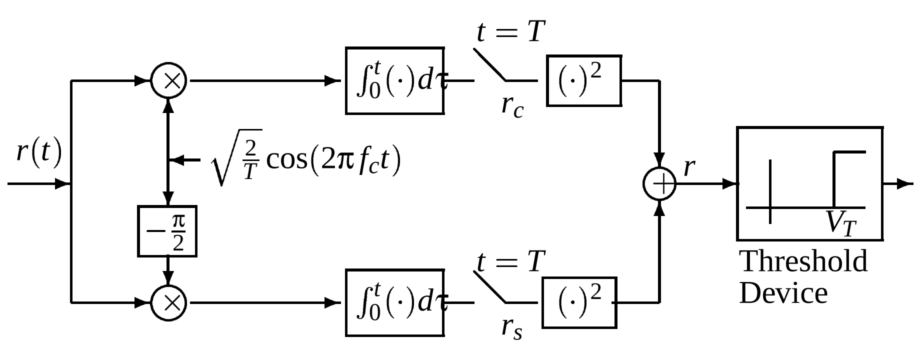
2) In order to obtain orthogonal signals at the demodulator, the expected value of r_m, $E[r_m]$, should be equal to zero for every $m \neq 0$. Since $E[n] = 0$, the latter implies that

$$\int_0^T \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)dt = 0, \qquad \forall m \neq 0$$

The equality is true when $m\Delta f$ is a multiple of $\frac{1}{T}$. Since the smallest value of m is 1, the necessary condition for orthogonality is

$$\Delta f = \frac{1}{T}$$

3. Problem 7.53 of text **1)** The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



2) If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c, r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by

$$p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned}
r_c &= \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\
&= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b} \cos(2\pi f_c t + \phi) \cos(2\pi f_c t) dt + \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\
&= \frac{2\sqrt{\mathcal{E}_b}}{T_b} \int_0^T \frac{1}{2} \left(\cos(2\pi 2 f_c t + \phi) + \cos(\phi)\right) dt + n_c \\
&= \sqrt{\mathcal{E}_b} \cos(\phi) + n_c
\end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have

$$r_s = \sqrt{\mathcal{E}_b} \sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ is (see Problem 4.31)

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2+\mathcal{E}_b}{N_0}} I_0\left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0}\right)$$

that is a Rician PDF.

3) For equiprobable signals the probability of error is given by

$$P(\text{error}) = \frac{1}{2} \int_{-\infty}^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned}
P(\text{error}) &= \frac{1}{2} \int_0^{V_T} p(r|s_1(t)) dr + \frac{1}{2} \int_{V_T}^{\infty} p(r|s_0(t)) dr \\
&= \frac{1}{2} \int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr + \frac{1}{2} \int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr
\end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x, that is for high SNR. In this case

$$\frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr \approx \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore,

$$\begin{aligned}
\frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2}\int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\
&= \frac{1}{2}Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right]
\end{aligned}$$

Finally

$$\begin{aligned}
P(\text{error}) &= \frac{1}{2}Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}\int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\
&\leq \frac{1}{2}Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}e^{-\frac{\mathcal{E}_b}{4N_0}}
\end{aligned}$$

4. Problem 7.54 of text (a) Four phase PSK
   If we use a pulse shape having a raised cosine spectrum with a rolloff α, the symbol rate is determined from the relation

$$\frac{1}{2T}(1+\alpha) = 50000$$

   Hence,

$$\frac{1}{T} = \frac{10^5}{1+\alpha}$$

   where $W = 10^5$ Hz is the channel bandwidth. The bit rate is

$$\frac{2}{T} = \frac{2\times 10^5}{1+\alpha} \quad \text{bps}$$

(b) Binary FSK with noncoherent detection
In this case we select the two frequencies to have a frequency separation of $\frac{1}{T}$, where $\frac{1}{T}$ is the symbol rate. Hence

$$f_1 = f_c - \frac{1}{2T}$$

$$f_2 = f + c + \frac{1}{2T}$$

where f_c is the carrier in the center of the channel band. Thus, we have

$$\frac{1}{2T} = 50000$$

or equivalently

$$\frac{1}{T} = 10^5$$

Hence, the bit rate is 10^5 bps.
(c) $M = 4$ FSK with noncoherent detection
In this case we require four frequencies with adjacent frequencies separation of $\frac{1}{T}$. Hence, we select

$$f_1 = f_c - |\frac{1.5}{T}, \quad f_2 = f_c - \frac{1}{2T}, \quad f_3 = f_c + \frac{1}{2T}, \quad f_4 = f_c + \frac{1.5}{T}$$

where f_c is the carrier frequency and $\frac{1}{2T} = 25000$, or, equivalently,

$$\frac{1}{T} = 50000$$

Since the symbol rate is 50000 symbols per second and each symbol conveys 2 bits, the bit rate is 10^5 bps.