EECS 455 Solutions to Problem Set 8

1. Problem 7.55 of text

   a) For n repeaters in cascade, the probability of i out of n repeaters to produce an error is given by the binomial distribution

$$P_i = \binom{n}{i} p^i (1-p)^{n-i}$$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i}$$

Let P_{even} be the probability that an even number of repeaters produces an error. Then

$$P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore,

$$P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p+1-p)^n = 1$$

One more relation between P_{even} and P_{odd} can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$\begin{aligned}
P_{\text{even}} - P_{\text{odd}} &= \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} \\
&\stackrel{a}{=} \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} \\
&= (1-p-p)^n = (1-2p)^n
\end{aligned}$$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for i even and -1 when i is odd. Solving the system

$$\begin{aligned}
P_{\text{even}} + P_{\text{odd}} &= 1 \\
P_{\text{even}} - P_{\text{odd}} &= (1-2p)^n
\end{aligned}$$

we obtain

$$P_n = P_{\text{odd}} = \frac{1}{2}(1-(1-2p)^n)$$

b) Expanding the quantity $(1-2p)^n$, we obtain

$$(1-2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \cdots$$

Since, $p \ll 1$ we can ignore all the powers of p which are greater than one. Hence,

$$P_n \approx \frac{1}{2}(1-1+n2p) = np = 100 \times 10^{-6} = 10^{-4}$$

2. Problem 7.56 of text

   The overall probability of error is approximated by

$$P(e) = KQ\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]$$

   Thus, with $P(e) = 10^{-6}$ and $K = 100$, we obtain the probability of each repeater $P_r = Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right] = 10^{-8}$. The argument of the function $Q[\cdot]$ that provides a value of 10^{-8} is found from tables to be

$$\sqrt{\frac{\mathcal{E}_b}{N_0}} = 5.61$$

   Hence, the required $\frac{\mathcal{E}_b}{N_0}$ is $5.61^2 = 31.47$

3. Problem 7.57 of text

   a) The antenna gain for a parabolic antenna of diameter D is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

   If we assume that the efficiency factor is 0.5, then with

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^9} = 0.3 \text{ m} \qquad D = 3\times 0.3048 \text{ m}$$

   we obtain

$$G_R = G_T = 45.8458 = 16.61 \text{ dB}$$

   b) The effective radiated power is

$$\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}$$

   c) The received power is

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995\times 10^{-9} = -85.23 \text{ dB } = -55.23 \text{ dBm}$$

   Note that

$$\text{dBm } = 10\log_{10}\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10\log_{10}(\text{power in Watts })$$

4. Problem 7.58 of text

   a) The antenna gain for a parabolic antenna of diameter D is

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2$$

   If we assume that the efficiency factor is 0.5, then with

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^9} = 0.3 \text{ m} \qquad \text{and} \qquad D = 1 \text{ m}$$

   we obtain

$$G_R = G_T = 54.83 = 17.39 \text{ dB}$$

   b) The effective radiated power is

$$\text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \text{ dB}$$

   c) The received power is

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \text{ dB } = -67.20 \text{ dBm}$$

5. Problem 7.59 of text

   The wavelength of the transmitted signal is

$$\lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m}$$

   The gain of the parabolic antenna is

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.6 \left(\frac{\pi 10}{0.03}\right)^2 = 6.58 \times 10^5 = 58.18 \text{ dB}$$

   The received power at the output of the receiver antenna is

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{3 \times 10^{1.5} \times 6.58 \times 10^5}{(4 \times 3.14159 \times \frac{4 \times 10^7}{0.03})^2} = 2.22 \times 10^{-13} = -126.53 \text{ dB}$$

6. Problem 7.60 of text

   a) Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.5 \left(\frac{3.14159 \times 50}{\frac{3\times 10^8}{2\times 10^9}}\right)^2 = 5.483 \times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is

$$P_R = \frac{P_T G_T G_R}{(4\pi\frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^8}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

b) If $\frac{\mathcal{E}_b}{N_0} = 10 \text{ dB } = 10$, then

$$R = \frac{P_R}{N_0}\left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

7. Problem 7.62 of text

The wavelength of the transmission is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$

If 1 MHz is the passband bandwidth, then the rate of binary transmission is $R_b = W = 10^6$ bps. Hence, with $N_0 = 4.1 \times 10^{-21}$ W/Hz we obtain

$$\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \Longrightarrow 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}$$

The transmitted power is related to the received power through the relation

$$P_R = \frac{P_T G_T G_R}{(4\pi\frac{d}{\lambda})^2} \Longrightarrow P_T = \frac{P_R}{G_T G_R}\left(4\pi\frac{d}{\lambda}\right)^2$$

Substituting in this expression the values $G_T = 10^{0.6}$, $G_R = 10^5$, $d = 36 \times 10^6$ and $\lambda = 0.75$ we obtain

$$P_T = 0.1185 = -9.26 \text{ dBW}$$

Alternate Solution. Since it mentions reliable communication we assume that the data rate is the “capacity achieving” data rate. So (see lecture notes 1)

$$R/W = \log 2(1 + \frac{E_b}{N_0}\frac{R}{W})$$

Since $E_b/N_0 = 15$ dB or $E_b/N_0 = 31.62$ This correspondes to $R/W = 8$. So R=8Mbps. Now using $R = 8$ Mbps and $N_0 = 4.1 \times 10^{-21}$ we get

$$\begin{aligned} \frac{P_r/R}{N_0} &= 31.62 \\ P_r &= 31.62(RN_0) \\ P_r &= 31.62(32 \times 10^{-15}) \\ P_r &= 1.01 \times 10^{-12} \end{aligned}$$

$$
\begin{aligned}
P_T &= \frac{P_R}{G_T G_R}\left(4\pi\frac{d}{\lambda}\right)^2 \\
&= \frac{10^{-12}}{3}\left(4\pi\frac{36\times 10^6}{.75}\right)^2 \\
&= 117dB = 500GWatts
\end{aligned}
$$

This is obviously not a possible solution. However, from the problem statement this is a reasonable approach to take to solve the problem. However modulation techniques transmitting 8Mbps in a 1 MHz bandwidth would require a very linear (and thus very inefficient) amplifier. So the restriction that the communication use BPSK with data rate =bandwidth is resonable.

8. Problem 7.63 of text

   Since $T = 290^0 + 15^0 = 305^0 K$, it follows that

$$N_0 = kT = 1.38\times 10^{-23}\times 305 = 4.21\times 10^{-21}\ \text{W/Hz}$$

   The transmitting wavelength λ is

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{2.3\times 10^9} = 0.130\ \text{m}$$

   Hence, the gain of the receiving antenna is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2 = 0.55\left(\frac{3.14159\times 64}{0.130}\right)^2 = 1.3156\times 10^6 = 61.19\ \text{dB}$$

   and therefore, the received power level is

$$P_R = \frac{P_T G_T G_R}{(4\pi\frac{d}{\lambda})^2} = \frac{17\times 10^{2.7}\times 1.3156\times 10^6}{(4\times 3.14159\times \frac{1.6\times 10^{11}}{0.130})^2} = 4.686\times 10^{-12} = -113.29\ \text{dB}$$

   If $\mathcal{E}_b/N_0 = 6$ dB $= 10^{0.6}$, then

$$R = \frac{P_R}{N_0}\left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{4.686\times 10^{-12}}{4.21\times 10^{-21}}\times 10^{-0.6} = 4.4312\times 10^9 = 4.4312\ \text{Gbits/sec}$$