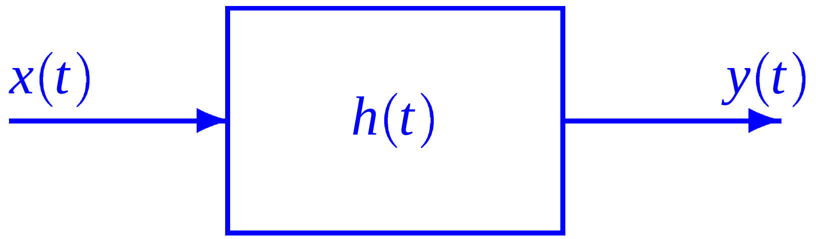
Lecture 2

Goals:

- Be able to calculate the output of a linear system (convolution) when the input is a specified function of time (or frequency).
- Be able to work in both time domain and frequency domain.
- Be able to determine the noise variance out of a linear system.

Filtering, Convolution, Correlation and Noise

In most receivers in a digital communication system the received signal is filtered before a decision is made as to the data bit that is transmitted. The purpose of filtering is to remove as much of the noise as possible without removing any of the signal.



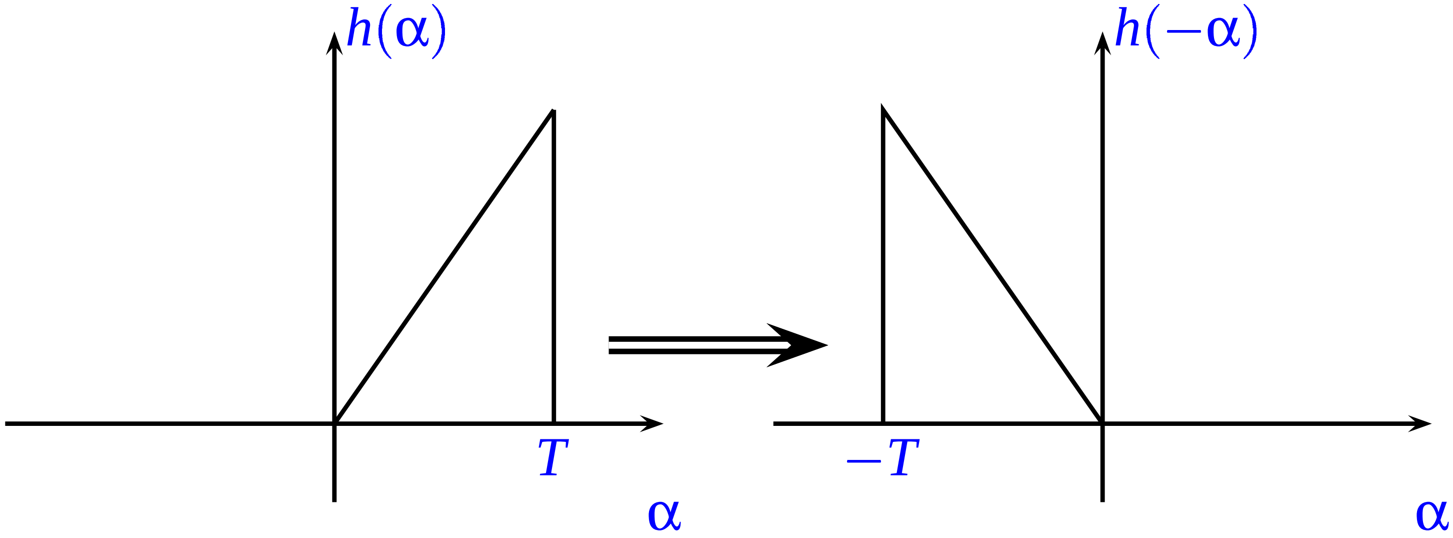
Convolution

Mathematically, filtering is the convolution of the input signal to the filter and the impulse response of the filter. That is, if the input to the filter is the signal $x(t)$ and the impulse response of the filter is $h(t)$ the output of the filter $y(t)$ is given by

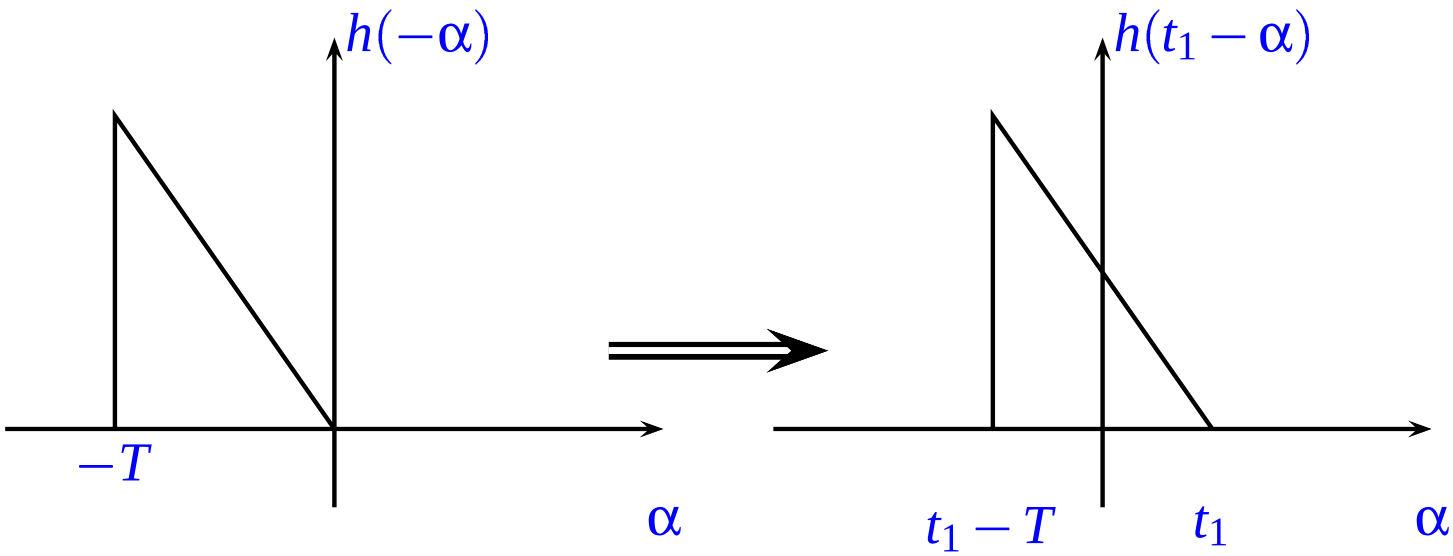
$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(t-\alpha)h(\alpha)d\alpha \\ &= \int_{-\infty}^{\infty} h(t-\alpha)x(\alpha)d\alpha \end{aligned}$$

The above mathematical operation on $x(t)$ and $h(t)$ is called convolution of h with x.

The convolution operation is best understood graphically. Consider the output of the convolution at time $t = t_1$. First the function $h(\alpha)$ is flipped right to left to yield $h(-\alpha)$.

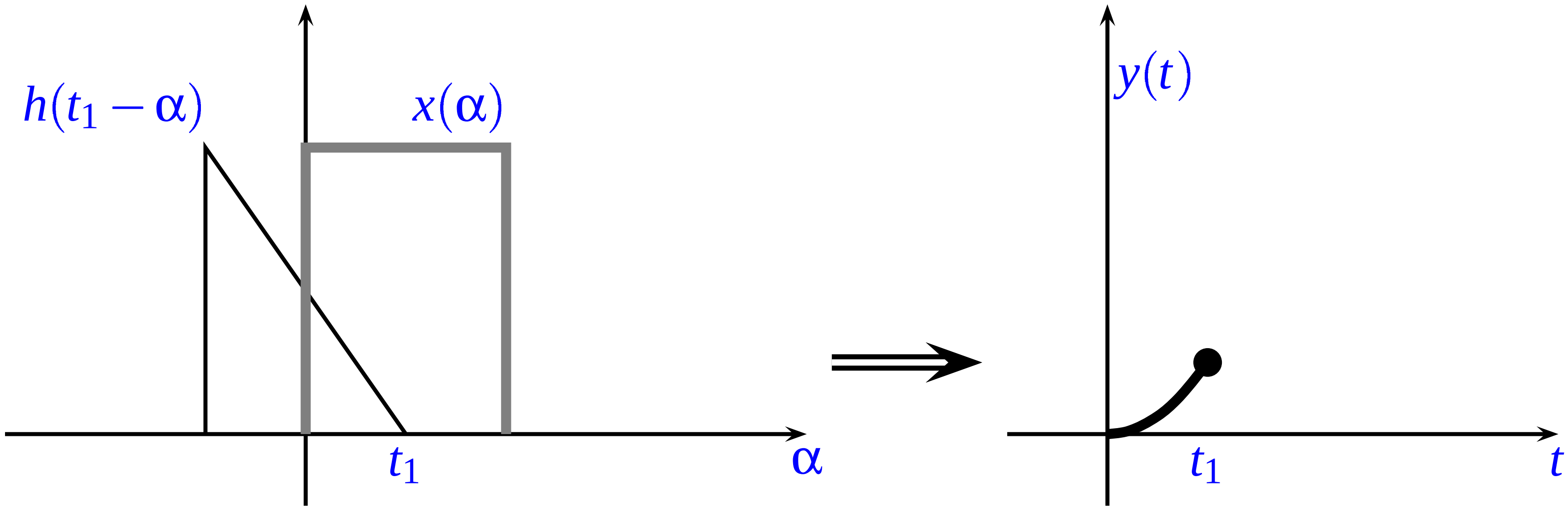


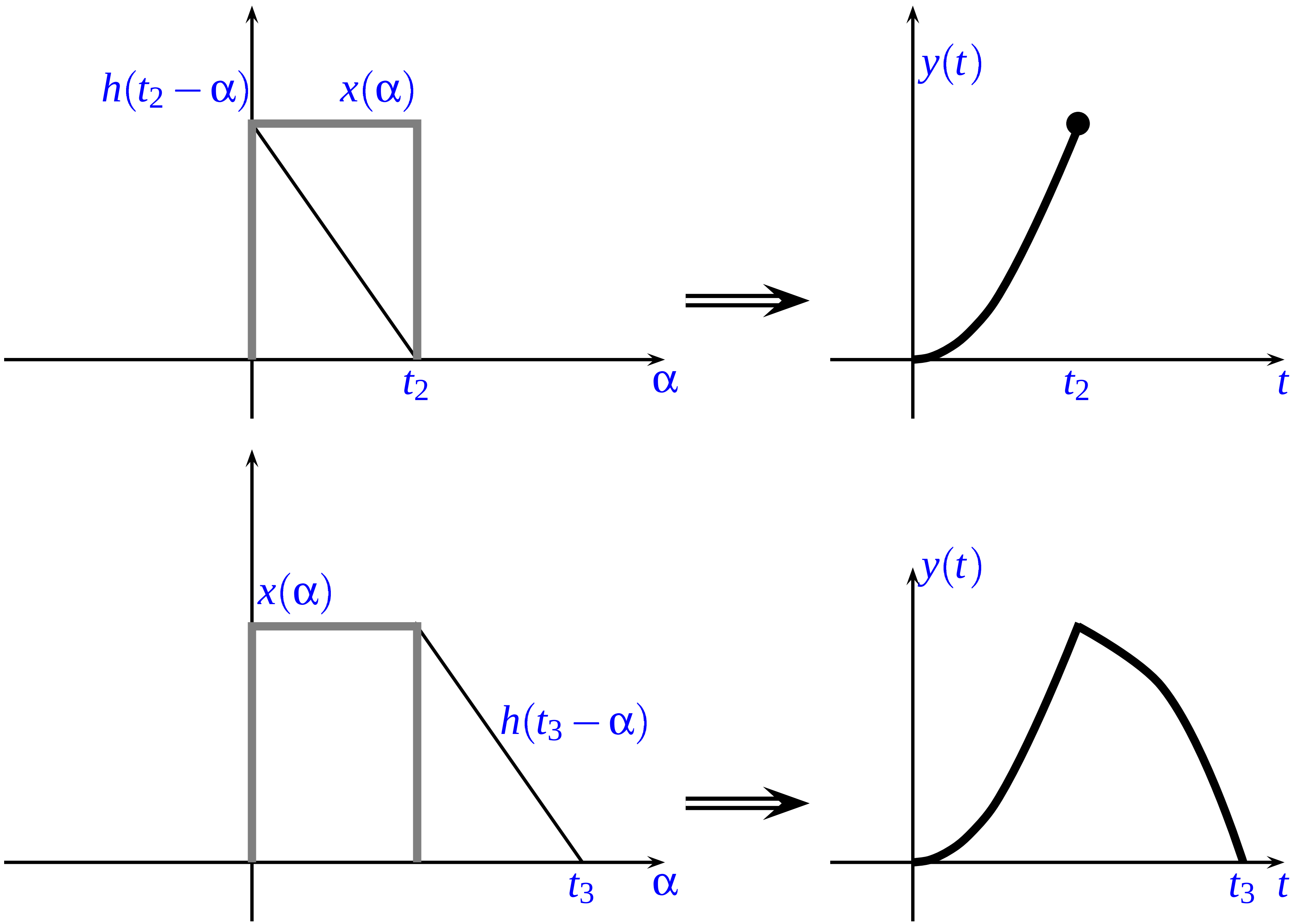
Second the function $h(-\alpha)$ is shifted to the right by t_1 seconds



$h(-\alpha)$
$-T$
α
$h(t_1 - \alpha)$
$t_1 - T$
t_1
α

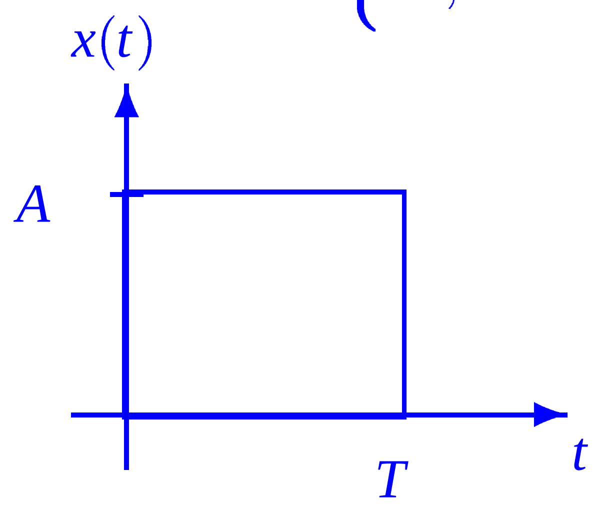
Third the flipped, shifted function h is correlated with the input x.



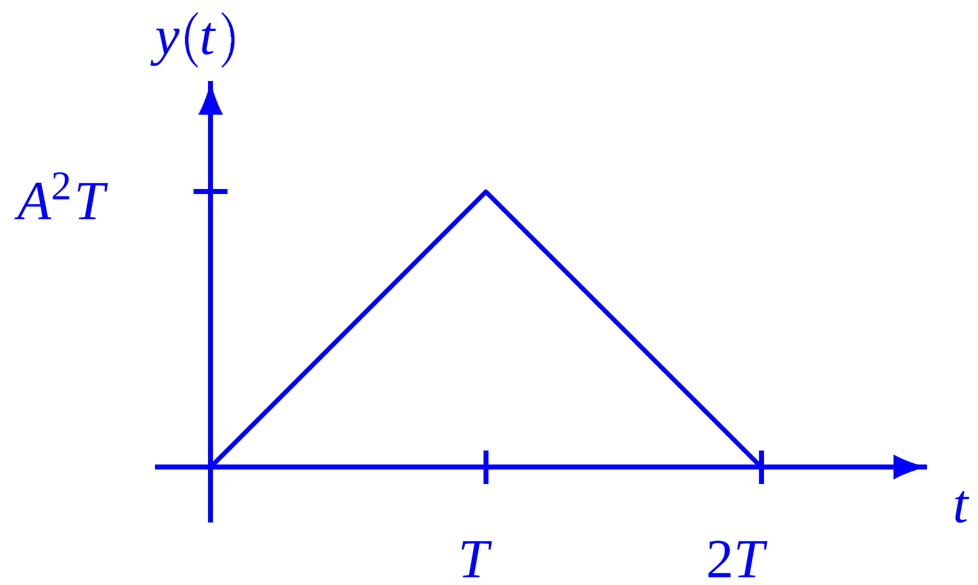

$h(t_2 - \alpha)$
$x(\alpha)$
t_2
α
$y(t)$
t_2
t
$x(\alpha)$
$h(t_3 - \alpha)$
t_3
α
$y(t)$
t_3
t

For example if $x(t)$ and $h(t)$ are rectangular pulses of amplitude A and duration T beginning at $t = 0$.

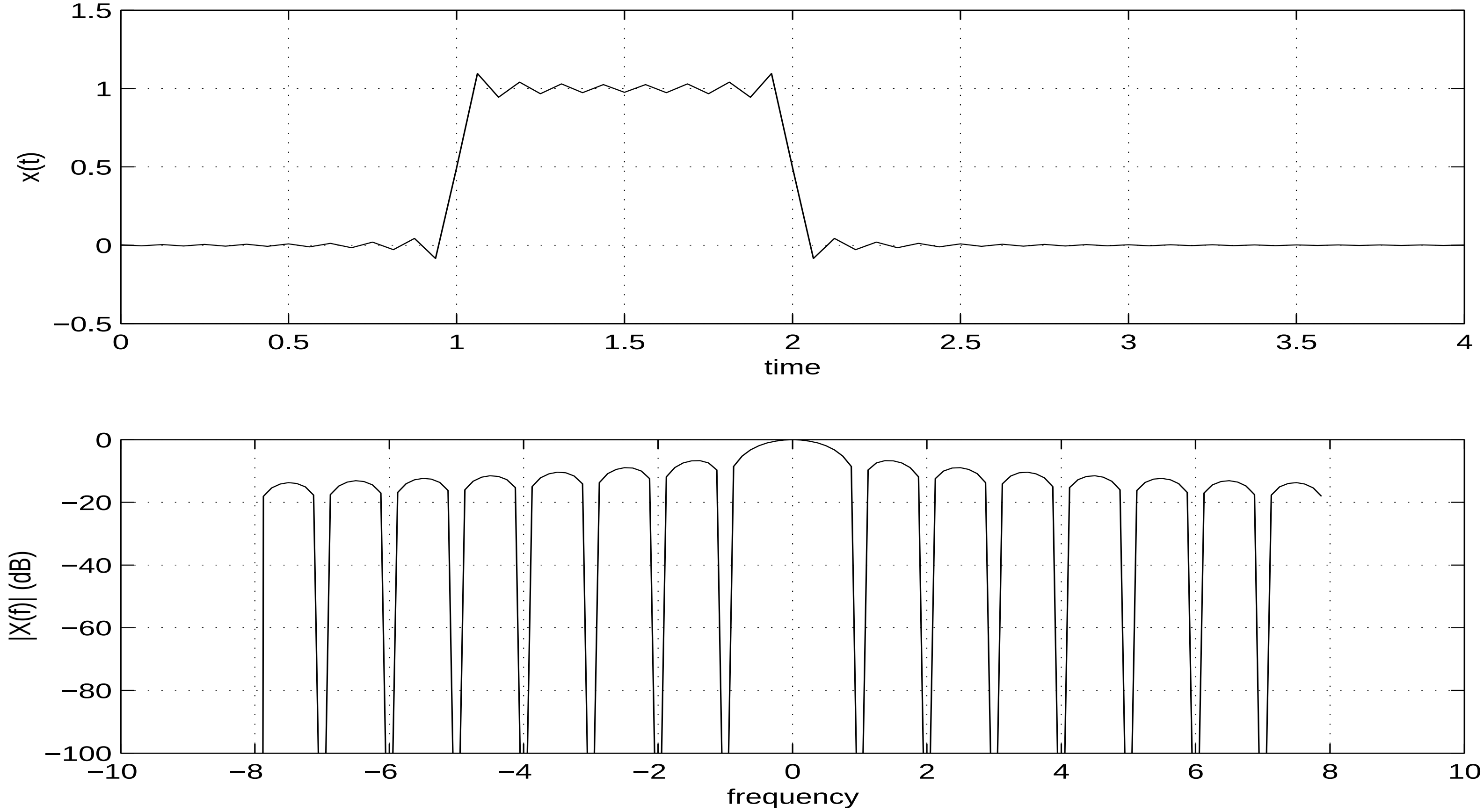
$$x(t) = h(t) = Ap_T(t) = \begin{cases} A, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$$



then the output of the filter is a triangular pulse of duration $2T$.


$y(t)$
A^2T
t
T
$2T$

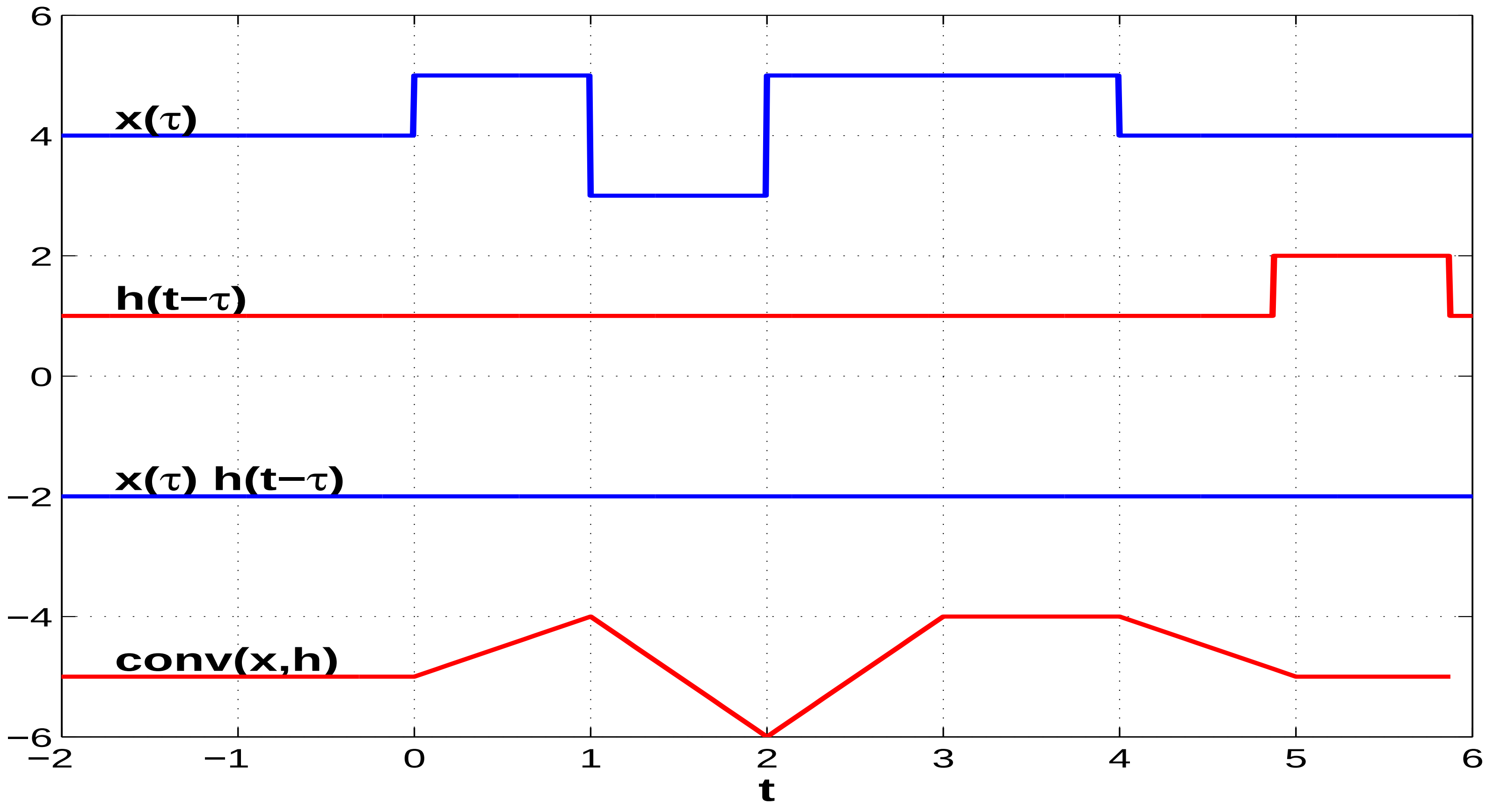
Rectangular Pulse



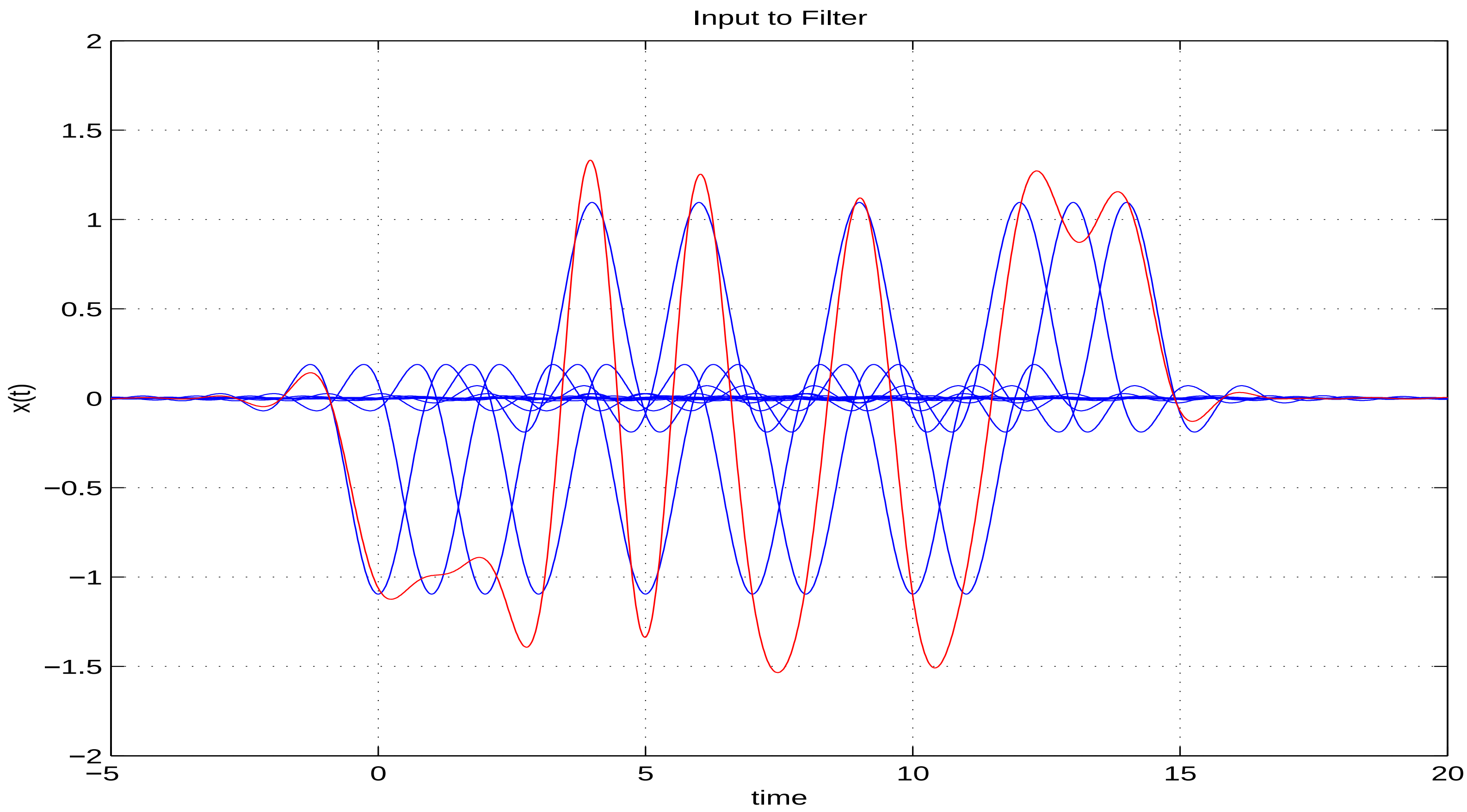
Properties of Linear Systems

- If the output of a linear system is $y_1(t)$ when $x_1(t)$ is the input and the output of is $y_2(t)$ when $x_2(t)$ is the input then the output due to $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is $\alpha_1 y_1(t) + \alpha_2 y_2(t)$
- If the output of a linear system is $y(t)$ when $x(t)$ is the input then the output due to $x(t-\tau)$ is $y(t-\tau)$

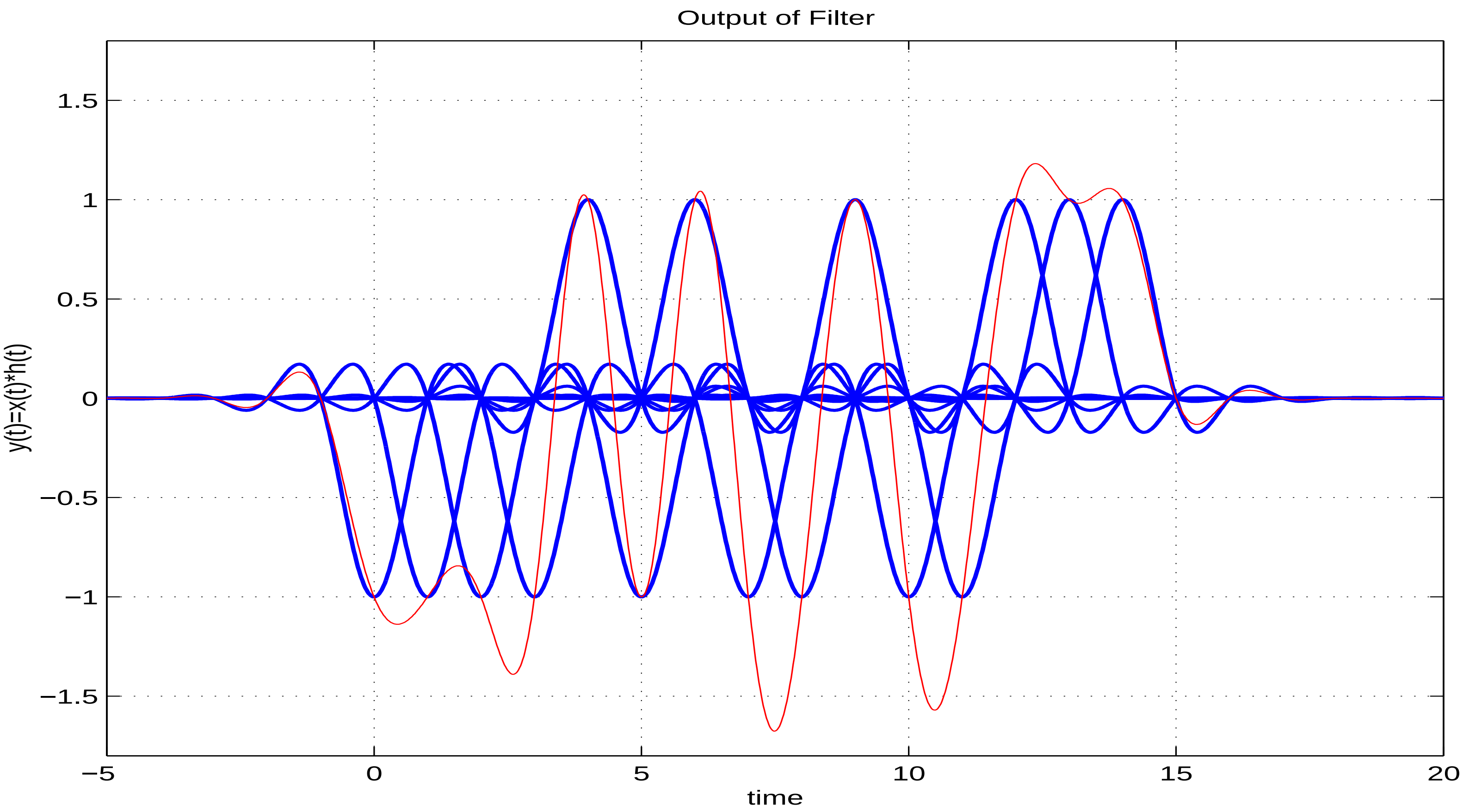
Three rectangular pulses



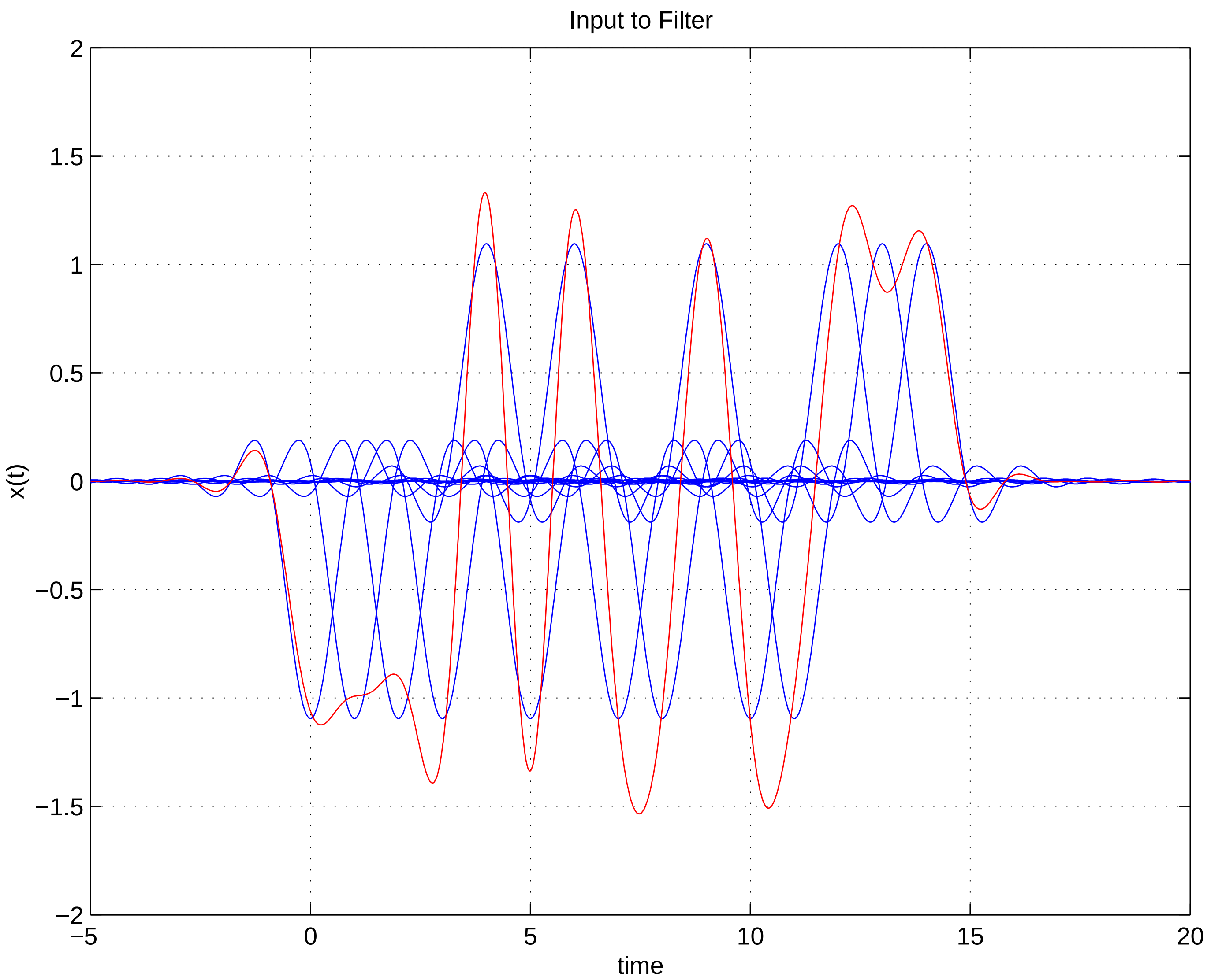
Other Input



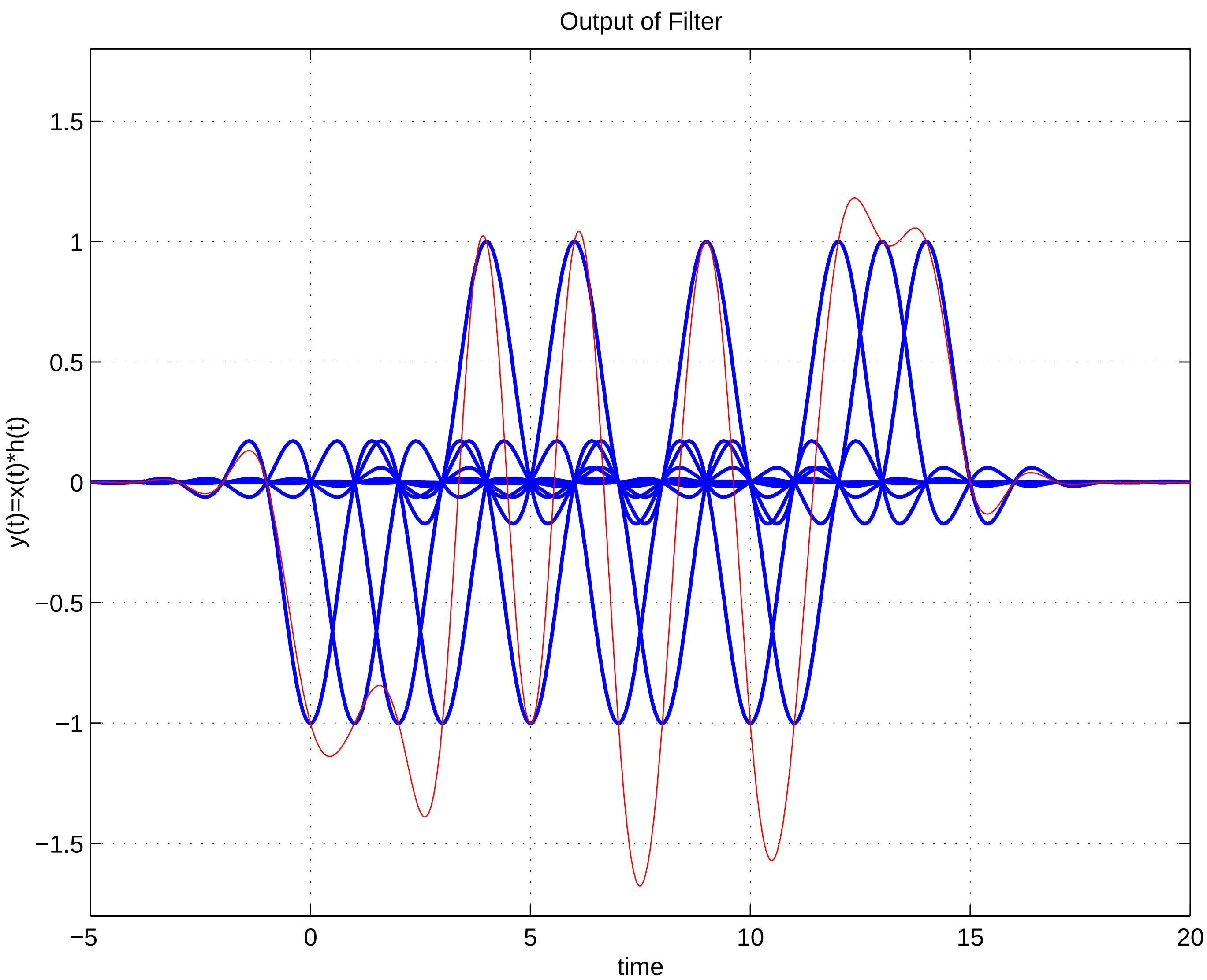
Other Output



Total Input



Total Output



Frequency Domain Analysis

Signals and filtering can also be described in the frequency domain. The frequency content of a signal is obtained via the Fourier Transform.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt.$$

Convolution in the time domain corresponds to multiplication in the frequency domain and thus

$$y(t) = x(t) * h(t) \Leftrightarrow Y(f) = H(f)X(f).$$

One useful relation between the frequency domain and time domain is Parseval's Theorem

$$\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt = \int_{-\infty}^{\infty} X_1(f) X_2^*(f) df.$$

As a special case

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

Ex. 1: Rectangular Filtering of Rectangular Pulses

Motivation: This is the simplest form of modulation. A single data bit is transmitted by sending either a positive pulse to represent a 0 or a negative pulse of duration T to represent a 1. The receiver decides which bit was transmitted by filtering the received signal with a filter matched to the transmitted signal and sampling the filter output.

$$\begin{aligned}
x(t) &= p_T(t). \\
h(t) &= p_T(t). \\
y(t) &= h(t) * x(t) = \Lambda_T(t) = \begin{cases} t, & 0 \leq t \leq T \\ (2 - \frac{t}{T})T, & T \leq t \leq 2T \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

$$
\begin{aligned}
X(f) &= T\mathrm{sinc}(fT)e^{-j\pi fT} = T\frac{\sin(\pi fT)}{\pi fT}e^{-j\pi fT}. \\
H(f) &= T\mathrm{sinc}(fT)e^{-j\pi fT} = T\frac{\sin(\pi fT)}{\pi fT}e^{-j\pi fT}. \\
Y(f) &= H(f)X(f) = T^2\mathrm{sinc}^2(fT)e^{-j2\pi fT}.
\end{aligned}
$$

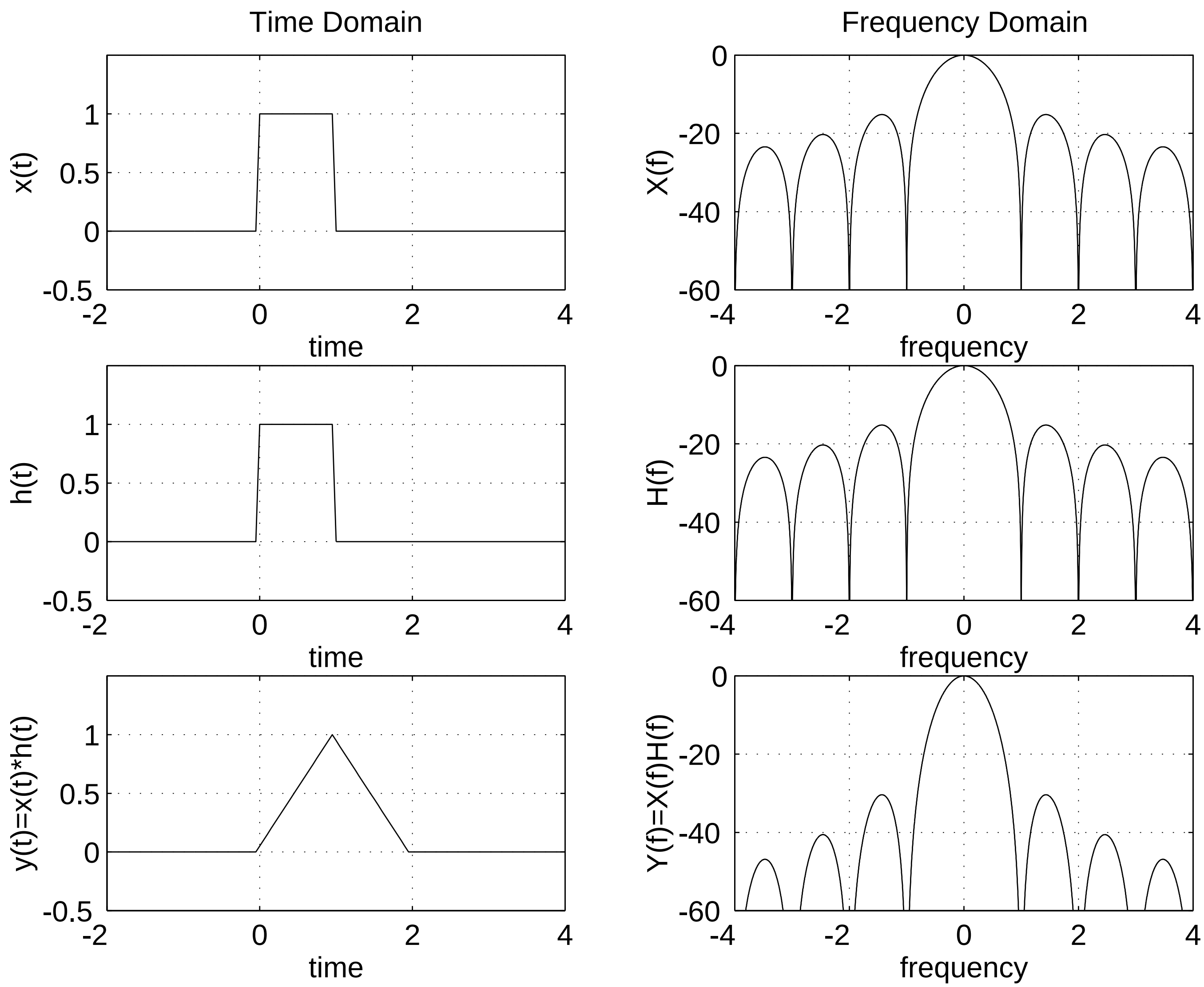


Figure 7: **Filtering Rectangular Pulses**

Ex. 2: Raised Cosine Filtering of Raised Cosine Pulses

$$x(t) = h(t) = \frac{\sin(\pi(1-\alpha)t/T) + 4\alpha t/T \cos(\pi(1+\alpha)t/T)}{\pi[1-(4\alpha t/T)^2]t/T}.$$

$$X(f) = H(f) = \begin{cases} \sqrt{T}, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \sqrt{\frac{T}{2}[1-\sin(\pi T(f-\frac{1}{2T})/\alpha)]}, & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise.} \end{cases}$$

$$y(t) = \frac{\sin(\pi t/T)}{\pi t/T} \frac{\cos(\alpha\pi t/T)}{1-4\alpha^2 t^2/T^2}.$$

$$Y(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}[1 - \sin(\pi T(|f| - \frac{1}{2T})/\alpha)], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & \text{otherwise}. \end{cases}$$

The parameter α is called the roll-off factor and is between 0 and 1. The (absolute) bandwidth is $W = (1+\alpha)/2T$. Notice that the output is zero at multiples of T except at $t = 0$.

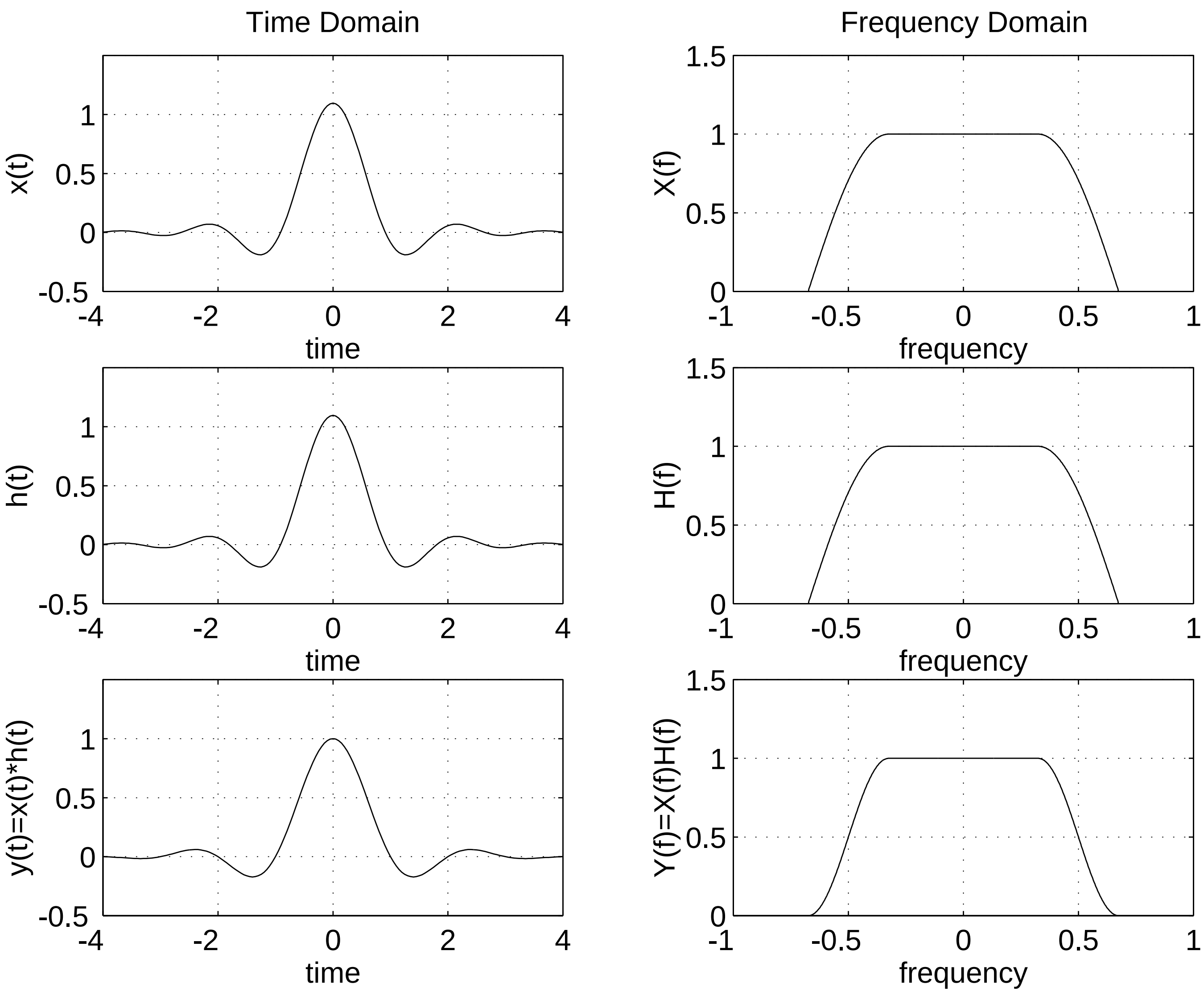


Figure 8: Filtering Raised Cosine Pulses

Ex. 3: Gaussian Filtering of a Rectangular Pulse

$$\begin{aligned} x(t) &= p_T(t) \\ h(t) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\{-t^2/(2\sigma^2)\} \end{aligned}$$

σ is related to the 3dB bandwidth B by

$$\sigma^2 = \frac{\ln(2)}{\pi^2 B}.$$

$$y(t) = \Phi(\frac{t}{\sigma}) - \Phi(\frac{t-T}{\sigma})$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du.$$

In the frequency domain

$$
\begin{aligned}
X(f) &= T\,\mathrm{sinc}(fT)e^{-j\pi fT} \\
H(f) &= \exp\{-2\pi^2\sigma^2 f^2\} \\
Y(f) &= X(f)H(f).
\end{aligned}
$$

This is a noncausal filter. In practice a delay must be added to make the filter implementable.

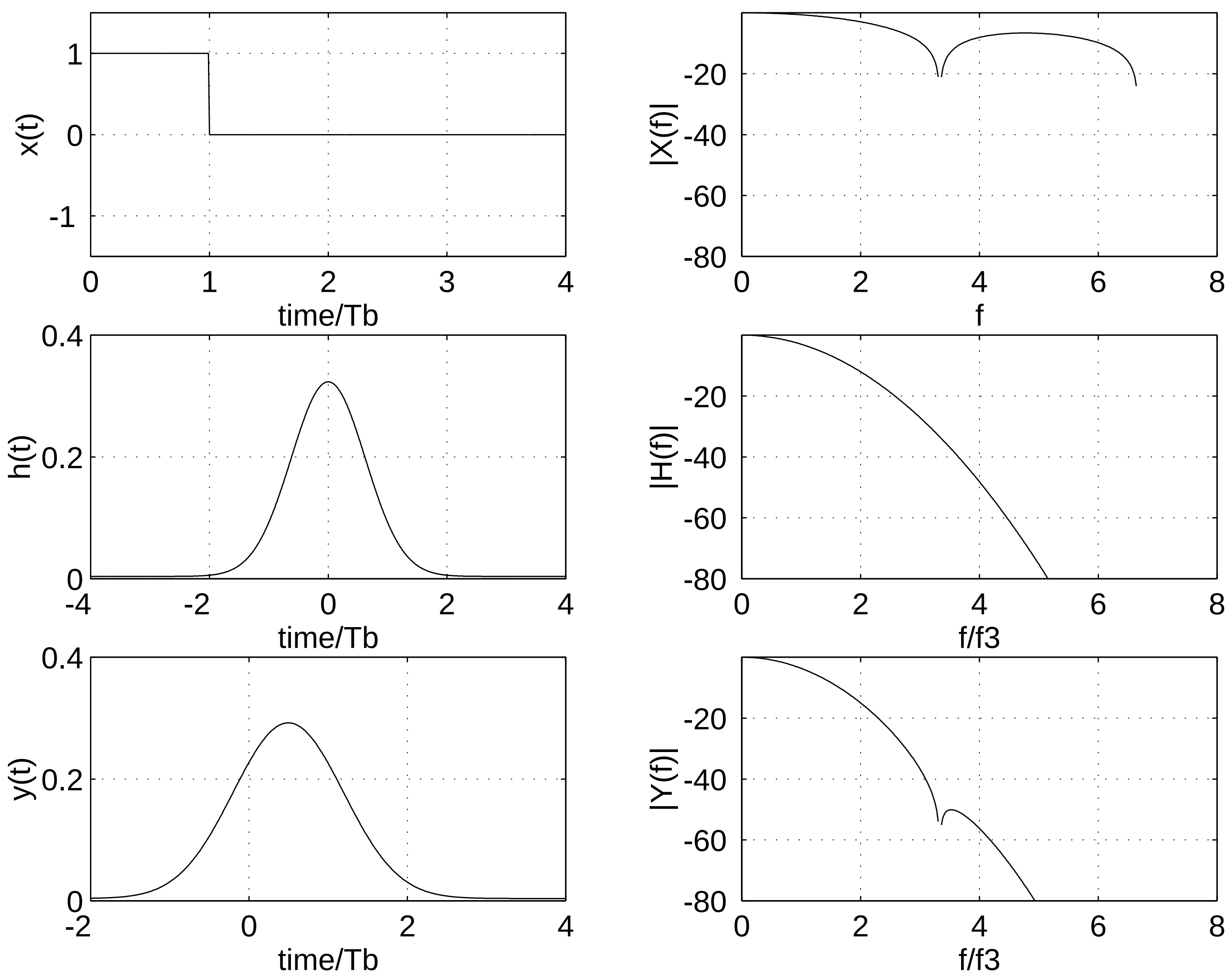


Figure 9: Gaussian Filtering of Rectangular Pulses

Example 4: Spread-spectrum signals

In this example the basic pulse shape has much larger bandwidth. The pulse shape consists of a sequence of shorter pulses (called chips). The filter is the time reverse (and delayed) version of the pulse. Notice that the output lasts for $2T$ seconds and is zero at time 0 and time $2T$. Notice also that the output is a piecewise linear function of time.

Motivation: As in the first example a transmitter can send a 0 by sending the T second waveform shown below and send a 1 by sending the same waveform but with opposite polarity. The receiver filters the signal (to remove out-of-band noise). The filter is matched to the transmitted signal (with a time reversal). The receiver decides 0 is sent if the filter output at time T is larger than 0. Otherwise the receiver decides 1.

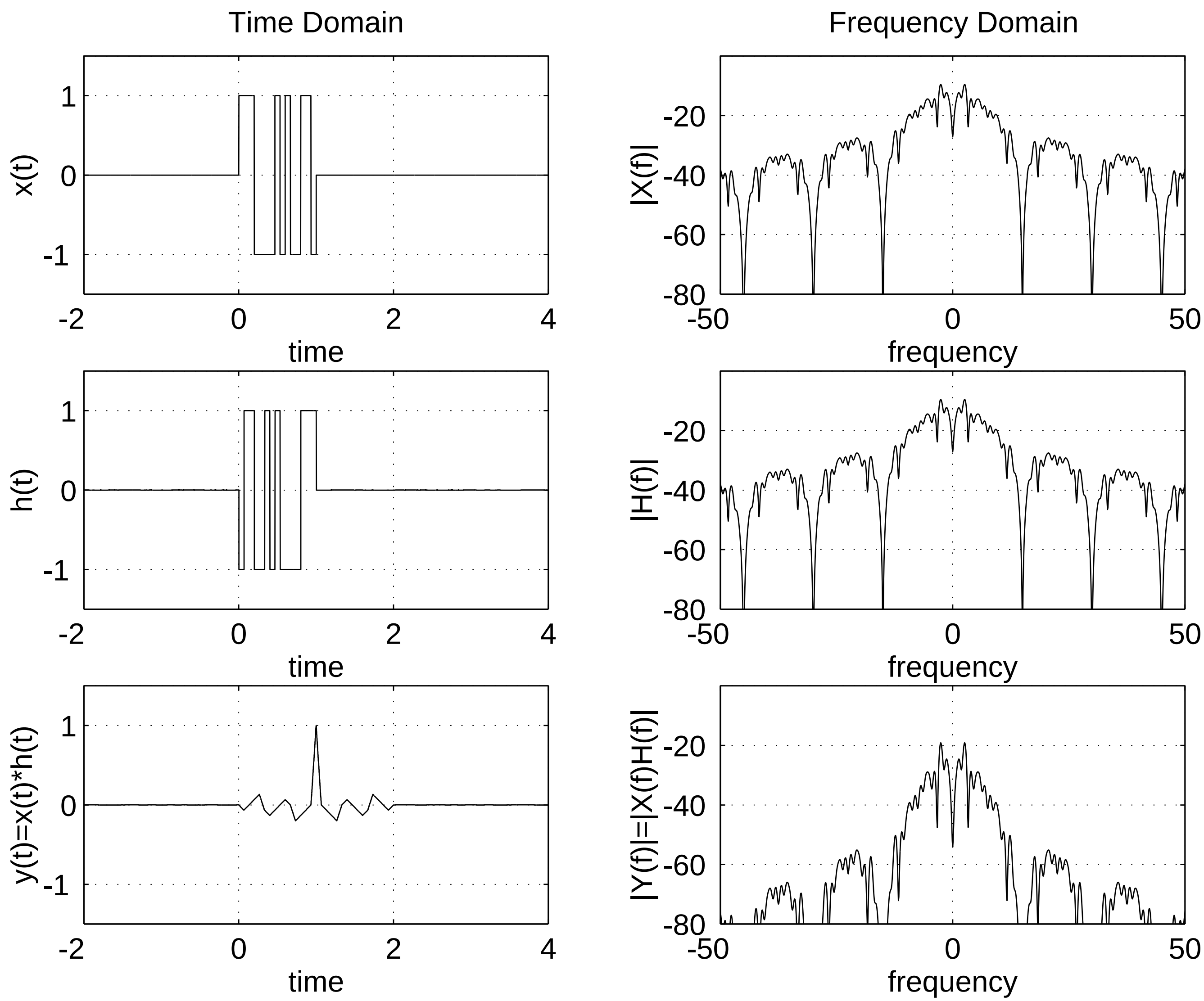


Figure 10: Filtering Spread Spectrum Pulses

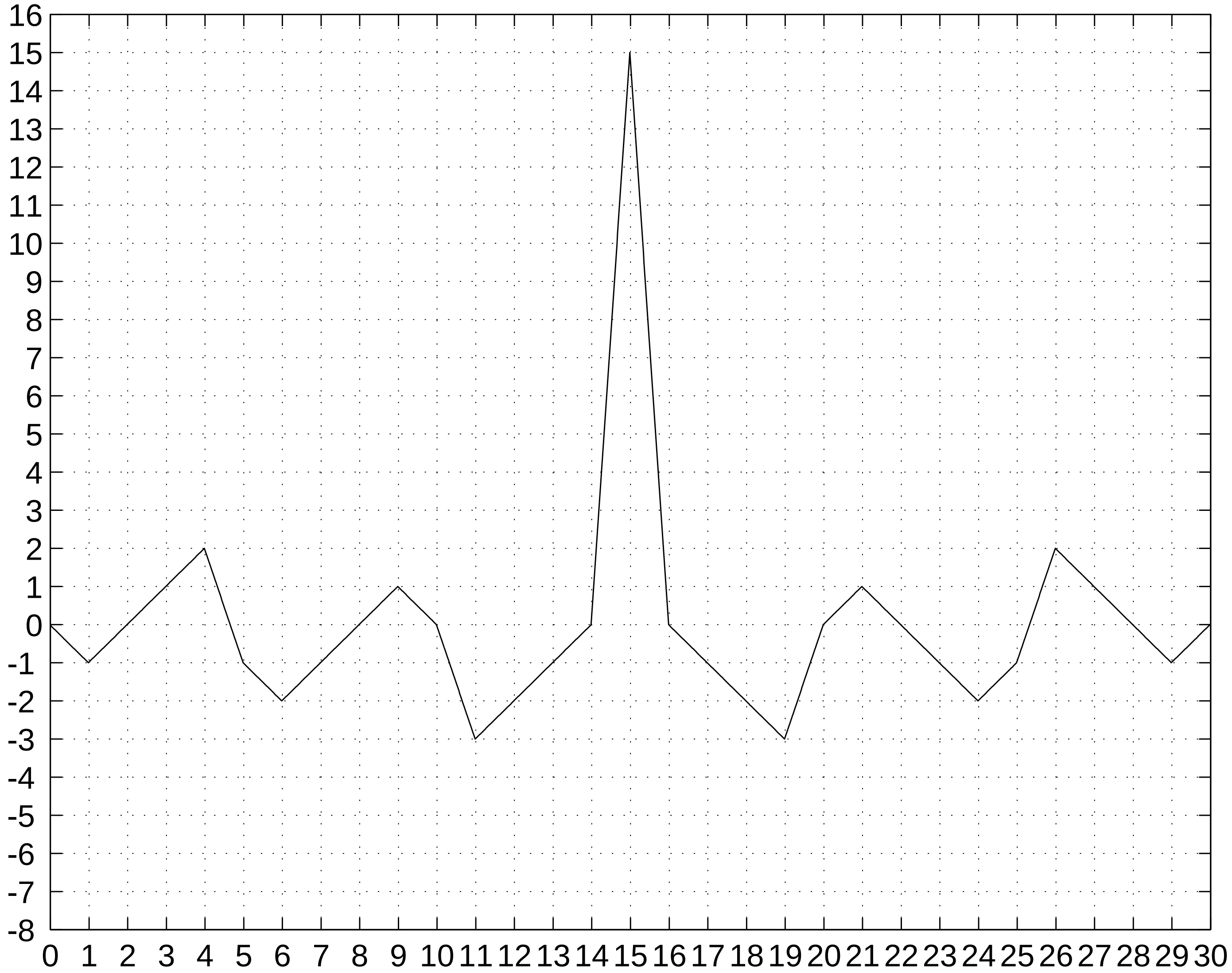


Figure 11: Filtering Spread Spectrum Pulses

Multiuser System

Now consider two users which have different basic signal waveforms. Consider a filter matched to the basic signal of the first user. The output due to the signal of the first user alone is shown as is the output due the second user alone. If these users both transmitted simultaneously then the output would be the sum of the two outputs.

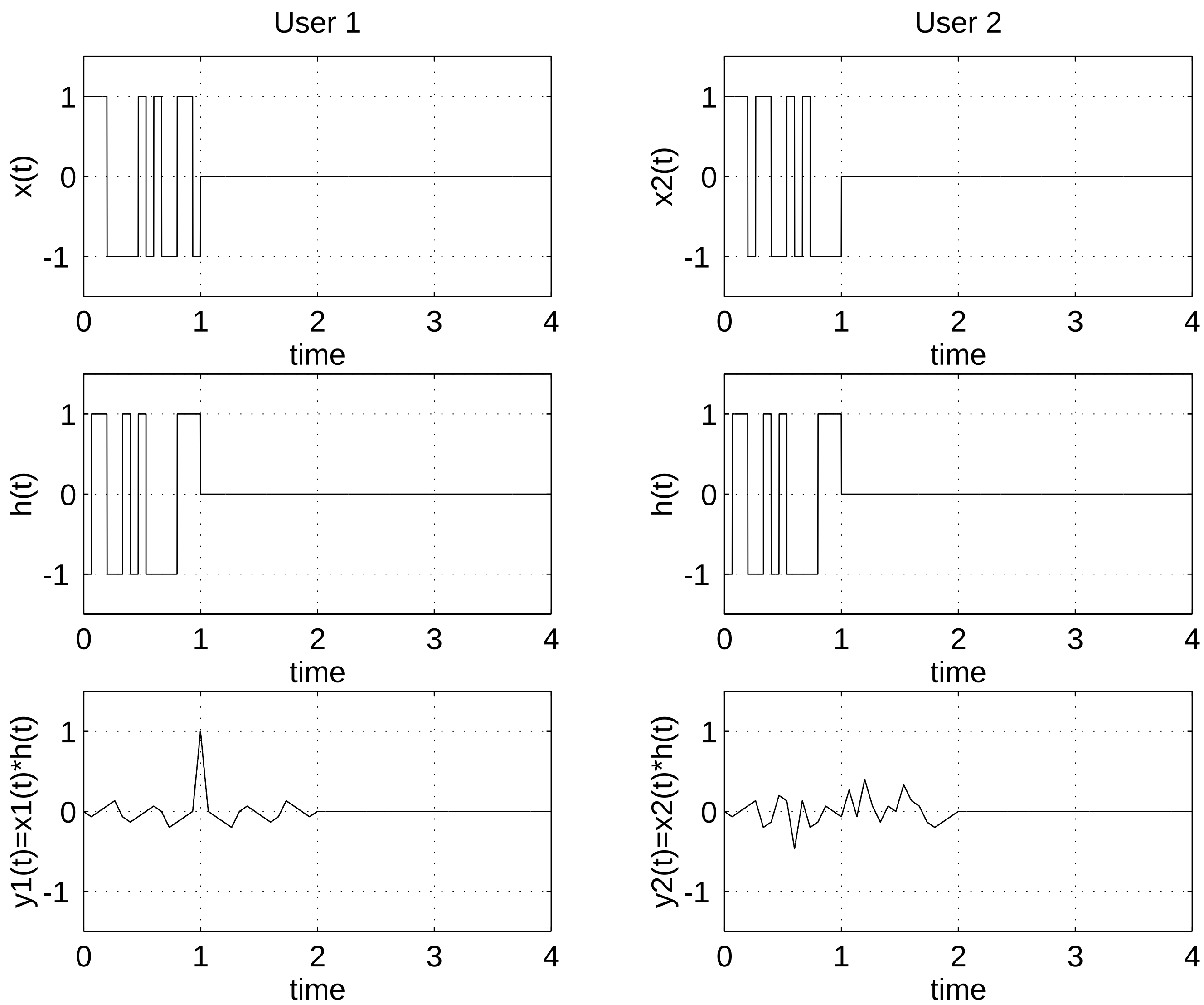


Figure 12: Filtering Spread Spectrum Pulses

In the next figure we expand the filter output to show more detail. From this it is evident that the output due to the interference is also simply a piecewise linear function.

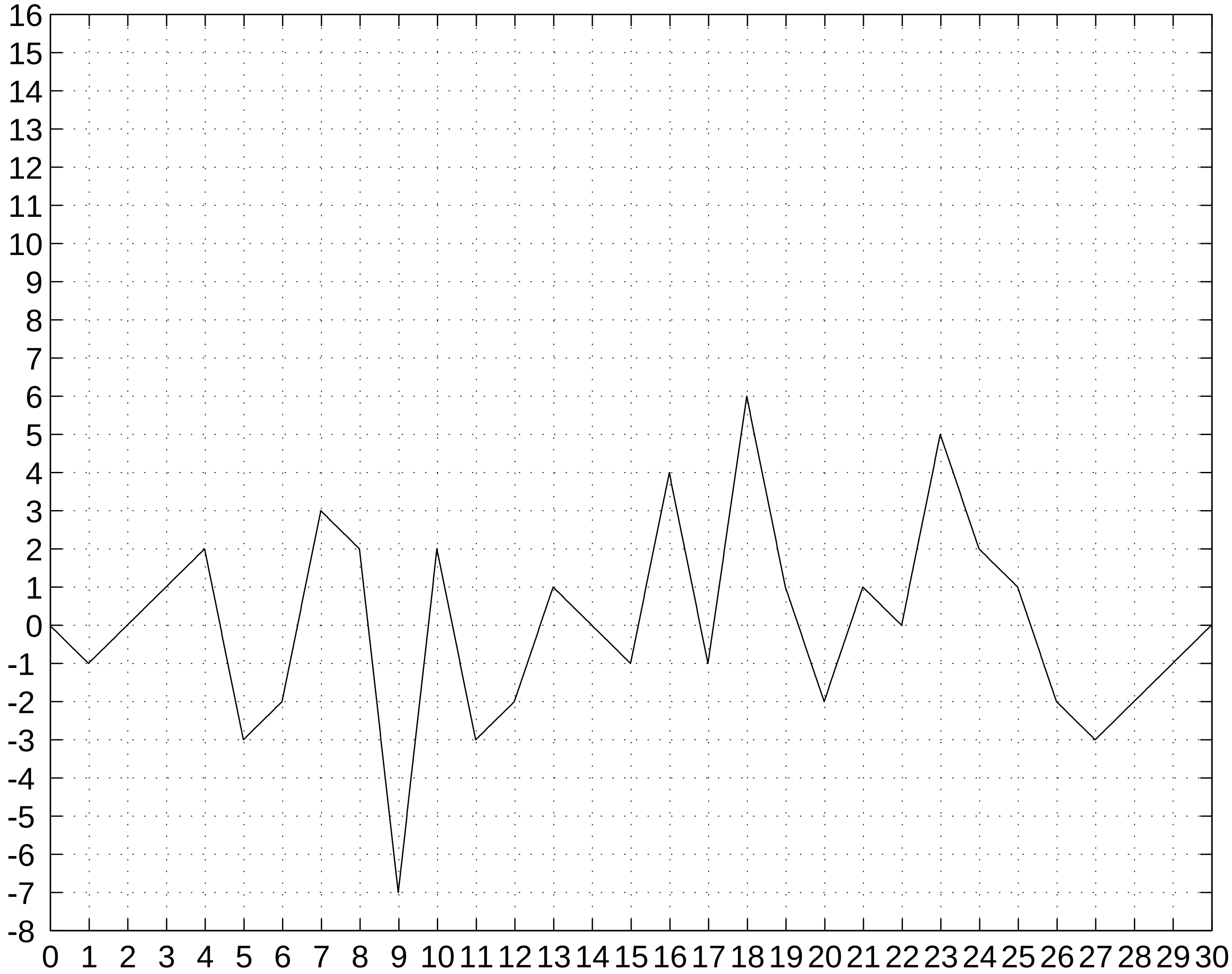


Figure 13: Filtering Spread Spectrum Pulses

If we just assume that the filter is sampled at time T and that the filter is causal $h(t) = 0$, $t < 0$ then

$$y(t) = \int_{-\infty}^{t} h(t-\alpha)x(\alpha)d\alpha$$

If the filter has finite response, say for T seconds then

$$y(t) = \int_{t-T}^{t} h(t-\alpha)x(\alpha)d\alpha$$

The desired signal is the output sampled at time T.

$$y(T) = \int_{0}^{T} h(T-\alpha)x(\alpha)d\alpha$$

This can be implemented with a correlator as shown below

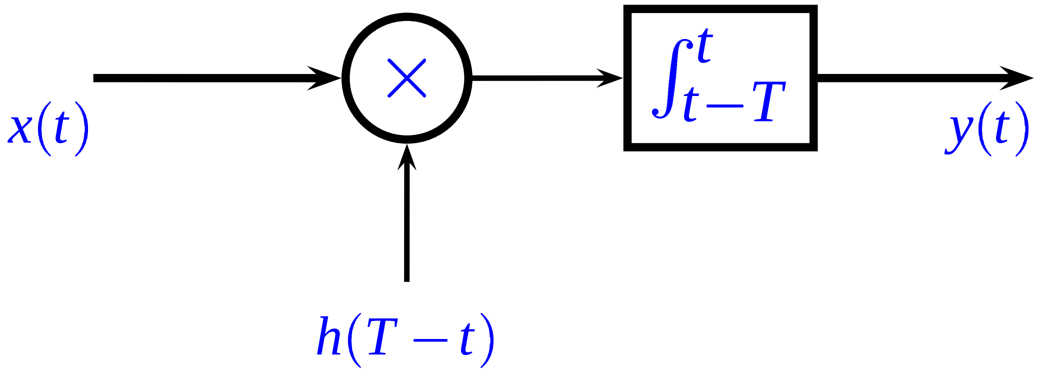


Figure 14: Correlator Structure

In a digital communication system it is usual for the received filter to be matched to the transmitted signal. In this case, if $s(t)$ is the transmitted signal and is of duration T beginning at 0, we sample the filter output at time T and $h(t) = s(T-t)$. This is called the matched filter. The filter output is

$$
\begin{aligned}
y(t) = \int_{-\infty}^{\infty} h(t-\alpha)s(\alpha)d\alpha &= \int_{t-T}^{t} h(t-\alpha)s(\alpha)d\alpha \\
&= \int_{t-T}^{t} s(T-(t-\alpha))s(\alpha)d\alpha \\
&= \int_{t-T}^{t} s(\alpha-(t-T))s(\alpha)d\alpha
\end{aligned}
$$

(This is the autocorrelation of the signal $s(t)$). The desired signal is the output sampled at time T.

$$
y(T) = \int_{0}^{T} h(T-\alpha)s(\alpha)d\alpha = \int_{0}^{T} s(\alpha)s(\alpha)d\alpha
$$

$$= \int_0^T s^2(\alpha)\,d\alpha$$

In a spread-spectrum system the signal has the form

$$s(t) = \sum_{l=0}^{N-1} a_l \psi(t - lT_c)$$

so the impulse response of the matched filter has the following form

$$h(t) = s(T - t) = \sum_{l=0}^{N-1} a_l \psi(T - t - lT_c)$$

where N is the number of "chips" per bit, $1/T_c$ is the chip rate, and $NT_c = T$ is the data bit duration or the inverse data rate. In this case the implementation of the matched filter can be simplified as follows. Let $s(t)$ be the filter input then

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t-\alpha)s(\alpha)d\alpha \\ &= \int_{-\infty}^{\infty} \sum_{l=1}^{N-1} a_l \psi(T - t + \alpha - lT_c)s(\alpha)d\alpha \end{aligned}$$

$$= \sum_{l=0}^{N-1} a_l \int_{-\infty}^{\infty} \psi(T - t + \alpha - lT_c) s(\alpha) d\alpha.$$

Let $x(t)$ be the output of a filter with impulse response $\psi(T_c - t)$ then

$$x(t) = \int_{-\infty}^{\infty} \psi(T_c - t + \alpha) s(\alpha) d\alpha.$$

Now it is clear that

$$y(t) = \sum_{l=0}^{N-1} a_l x(t - (N - 1 - l)T_c).$$

Thus the matched filter can be implemented as a filter matched to the chip waveform followed by a weighted sum. Since we are interested in the sample only at time $t = mT$ we only need the samples of the chip matched filter at multiples of T_c. For example at time $t = T$ the output is

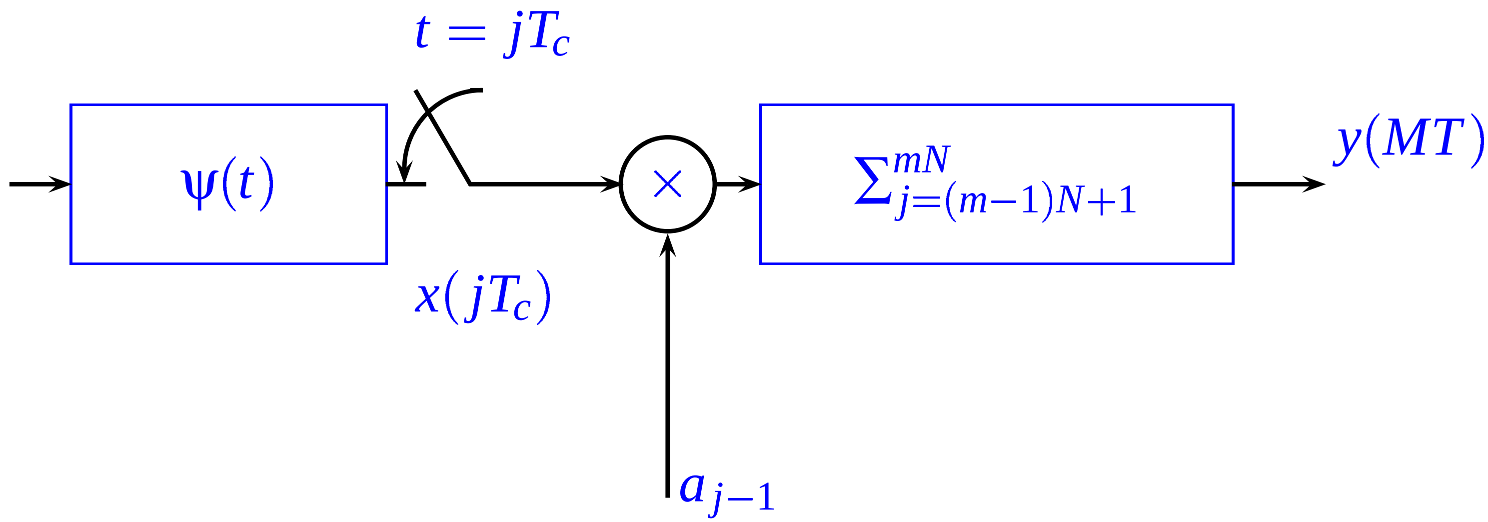
$$y(T) = \sum_{l=0}^{N-1} a_l x((l+1)T_c)$$

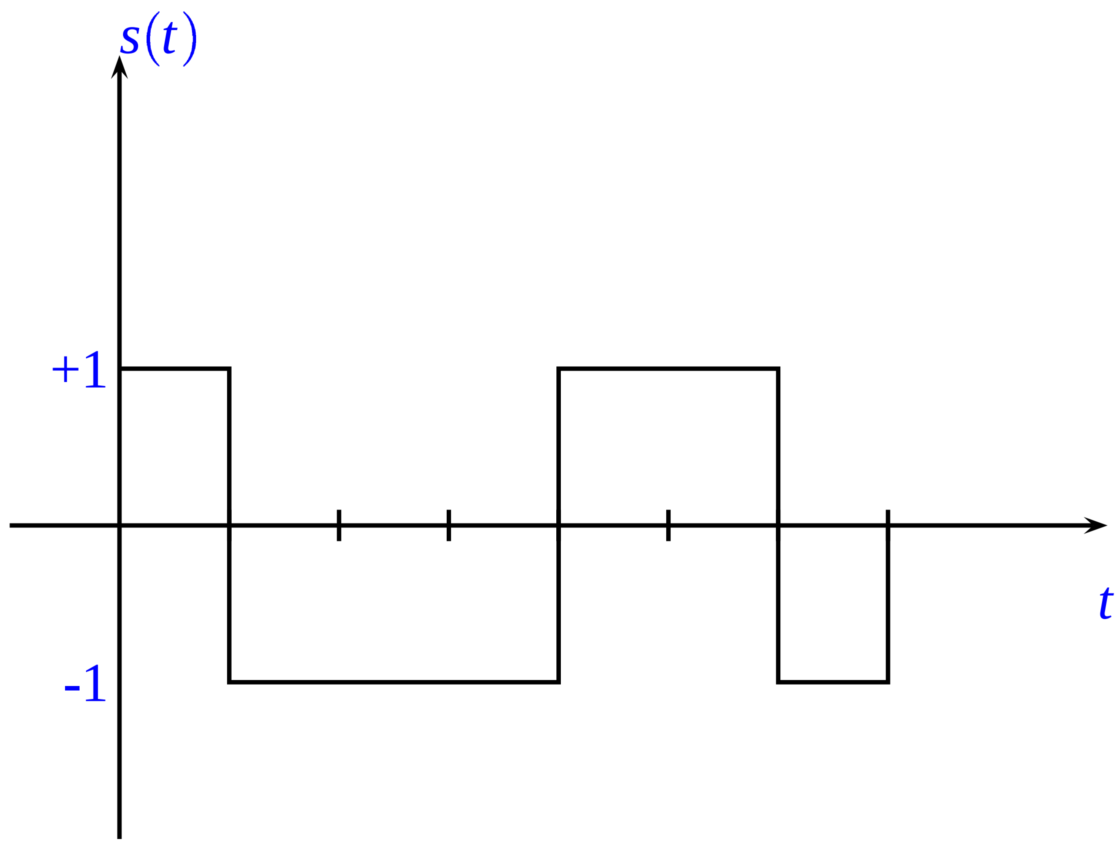
At time $t = mT$ the output is

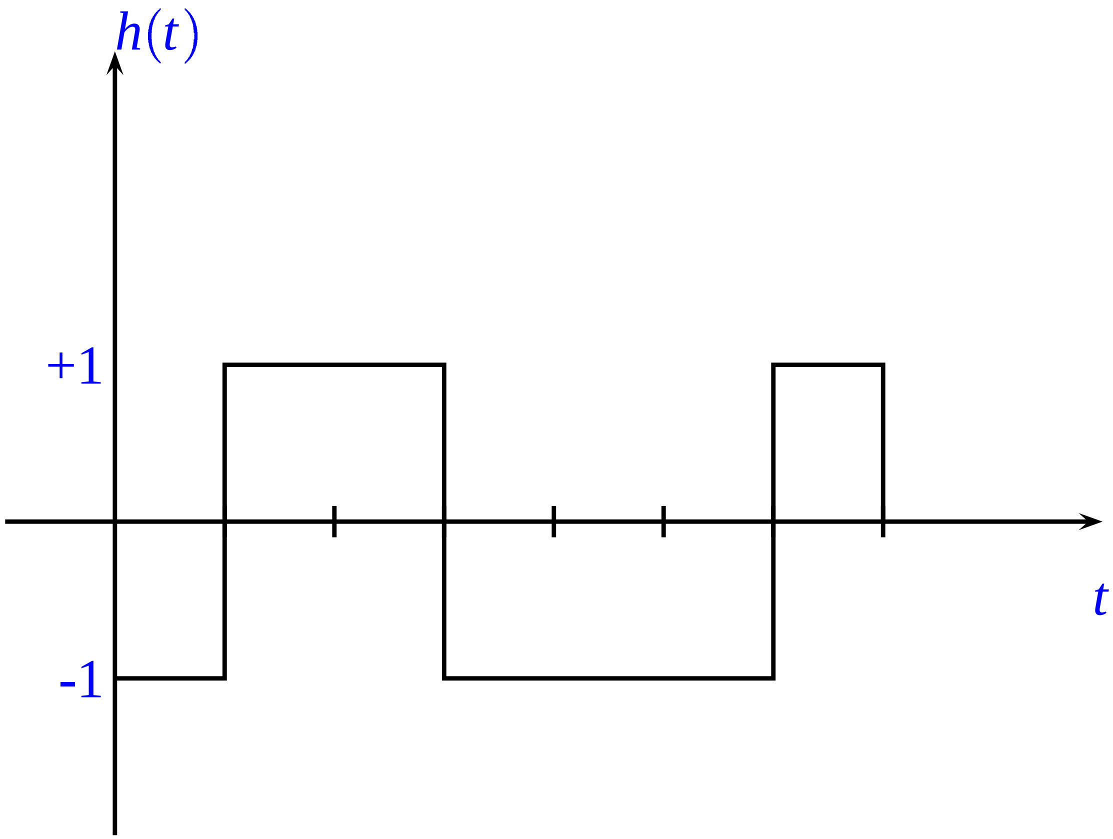
$$
\begin{aligned}
y(mT) &= \sum_{l=0}^{N-1} a_l x((m-1)T + (l+1)T_c) \\
y(mT) &= \sum_{j=(m-1)N+1}^{mN} a_{j-1-(m-1)N} x(jT_c)
\end{aligned}
$$

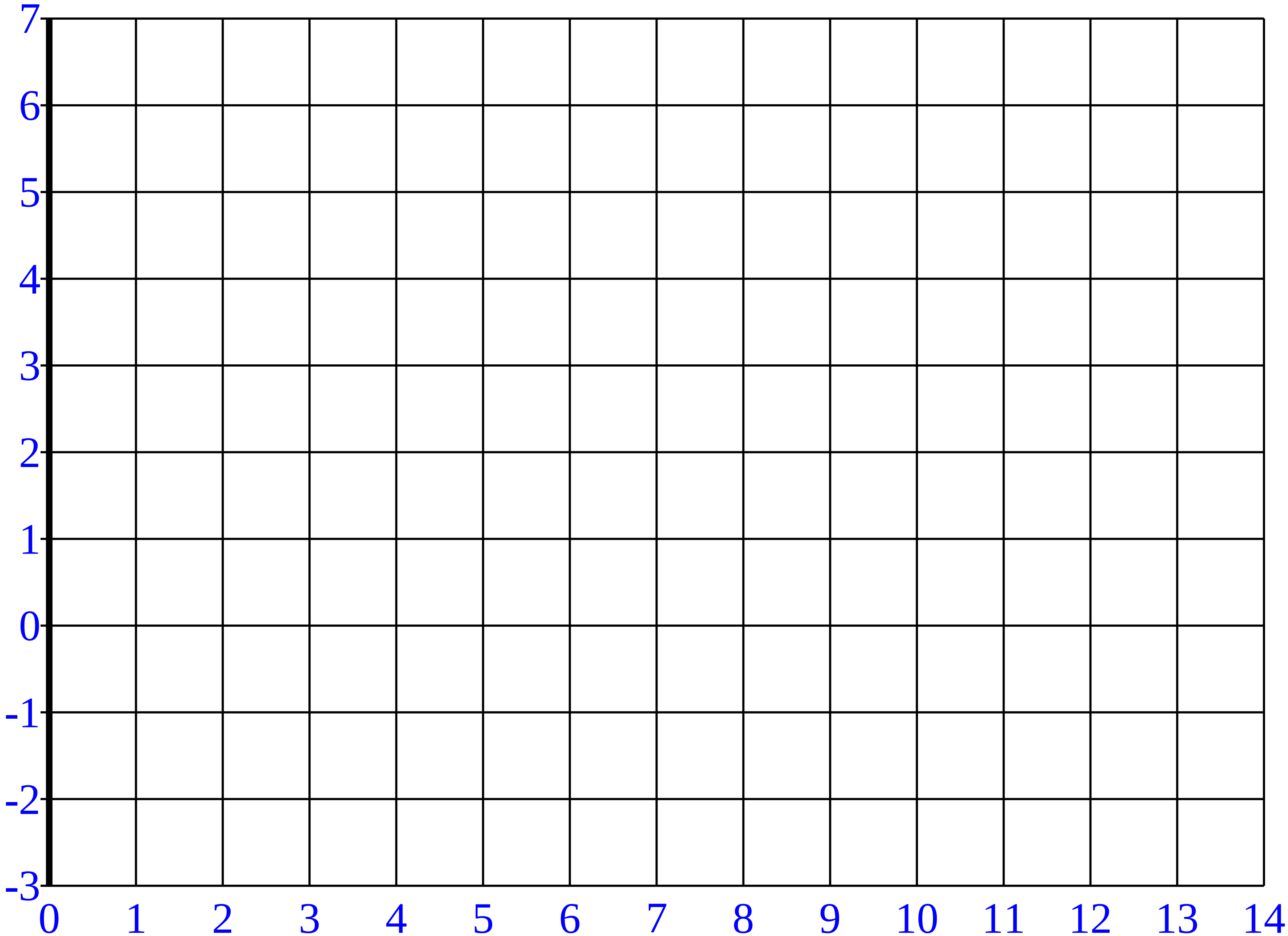
If the spreading sequence is periodic with period N so that $a_{j+N} = a_j$ then

$$
y(mT) = \sum_{j=(m-1)N+1}^{mN} a_{j-1} x(jT_c).
$$

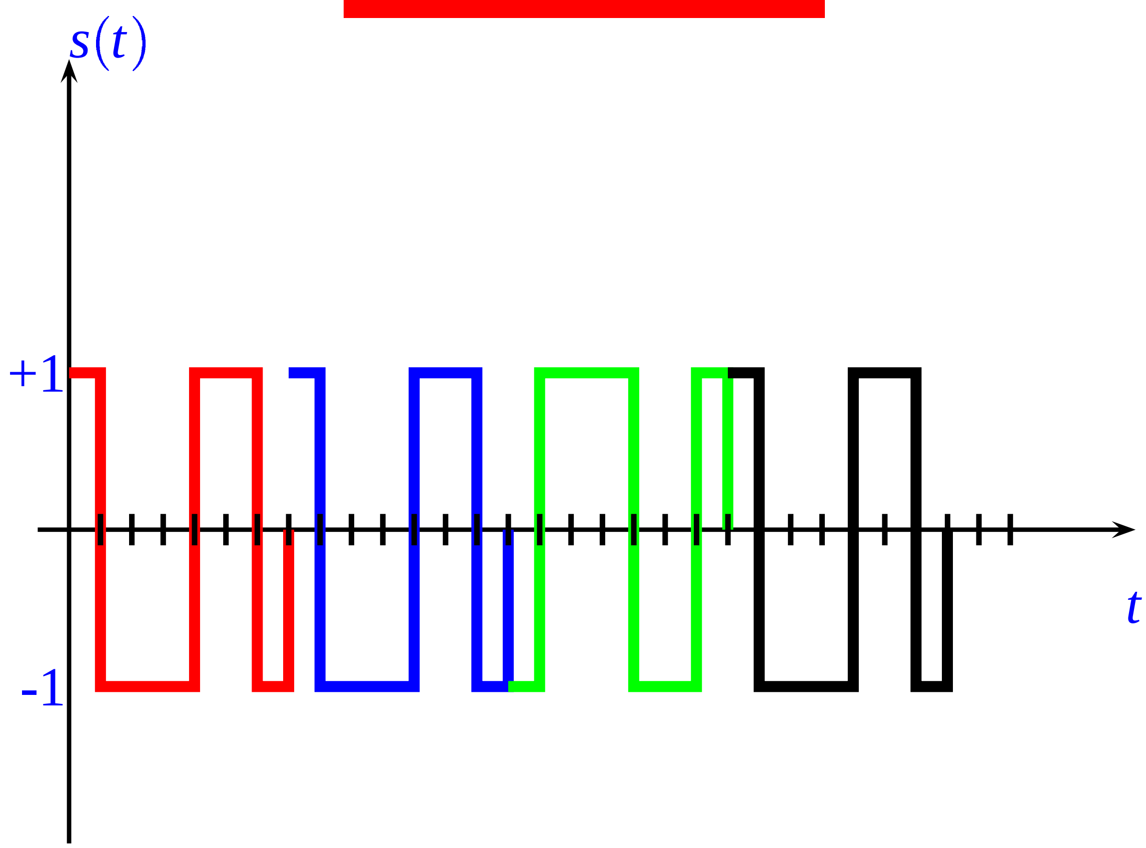

$t = jT_c$
$\psi(t)$
$x(jT_c)$
$\times$
$\sum_{j=(m-1)N+1}^{mN}$
$y(MT)$
a_{j-1}


$s(t)$
+1
-1
t

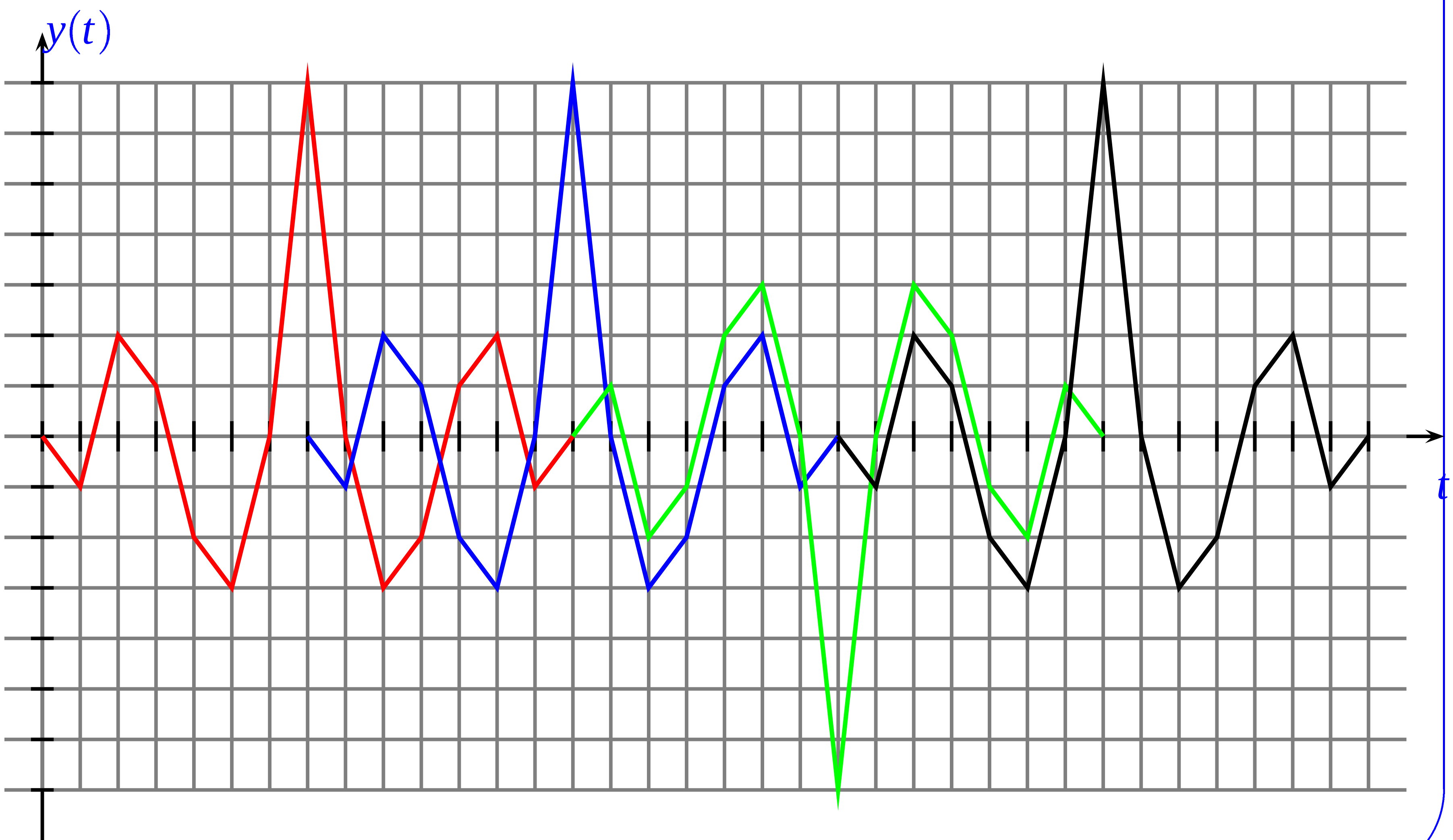

$h(t)$
+1
-1
t


7
6
5
4
3
2
1
0
-1
-2
-3
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

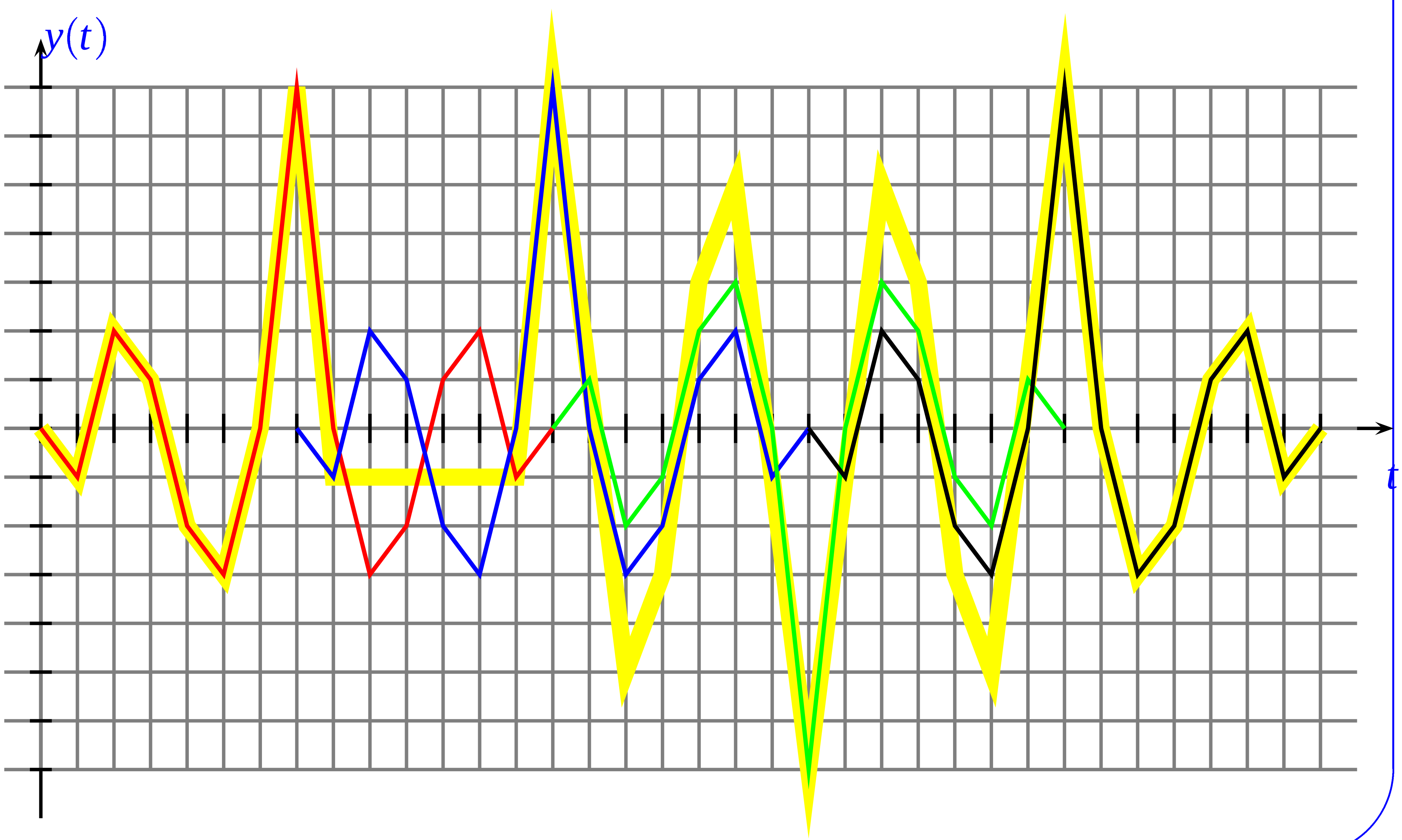
Multiple Bits



Multiple Bits



Multiple Bits



Noise

All communications systems has some amount of noise. All electrical systems have random thermal noise due to motion of electrons because the system is not at absolute zero temperature. This noise is usually modeled as having power at all frequencies but in actuality at very high frequencies the power decreases (in the optical range of frequencies).

The model widely used for thermal noise is that of zero mean white Gaussian noise. Since only the frequency band of the transmitted signal is of interest the noise outside this band is not important. For all systems considered here we model the noise as having equal power at all frequencies. In addition the noise will have zero mean or average.

The power spectral density function of a random signal is the amount of power in the signal as a function of frequency. The autocorrelation measures the correlation between the noise at different points in time. For noise like signals the autocorrelation does not depend on the time but just the time difference between two samples. In this case (and assuming zero mean) the process is called wide-sense stationary.

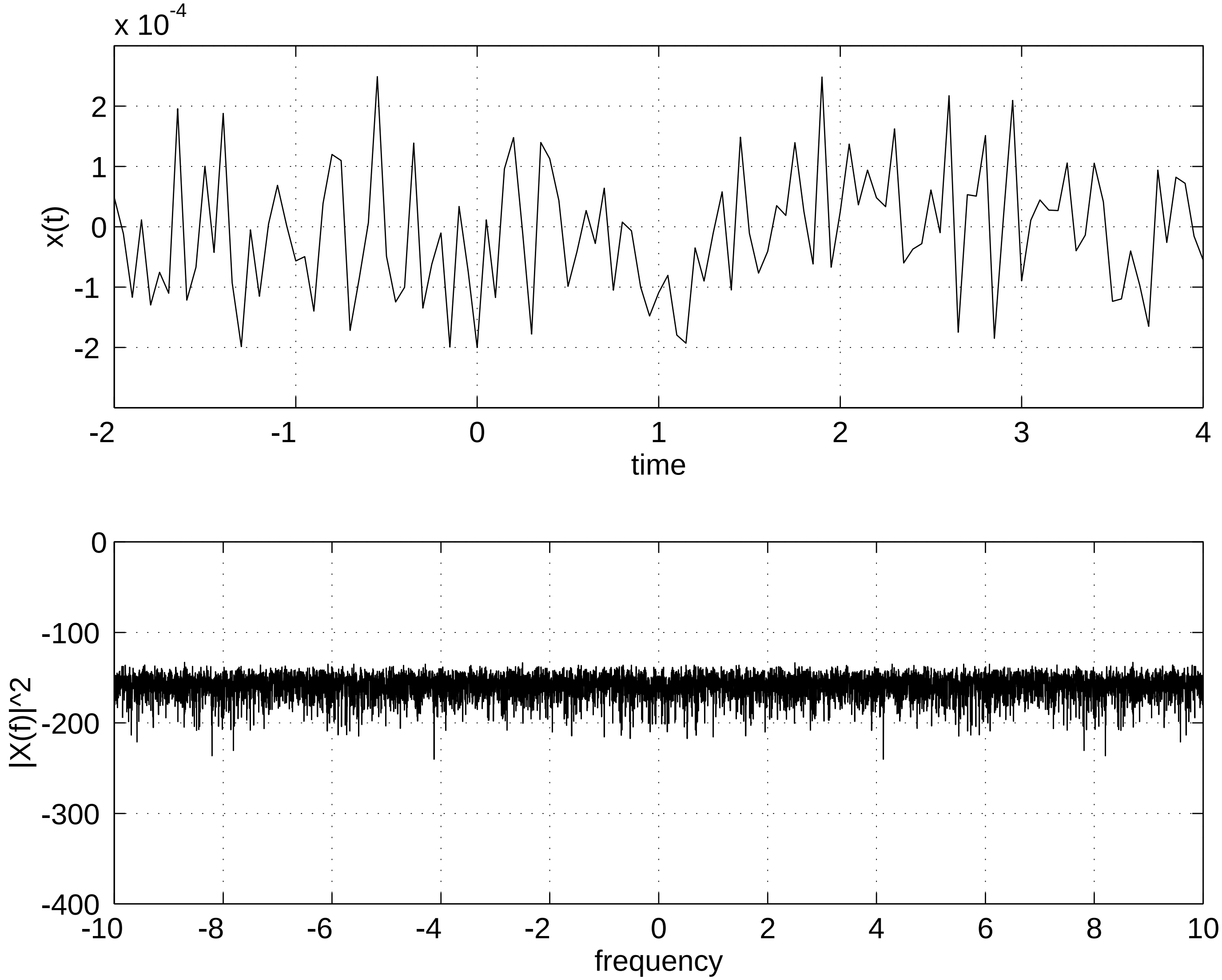


Figure 15: Sample Path and Frequency Content of Noise

Mathematical definitions:

$$R_N(\tau) = E[N(t)N(t+\tau)]$$

The power spectral density is the Fourier Transform of the autocorrelation function.

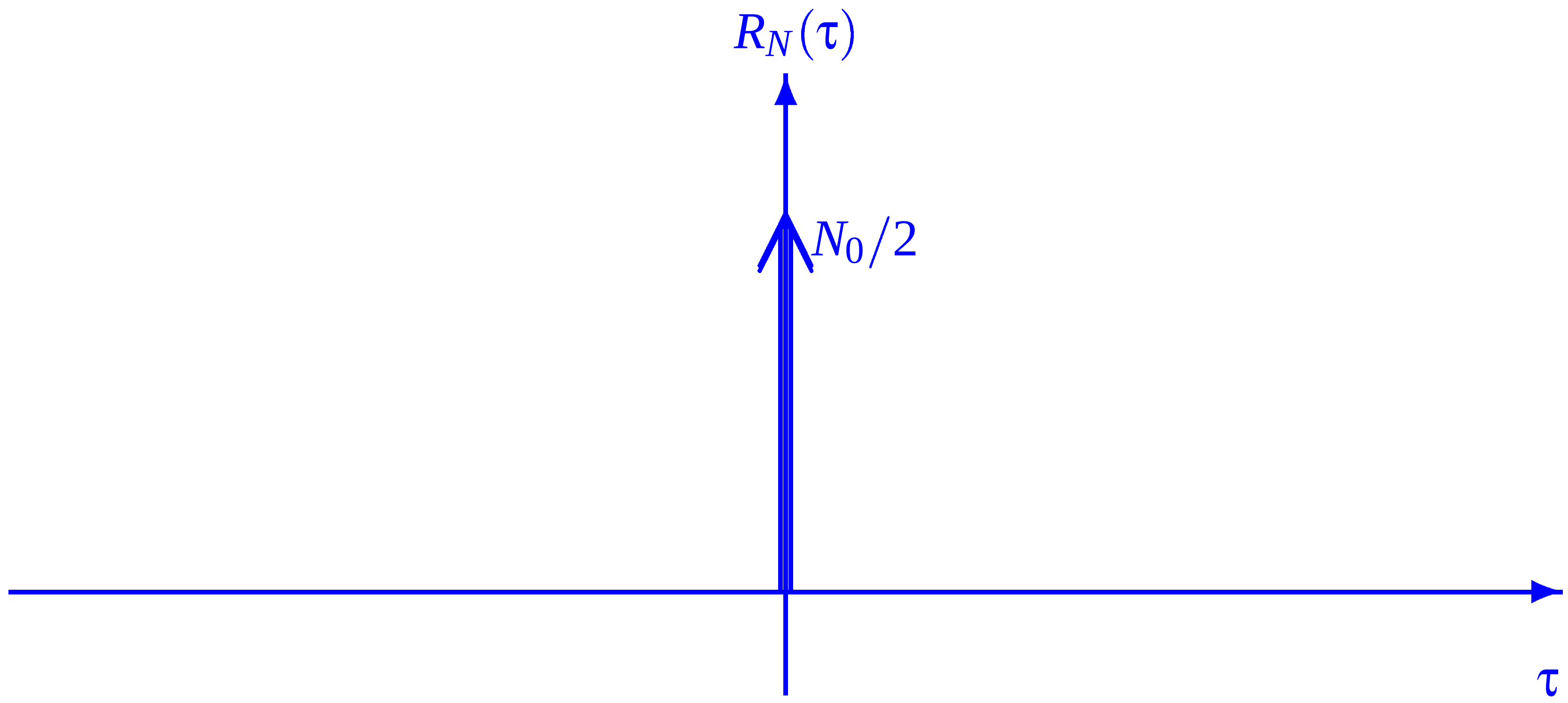
$$S_N(f) = \int_{\infty}^{\infty} R_N(\tau) e^{-j2\pi f\tau} d\tau$$

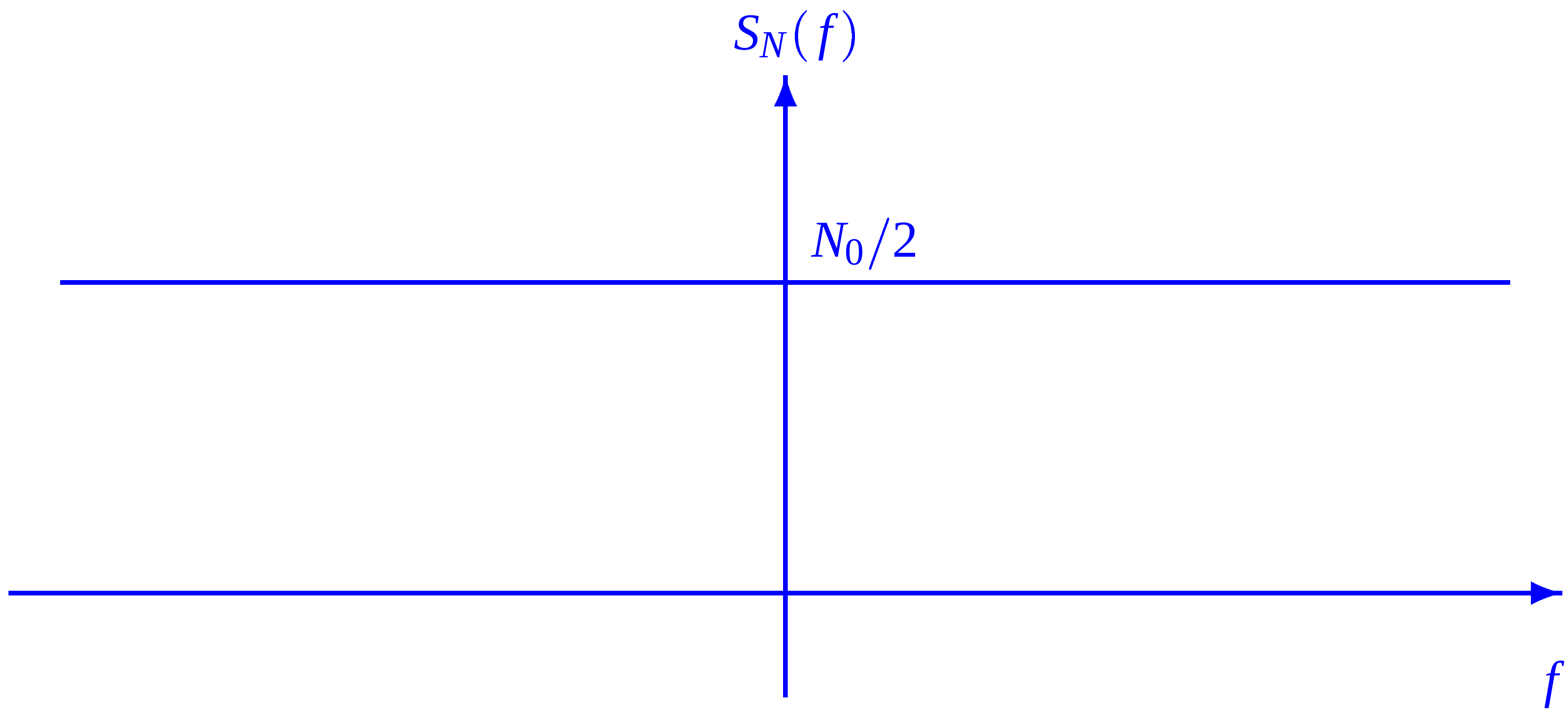
$$R_N(\tau) = \int_{\infty}^{\infty} S_N(f) e^{j2\pi f\tau} d\tau$$

Example

White Noise :

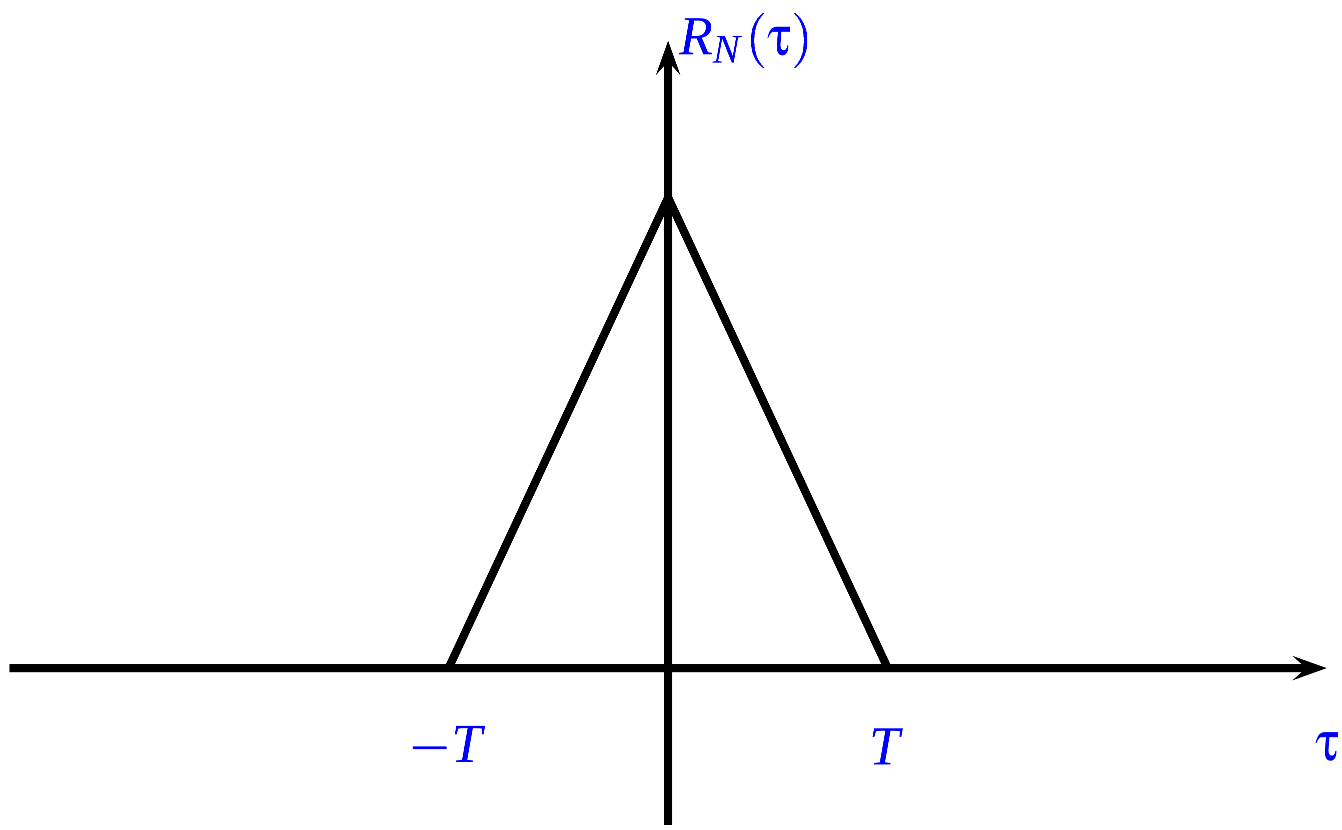
$$R_N(\tau) = \frac{N_0}{2}\delta(\tau), \qquad S_N(f) = \frac{N_0}{2}$$

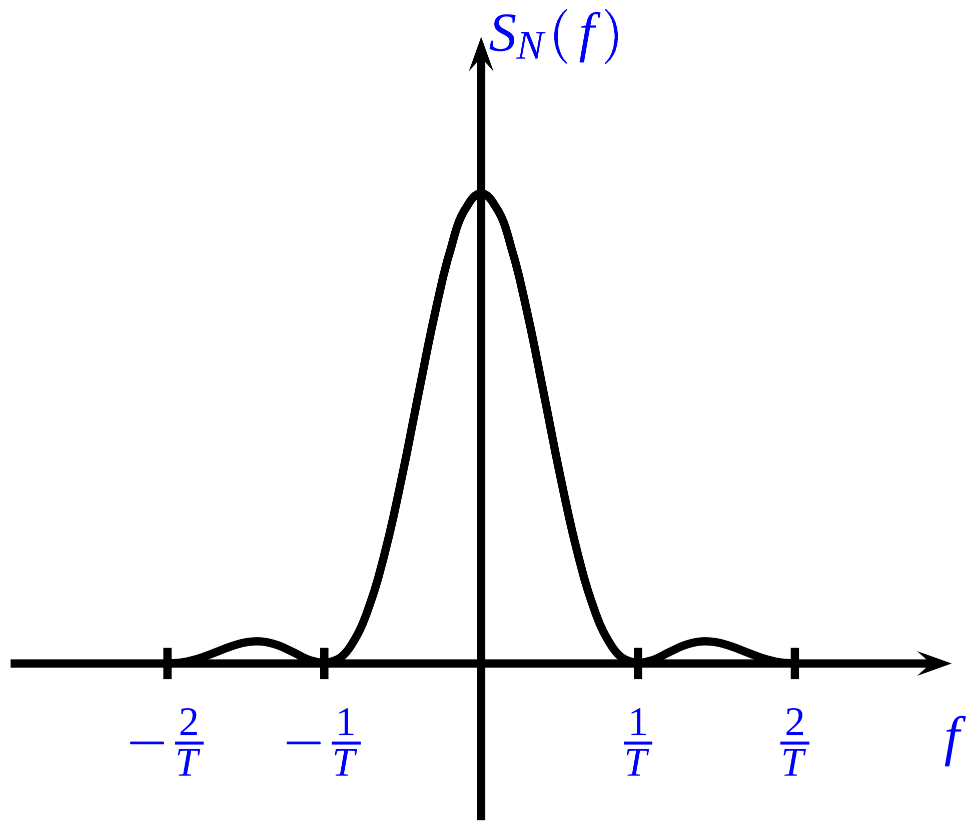



$S_N(f)$
$N_0/2$
f

Correlated Noise

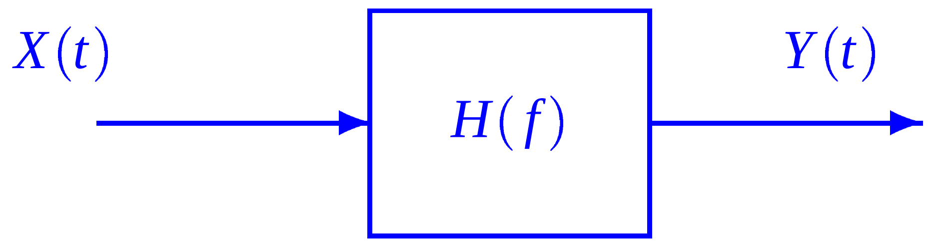
$$
\begin{aligned}
R_N(\tau) &= \Lambda(\tau/T) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| \le T \\ 0 & |\tau| > T \end{cases}, \\
S_N(f) &= T \operatorname{sinc}^2(fT) \\
&= T \frac{\sin^2(\pi f T)}{(\pi f T)^2}
\end{aligned}
$$


$R_N(\tau)$
$-T$
T
τ


$S_N(f)$
$-\frac{2}{T}$
$-\frac{1}{T}$
$\frac{1}{T}$
$\frac{2}{T}$
f

Noise into linear systems

Now consider noise at the input to the receiver.



The power spectral density of the output of the filter is determined from the power spectral density at the input to the filter and the transfer function of the filter.

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

The autocorrelation is given by

$$
\begin{aligned}
R_Y(\tau) &= E[Y(t)Y(t+\tau)] \\
&= E[\int_{-\infty}^{\infty} X(t-\alpha)h(\alpha)d\alpha \int_{-\infty}^{\infty} X(t+\tau-\beta)h(\beta)d\beta] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t-\alpha)X(t+\tau-\beta)]h(\alpha)h(\beta)d\alpha d\beta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\gamma-\beta)h(\gamma)h(\beta)d\gamma d\beta
\end{aligned}
$$

$$\boxed{R_Y(\tau) = R_X(\tau) * h * \tilde{h},}$$

where $\tilde{h}(t) = h(-t)$.

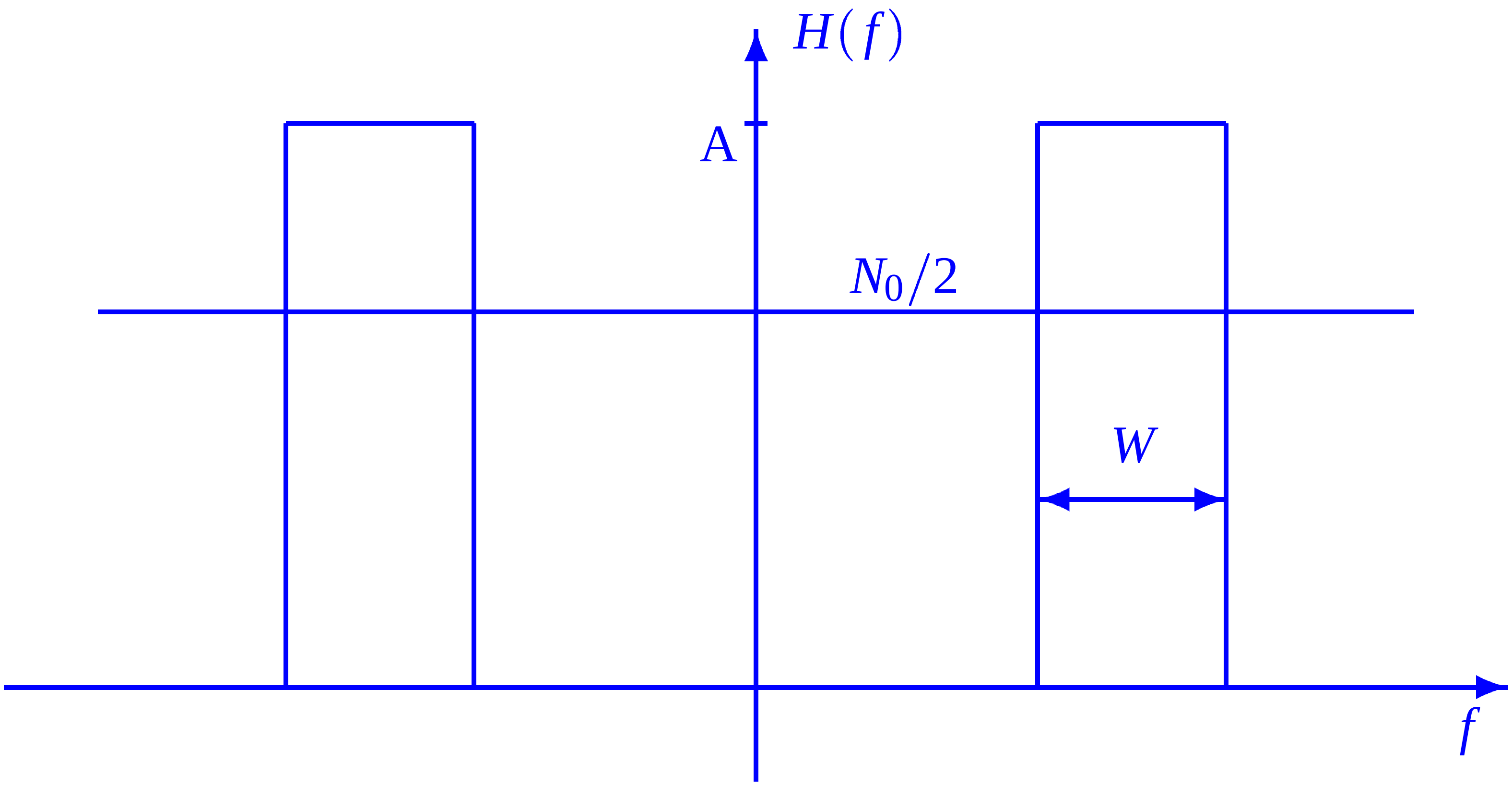
At any particular time the output due to noise alone is a random variable with a certain density function. The mean of the output is the convolution of the mean of the input signal with the impulse response of the system. The variance of the output is

$$\sigma^2 = \mathrm{Var}[Y(t)] = R_Y(0) = \int_{-\infty}^{\infty} R_X(\beta-\gamma)h(\gamma)h(\beta)d\gamma d\beta$$
$$= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)df$$

For the case when the noise is white with power spectral density $N_0/2$ the variance of the output is

$$\boxed{\sigma^2 = \mathrm{Var}[Y(t)] = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(\gamma)d\gamma = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df}$$

Example: Ideal Brickwall Filter



$$\sigma^2 = A^2 W N_0$$

(A filter for which the noise variance is σ^2 but does not have the brickwall shape is said to have noise bandwidth $\sigma^2/(A^2 N_0)$ where A is the peak output).

Example

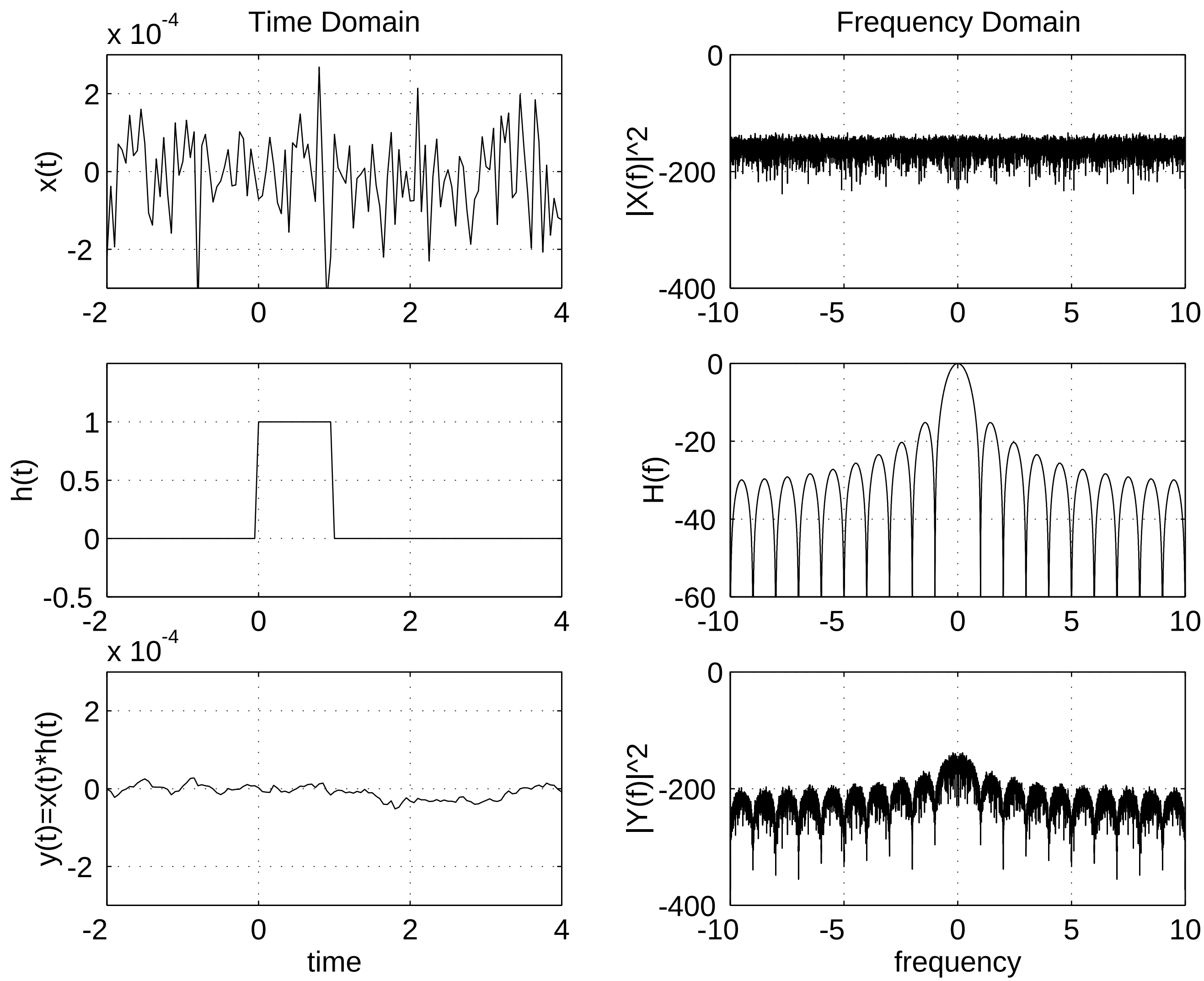


Figure 16: Filtering of Noise in Time and Frequency Domain.

Gaussian Density

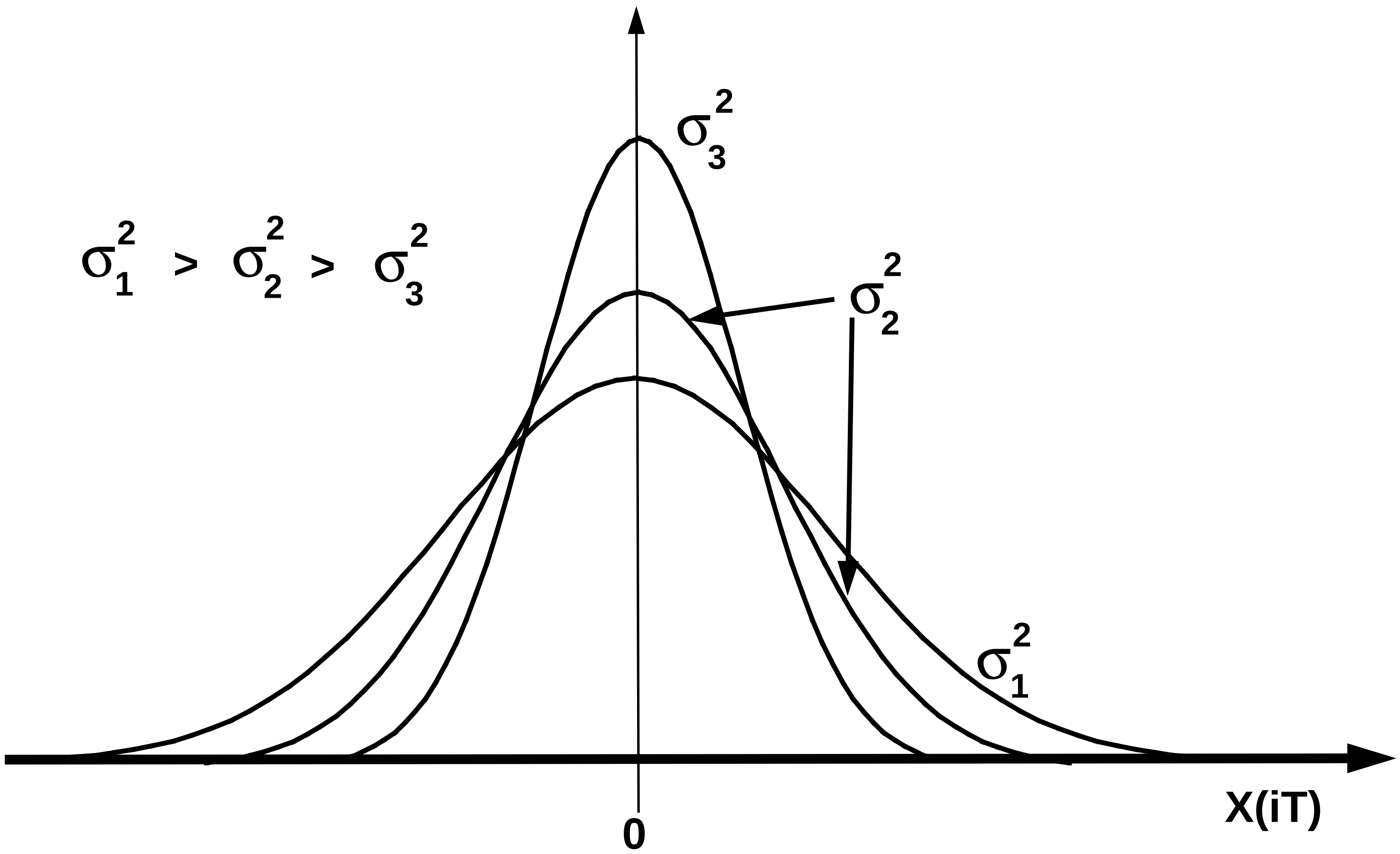


Figure 17: Probability Density of Noise with Different Variances.

Gaussian Density

If η is a Gaussian distributed random variable with mean μ and variance σ^2 then we can calculate various probabilities involving η. In particular

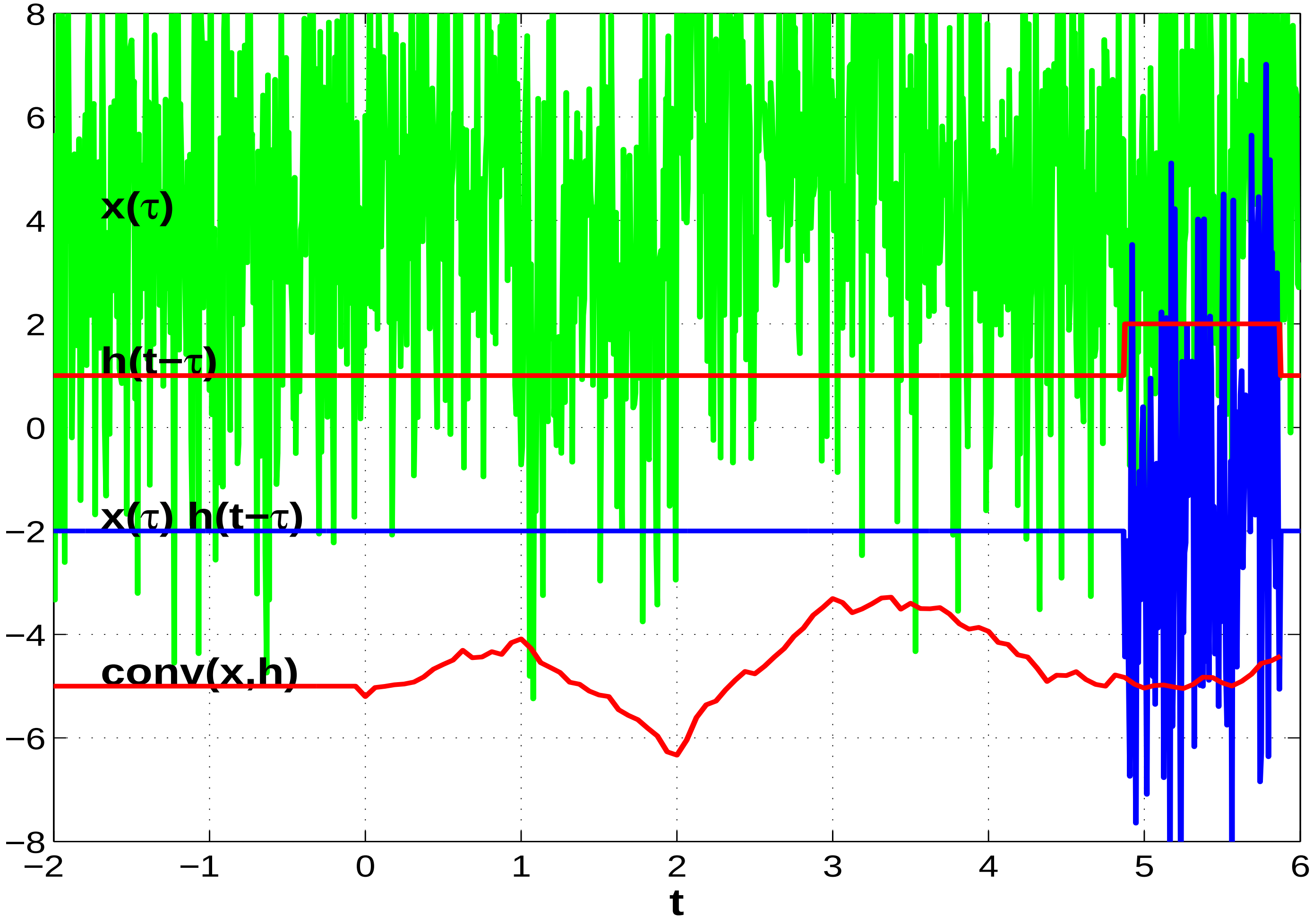
$$\begin{aligned} P\{\eta < x\} &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(w-\mu)^2/2\sigma^2} dw \\ &= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw \\ &= \Phi(\frac{x-\mu}{\sigma}) \\ &= Q(-\frac{x-\mu}{\sigma}) \end{aligned}$$

where

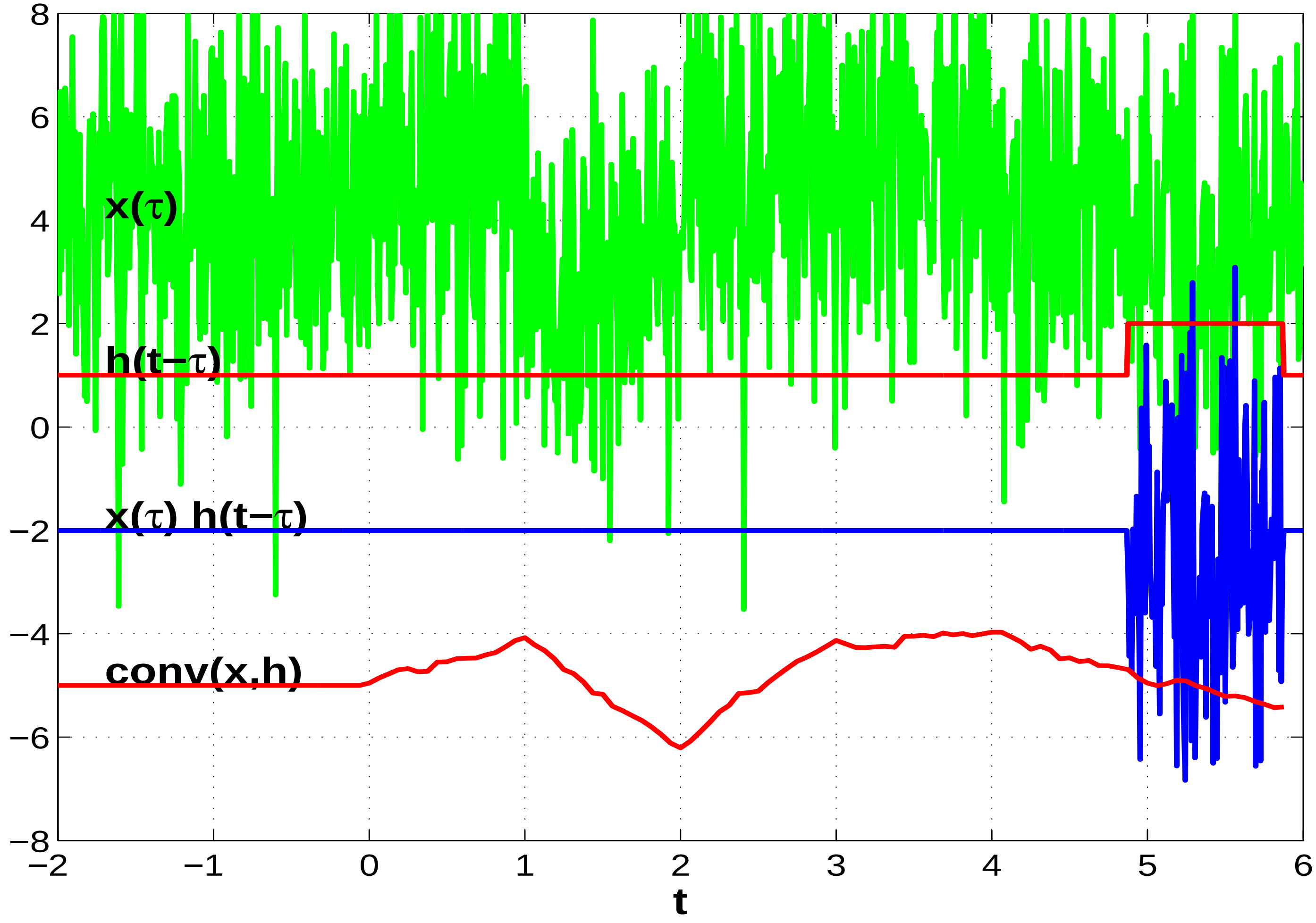
$$\Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw$$

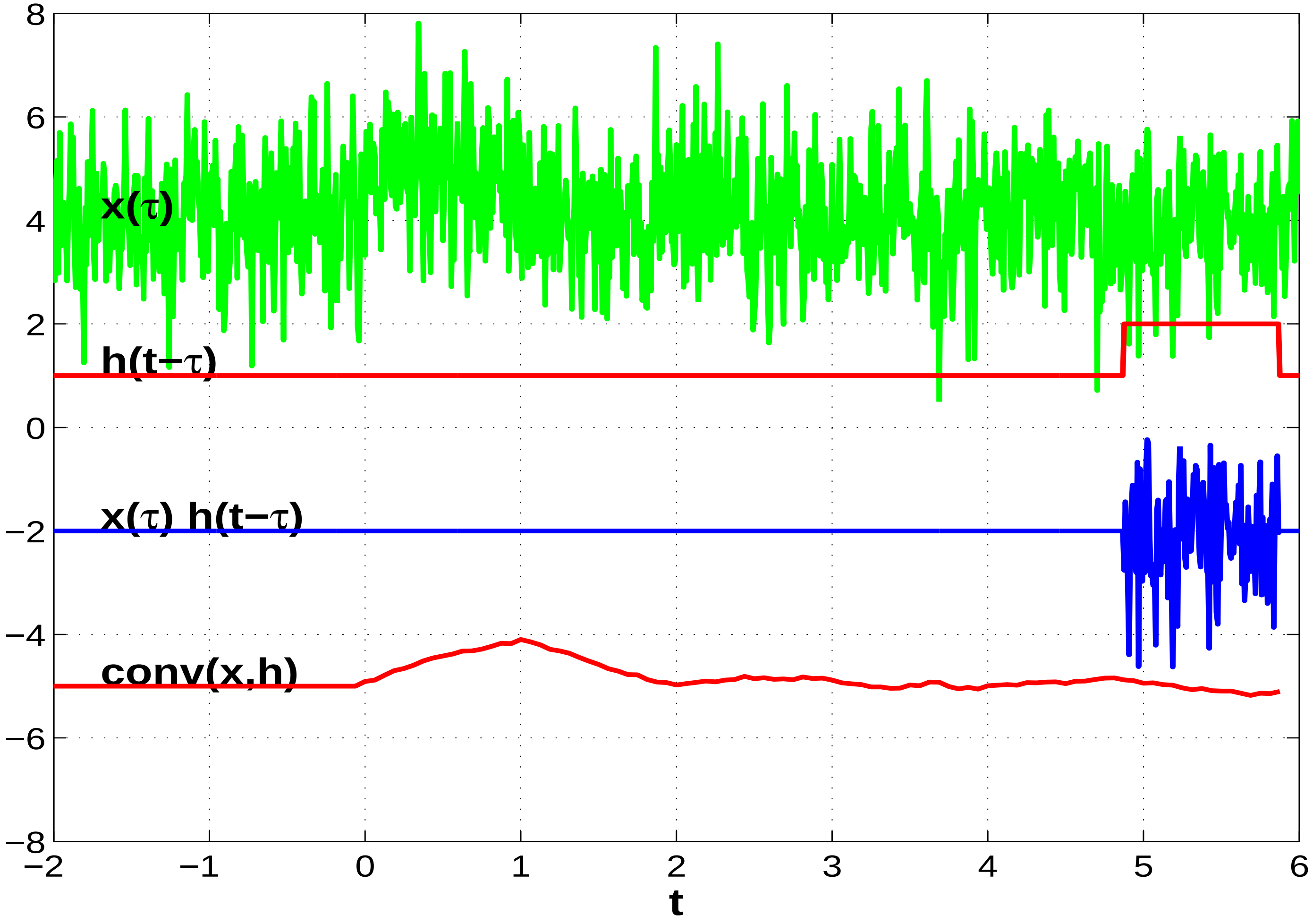
Example 1



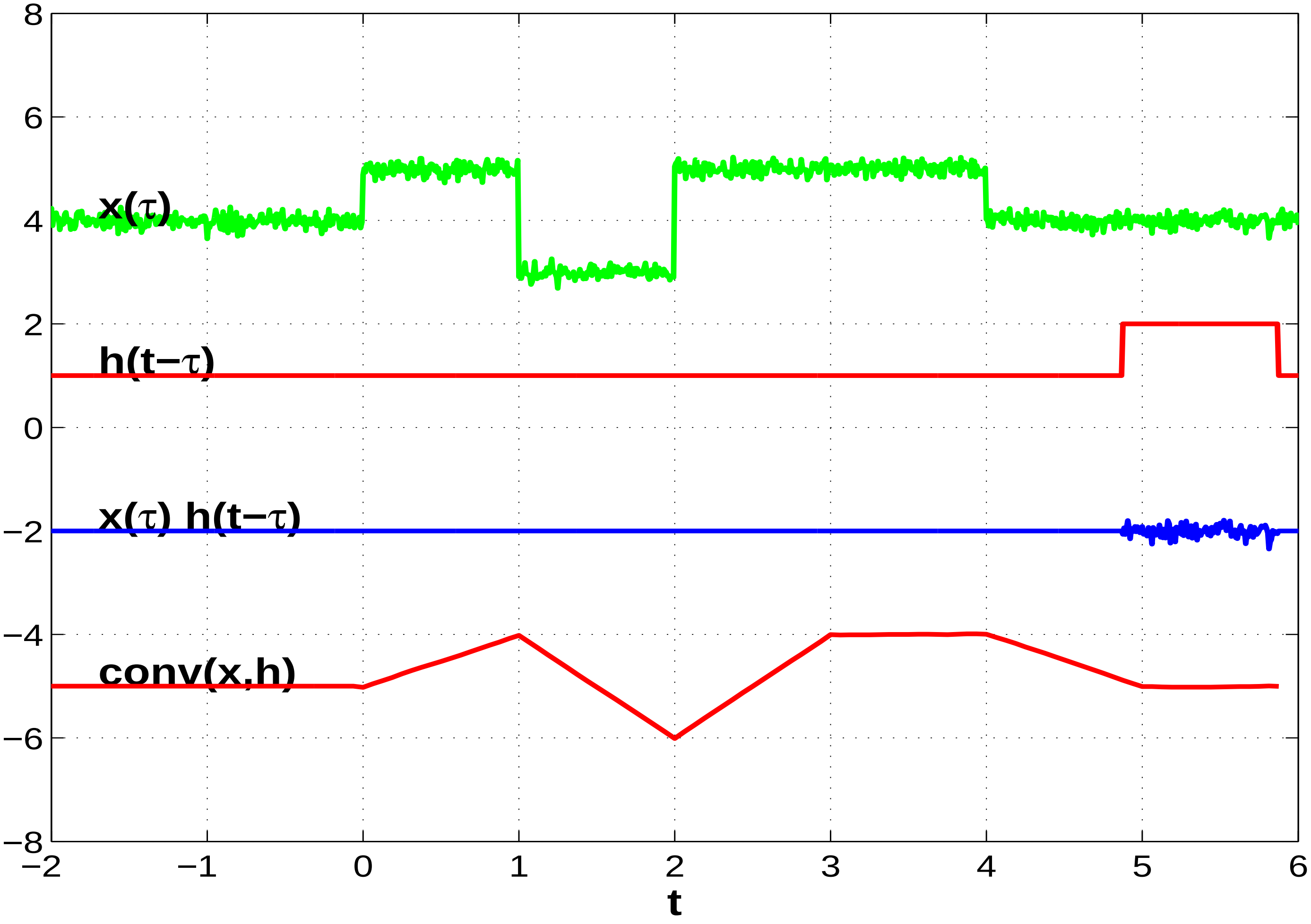
Example 2



Example 3



Example 4



Example 5