Lecture 3

Goals:

- Be able to determine the error probability in a system with two signals and a receiver filter in the presence of additive white Gaussian noise.

Baseband Modulation

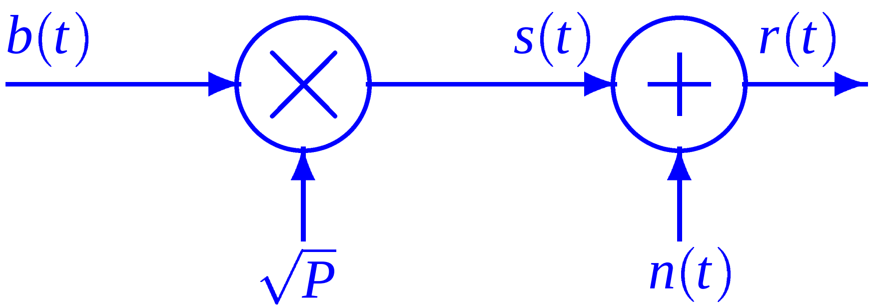


Figure 18: Baseband Communication System

$$b(t) = \sum_{l=-\infty}^{\infty} b_l \, p_T(t - lT), \quad b_l \in \{+1, -1\}.$$

$$p_T(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise.} \end{cases}$$

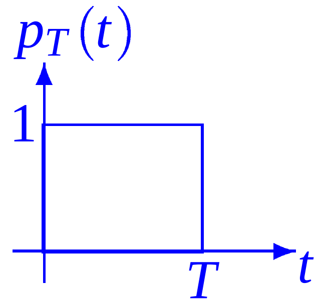
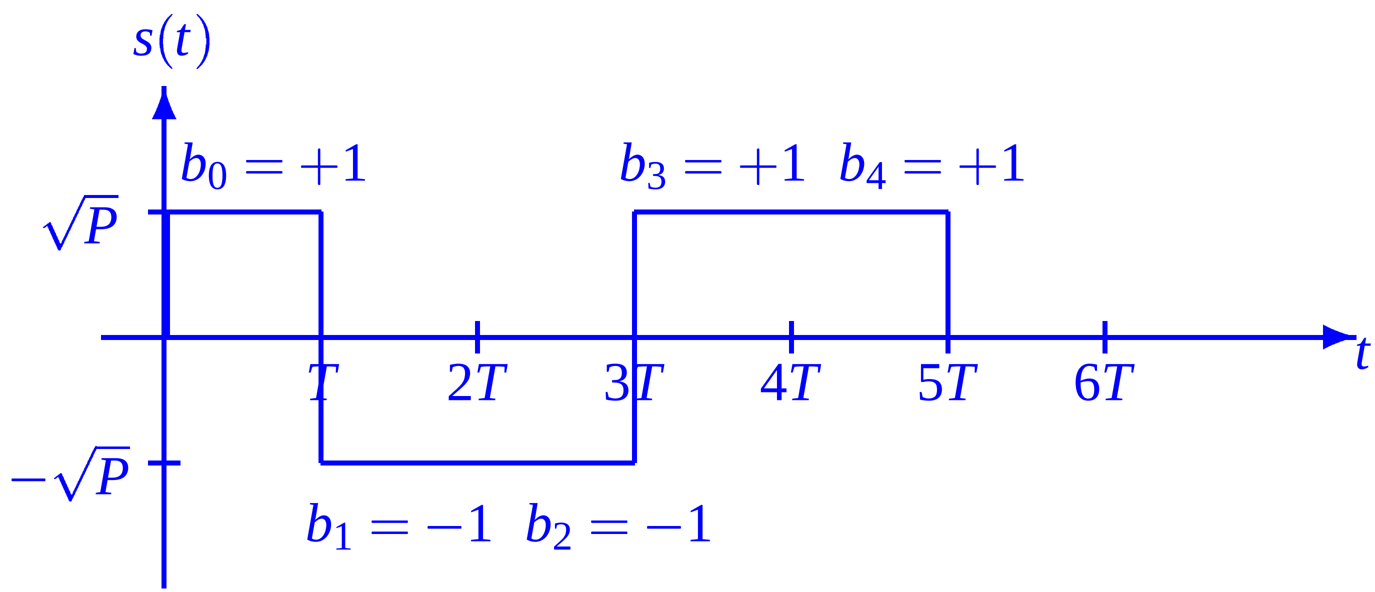


Figure 19: Pulse waveform

$$s(t) = \sqrt{P}\, b(t).$$

The signal has power P. The energy of the transmitted bit is $E = PT$.


$s(t)$
$b_0 = +1$
$b_3 = +1$ $b_4 = +1$
$\sqrt{P}$
t
T $2T$ $3T$ $4T$ $5T$ $6T$
$-\sqrt{P}$
$b_1 = -1$ $b_2 = -1$

Demodulator

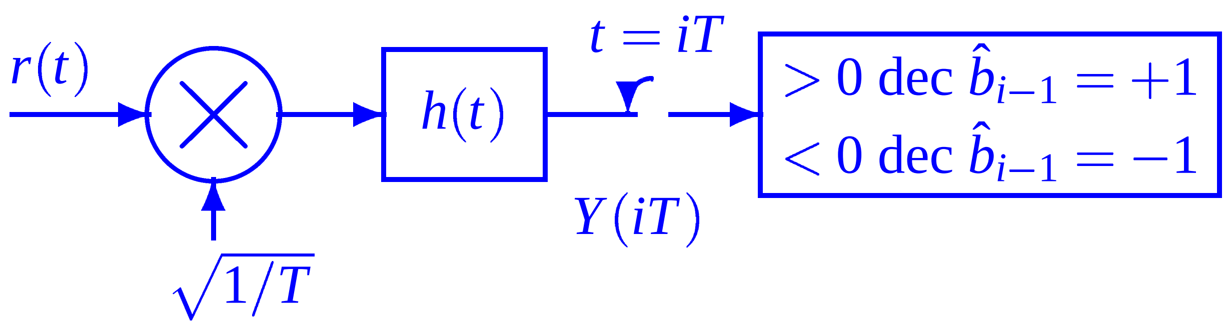


Figure 20: Demodulator for Baseband System

The output of the filter is used to decide the data bit transmitted. We let $\hat{b}_i$ denote the decision as to which bit was transmitted (during the interval $(iT, (i+1)T])$. If $\hat{b}_i = b_i$ then the correct decision was made. If $\hat{b}_i \neq b_i$ then an error was made.

The filter is "matched" to the baseband signal being transmitted. For simple rectangular type signals this is just a rectangular pulse of duration T. The impulse response is $h(t) = p_T(t)$. The output of the filter due only to the transmitted signal is given by

$$\begin{aligned}
\hat{s}(t) &= \int_{-\infty}^{\infty} \sqrt{1/T}\, h(t-\tau) r(\tau) d\tau \\
&= \int_{-\infty}^{\infty} \sqrt{1/T}\, p_T(t-\tau) r(\tau) d\tau \\
&= \int_{t-T}^{t} \sqrt{1/T}\, r(\tau) d\tau
\end{aligned}$$

That is, the filter is essentially a sliding integrator, integrating over the last T seconds.

Consider first transmitting just a single pulse (beginning at $t = 0$ and ending at $t = T$) with amplitude $\sqrt{P}\, b_0$, $b_0 \in \{+1, -1\}$. The output of the filter would have the form

$$\hat{s}(t) = \int_{t-T}^{t} \sqrt{P/T}\, b_0\, p_T(\tau) d\tau$$

That is, the filter is essentially integrating a square wave over T second intervals. The output would be maximum when the filter integrated from 0 to T, i.e. at $t = T$ in which case the output would be $(\sqrt{P/T})T = \sqrt{E}$. In general the output would look like a triangular function shown below (for $b_0 = +1$).

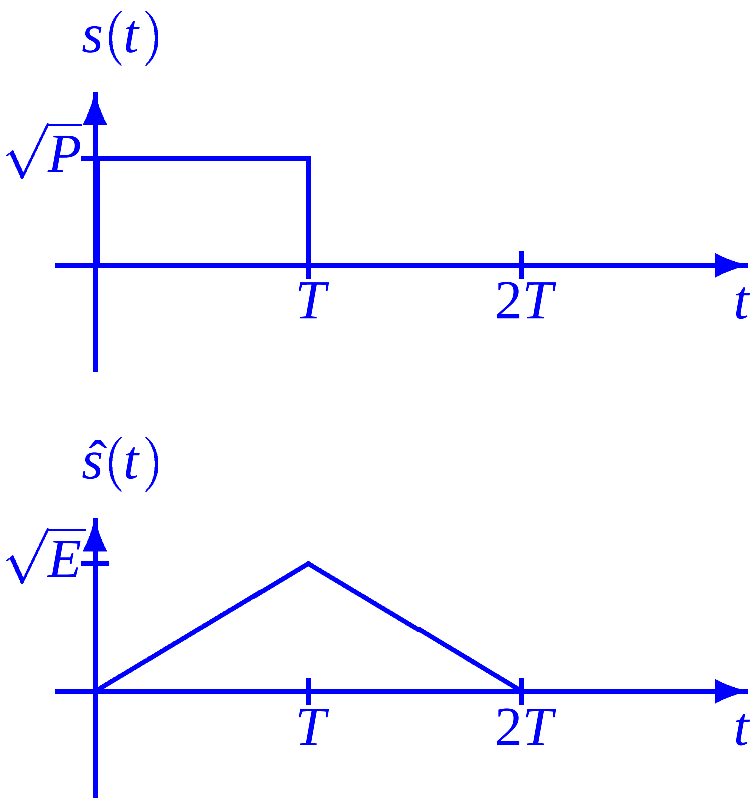
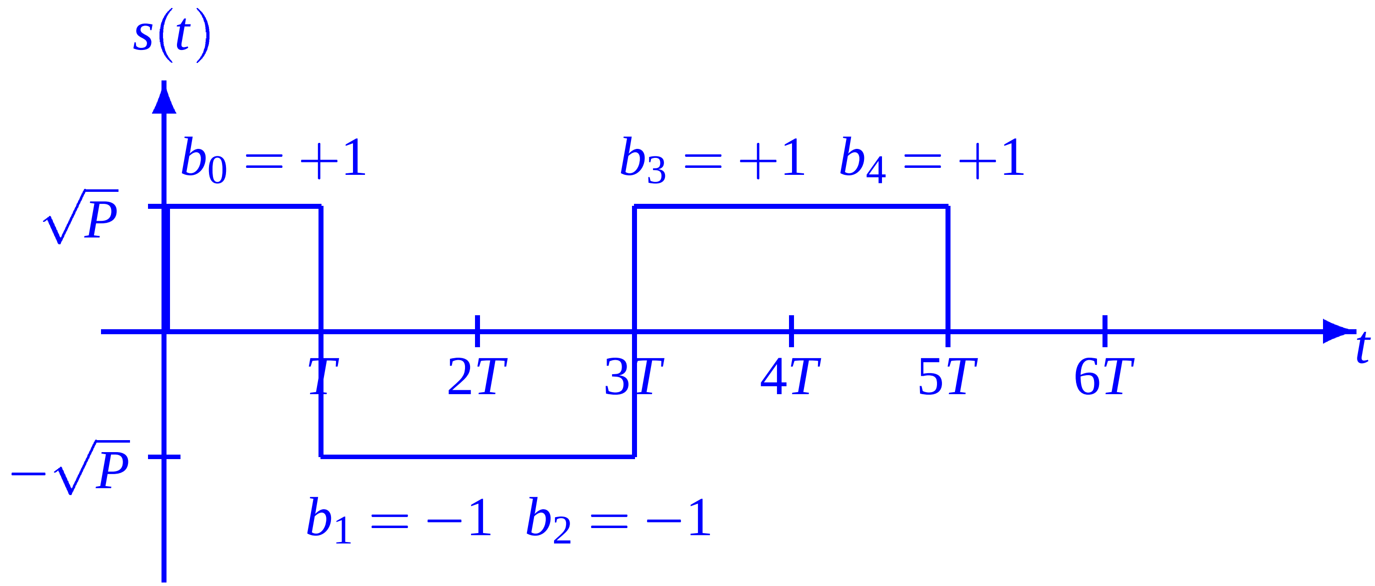
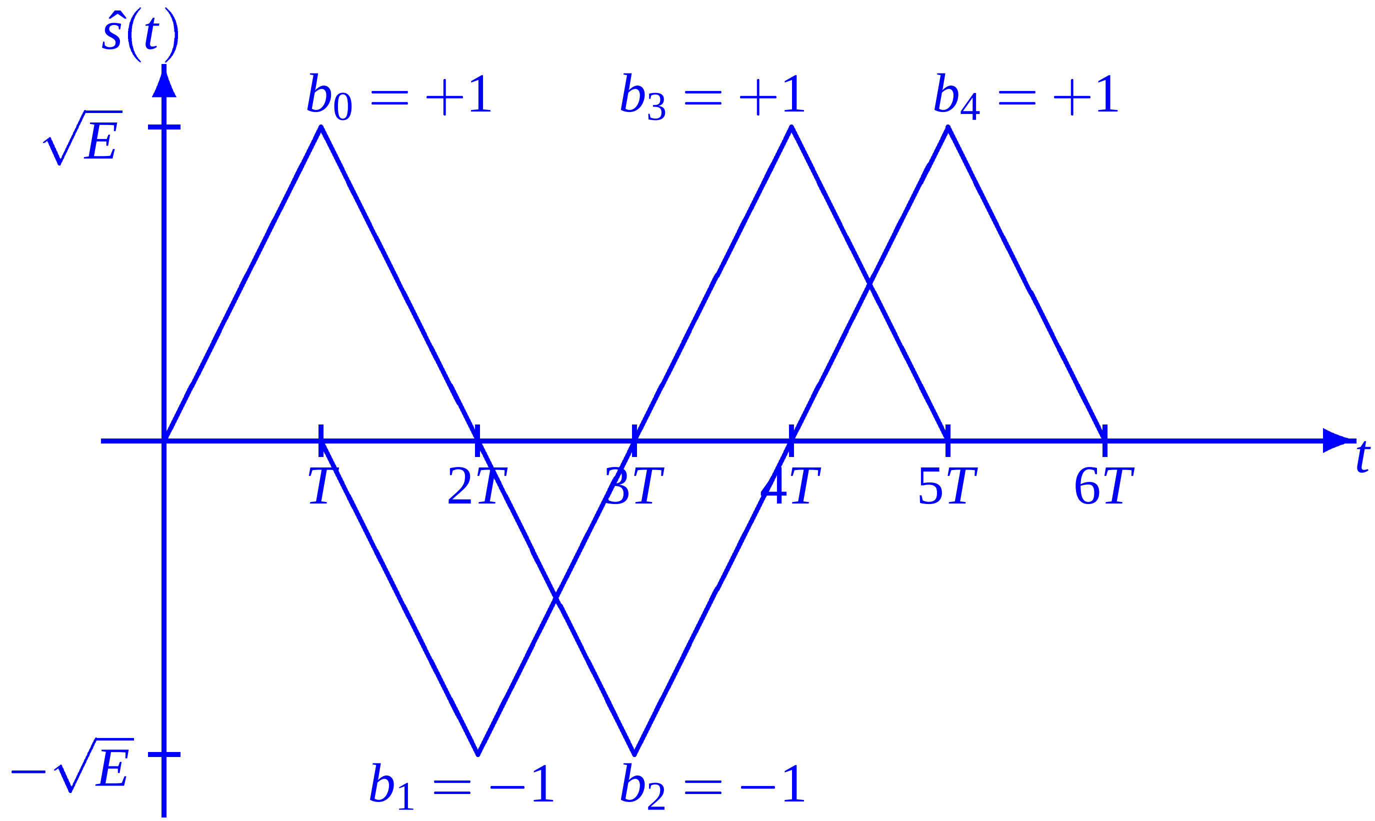


Figure 21: Matched Filter Outputs

We can construct a sequence of pulses by adding shifted versions of a single pulse. Because filtering is a linear and time-invariant operation the output due to the sum of shifted versions of the input is the corresponding sum of the shifted version of the output due to a single pulse.

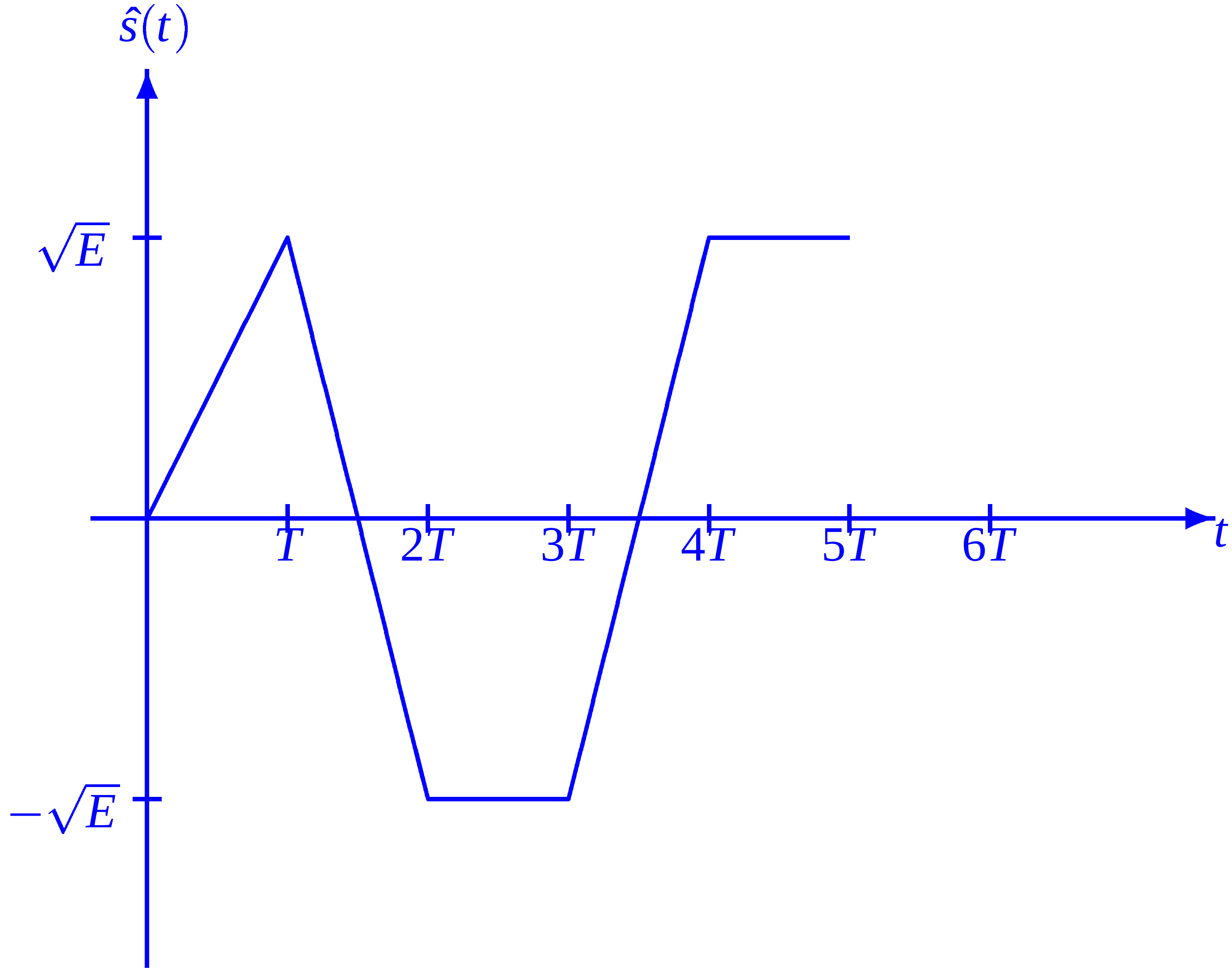
If now the transmitted signal was a sequence of pulses of various amplitudes the corresponding outputs would look like


$s(t)$
$b_0 = +1$
$b_3 = +1$
$b_4 = +1$
$\sqrt{P}$
t
T
$2T$
$3T$
$4T$
$5T$
$6T$
$-\sqrt{P}$
$b_1 = -1$
$b_2 = -1$



$\hat{s}(t)$
$\sqrt{E}$
$-\sqrt{E}$
$b_0 = +1$
$b_3 = +1$
$b_4 = +1$
$b_1 = -1$
$b_2 = -1$
T
$2T$
$3T$
$4T$
$5T$
$6T$
t

The actual output is the sum of the output due to each of the pulses. Notice that when one of the signals reaches its peak all other signals are zero. That is, there is no interference from adjacent symbols. We say there is no intersymbol interference.


$\hat{s}(t)$
$\sqrt{E}$
$-\sqrt{E}$
T
$2T$
$3T$
$4T$
$5T$
$6T$
t

Often the effect of imperfect filters and imperfect sampling is best illustrated with what is called the eye diagram. The eye diagram is obtained by examining the filter output and displaying many traces of finite duration (say $2T$). Imagine examining the filter output on an oscilloscope. Below we show the eye diagram for the optimal receiver.

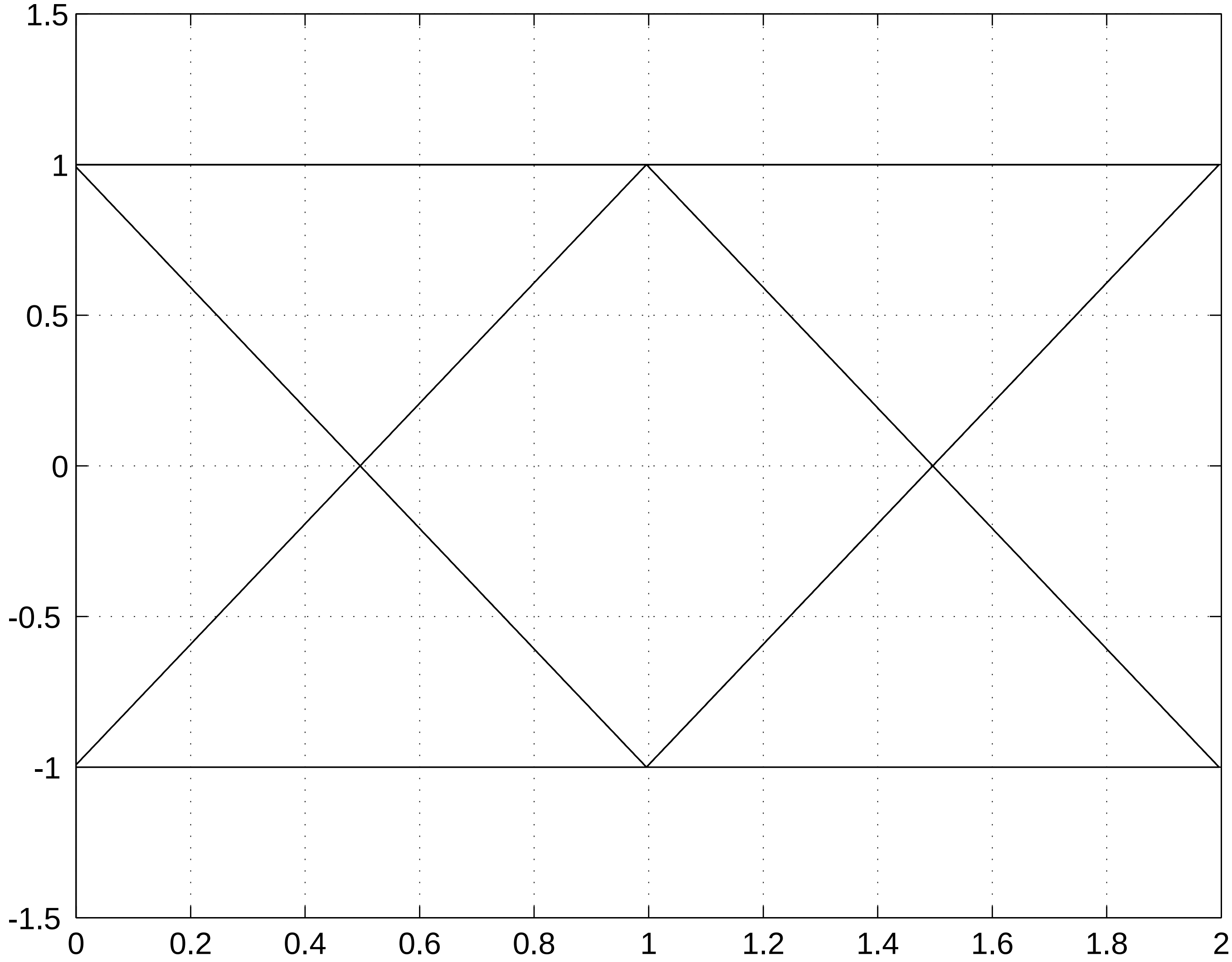


Figure 22: Eye Diagram for Optimal Matched Filter.

Now consider replacing the matched filter with a simple RC lowpass filter. The smaller the bandwidth of the filter the less noise that will appear at the filter output but with smaller filter bandwidth there will be more intersymbol interference.

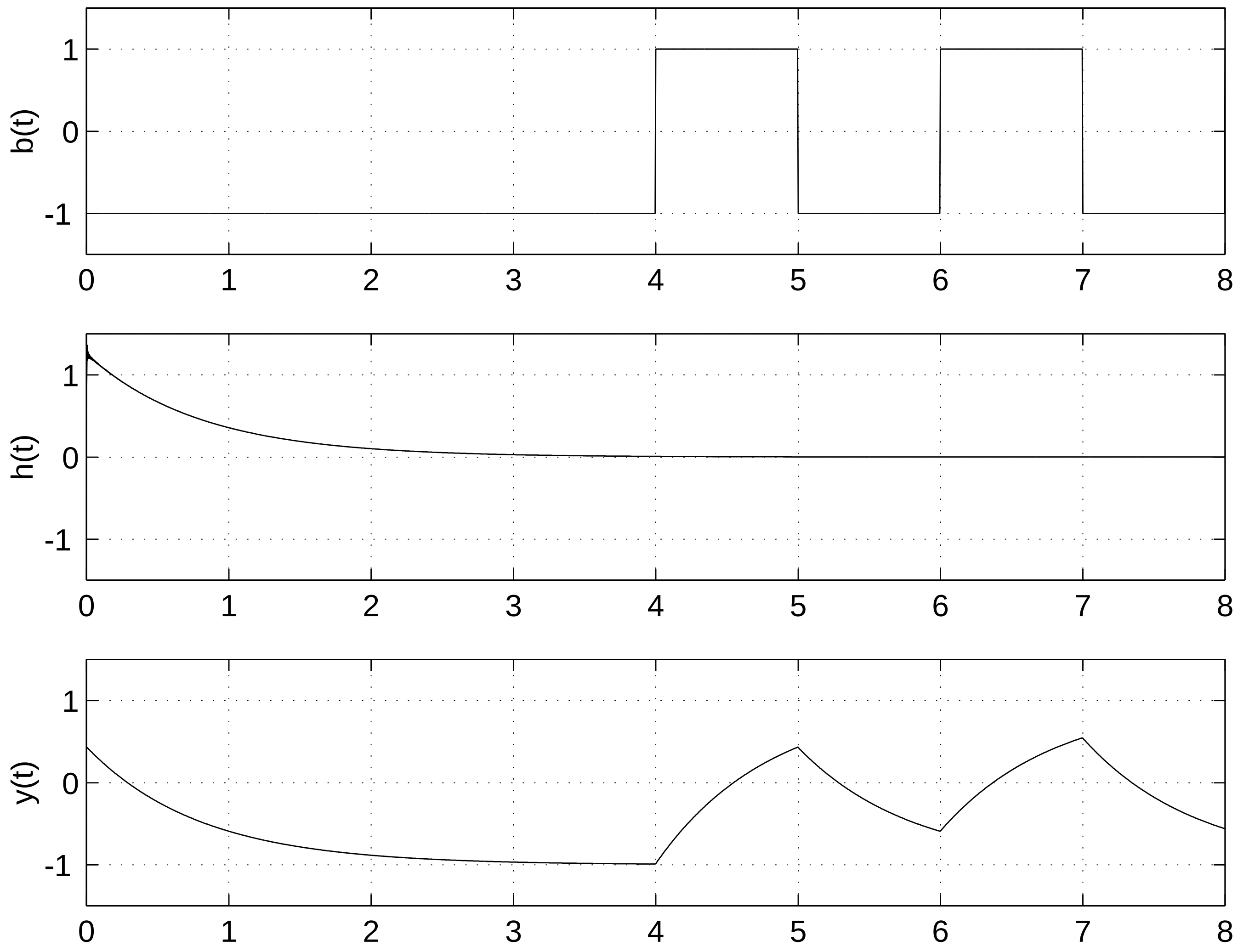


Figure 23: BPSK Signals Filtered with RC filter $RC = 1.256/T$.

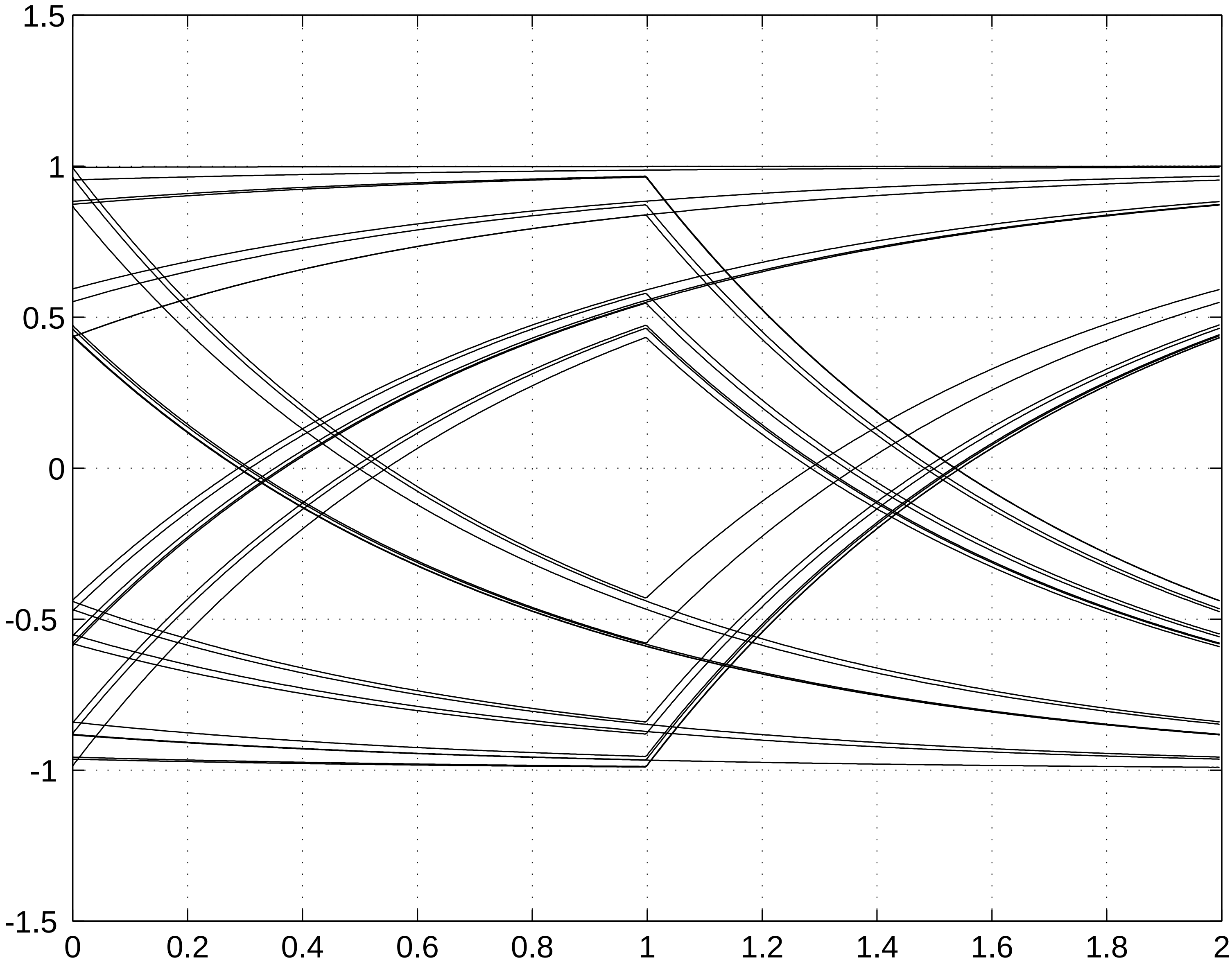


Figure 24: Eye Diagram for RC Filtered Signal.

To analyze performance we compute the output of the filter due to signal alone and the output of the filter due to noise alone. The overall output is the sum of these two outputs.

For the communication system described earlier the output of the filter $h(t)$ at time T due to signal has value either $\pm\sqrt{E}$. The noise at the output (due to the additive white Gaussian noise) has mean zero and variance

$$\mathrm{Var}[Y(T)] = \frac{1}{\sqrt{T}^2} \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0}{2}$$

The total output is the sum of the output due to signal alone plus the output due to noise alone.

If we just examine the output at multiples of the data symbol duration T then we can simply express the output of the filter as

$$Y(iT) = \sqrt{E} b_{i-1} + \eta_i$$

where η_i is the output due to the noise and has a Gaussian density with mean zero and variance $N_0/2$. This is often written as $\eta_i \sim N(0, \frac{N_0}{2})$ meaning that the variable η_i has a Normal or Gaussian distribution with zero mean and variance $N_0/2$.

The filter $h(t)$ that is matched to the signal is optimum in that it lets in the most signal and the least amount of noise.

Error Probability

The probability of error can be calculated as follows

$$
\begin{aligned}
P_{e,+1} &= P\{Y(iT) < 0 | b_{i-1} = +1\}, \\
P_{e,-1} &= P\{Y(iT) > 0 | b_{i-1} = -1\}, \\
\bar{P_e} &= P\{b_{i-1} = +1\} P_{e,+1} + P\{b_{i-1} = -1\} P_{e,-1}.
\end{aligned}
$$

Error Probability

Consider first $P_{e,+1}$.

$$
\begin{aligned}
P_{e,+1} &= P\{Y(iT) < 0 | b_{i-1} = +1\}, \\
&= P\{\sqrt{E} b_{i-1} + \eta_i < 0 | b_{i-1} = +1\}, \\
&= P\{\sqrt{E} + \eta_i < 0\}, \\
&= P\{\eta_i < -\sqrt{E}\}, \\
&= \Phi\left(\frac{-\sqrt{E}}{\sqrt{N_0/2}}\right), \\
&= \Phi\left(-\sqrt{\frac{2E}{N_0}}\right), \\
&= Q\left(\sqrt{\frac{2E}{N_0}}\right),
\end{aligned}
$$

The calculation of $P_{e,-1}$ is similar. The result is

$$\begin{aligned} P_{e,-1} &= P\{Y(iT) > 0 | b_{i-1} = -1\}, \\ &= Q(\sqrt{\frac{2E}{N_0}}), \end{aligned}$$

The average error probability is then

$$\begin{aligned} \bar{P_e} &= P\{b_{i-1} = +1\}Q(\sqrt{2E/N_0}) + P\{b_{i-1} = -1\}Q(\sqrt{2E/N_0}). \\ &= Q(\sqrt{2E/N_0}). \end{aligned}$$

Decision Statistic

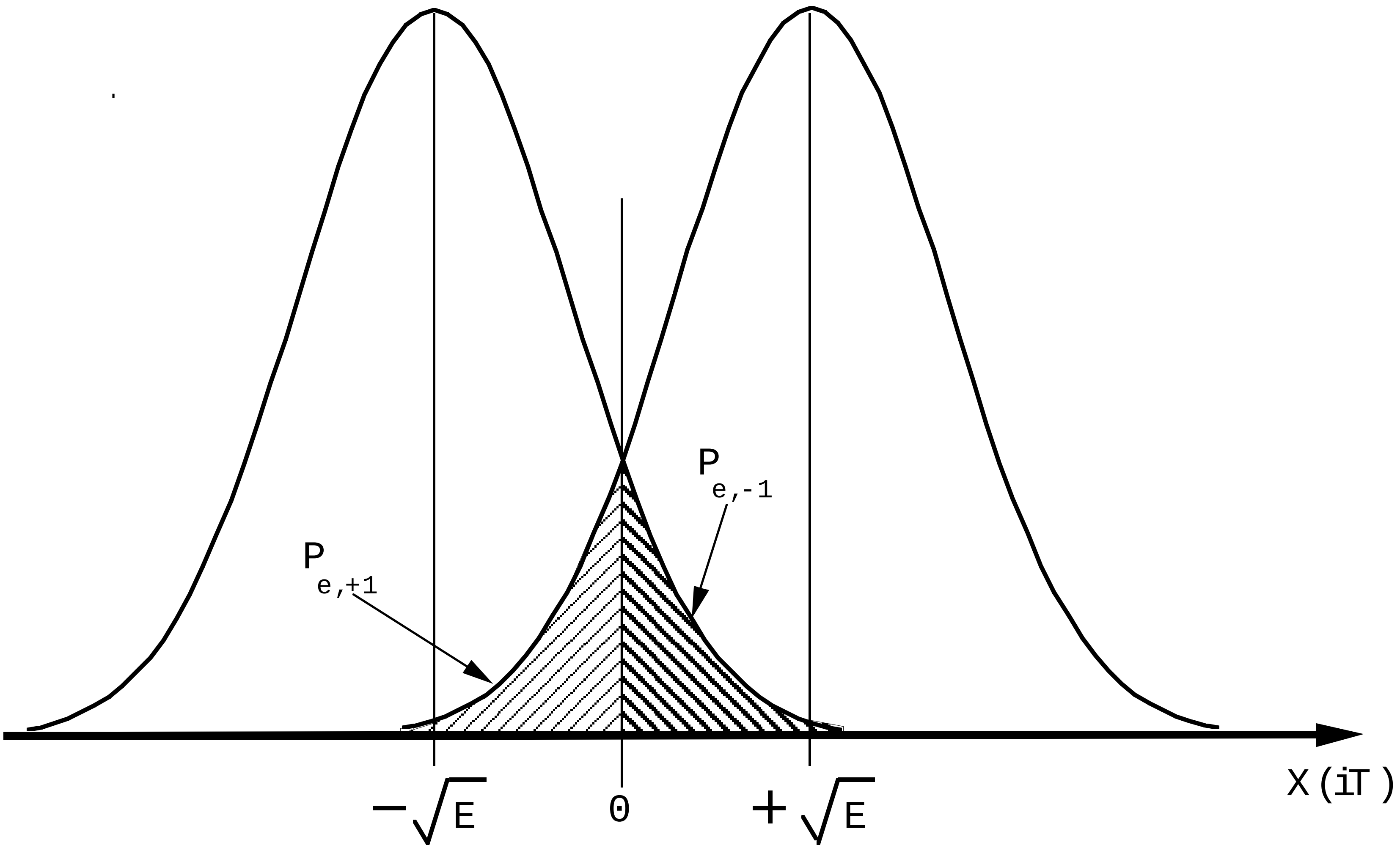


Figure 25: Probability Density of Decision Statistic for Antipodal Signalling

Bit Error Probability

The output due to signal alone is $\pm\sqrt{E}$. The error probability can be expressed in terms of the distance $d = 2\sqrt{E}$ between these two outputs and the noise variance σ^2. The output due to noise alone has variance $\sigma^2 = N_0/2$. If the noise is Gaussian then the probability of error is just the probability that the noise level causes the output to be on the opposite side of zero relative to the input. This is given by

$$P_{e,b} = Q\left(\frac{d}{2\sigma}\right) = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where $Q(x) = \int_x^\infty \frac{1}{2\pi} e^{-u^2/2} du$.

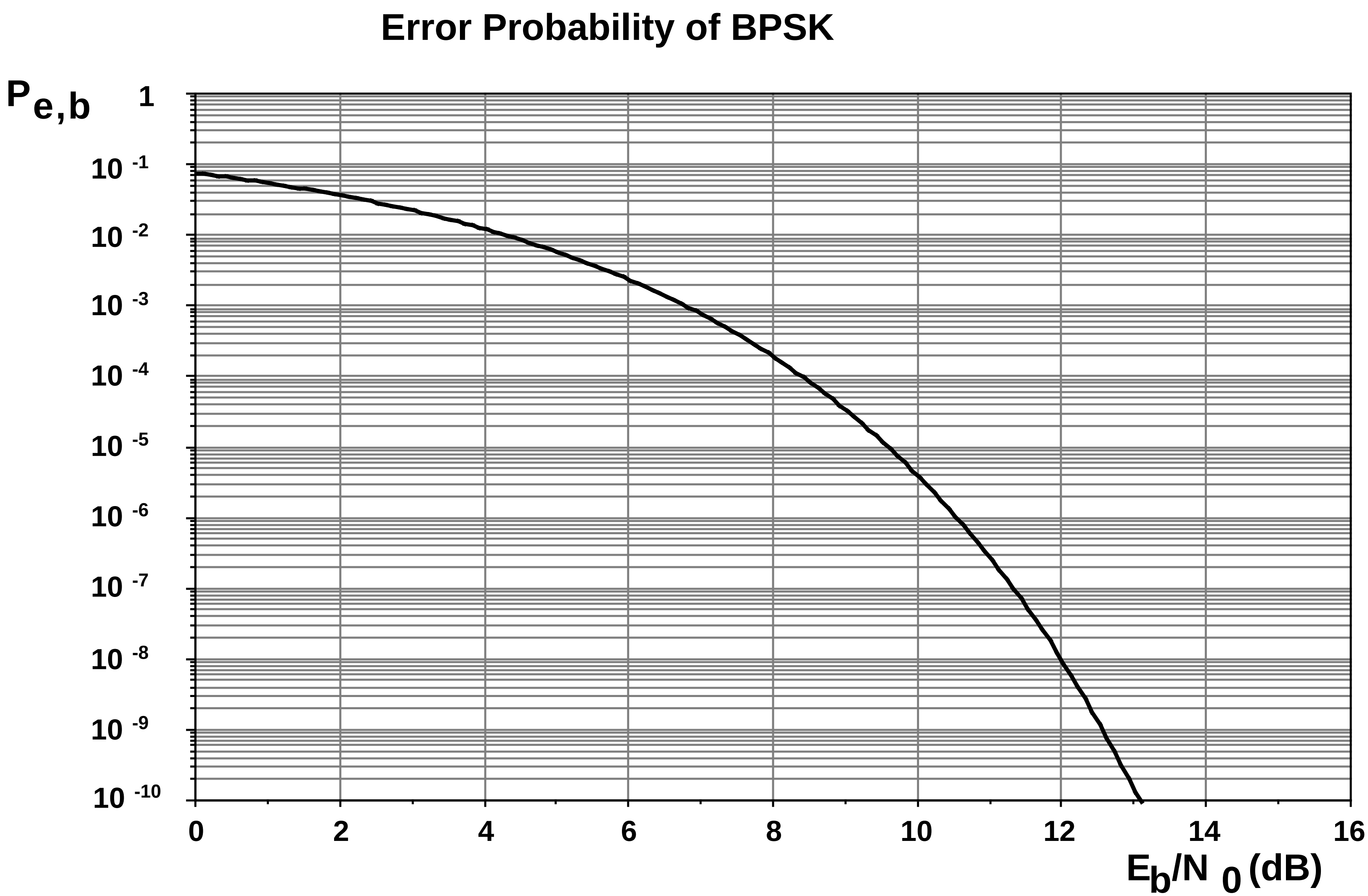


Figure 26: Error Probability for Antipodal Signalling.

For binary signals the energy transmitted per information bit E_b is equal to the energy per signal E. For $P_{e,b} = 10^{-5}$ we need a bit-energy, E_b to noise density N_0 ratio of $E_b/N_0 = 9.6$dB.

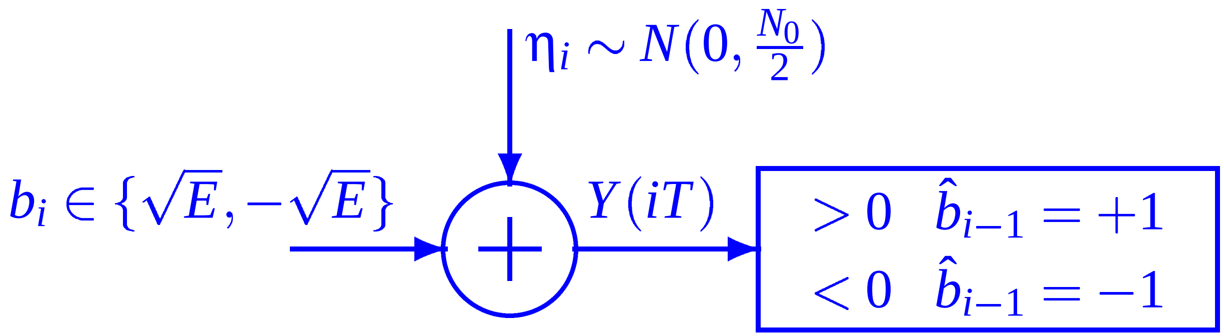
Note: $Q(x)$ is a decreasing function which is 1/2 at $x = 0$. There are efficient algorithms (based on Taylor series expansions) to calculate $Q(x)$. Since $Q(x) \leq e^{\{-x^2/2\}}/2$ the error probability can be upper bounded by

$$P_{e,b} \leq \frac{1}{2} e^{\{-E_b/N_0\}}$$

which decreases exponentially with signal-to-noise ratio.

For binary signals this is the smallest bit error probability, and the receiver shown above is optimum (in additive white Gaussian noise).

The models that we make for this modulation are the following



This channel is an additive Gaussian noise channel (with a decision device).

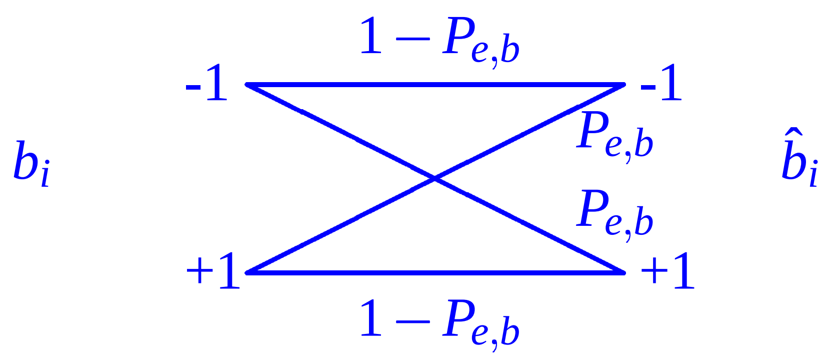


Figure 27: Binary Symmetric Channel (BSC)

This channel is a binary symmetric channel.