Lecture 4

Goals:

- Be able to determine bandwidth of digital signals
- Be able to convert a signal from baseband to passband and back

Bandwidth of Digital Data Signals

A digital data signal is modeled as a random process $Y(t)$ which is a stationary (wide sense) version of a process $X(t)$;

$$Y(t) = X(t-U)$$

where U is a random variable needed inorder to make $Y(t)$ wide sense stationary. In many digital communication systems $X(t)$ is an infinite sequence j pulses or waveforms i.e.

$$X(t) = \sum_{\ell=-\infty}^{\infty} b_\ell x(t-\ell T) \; .$$

In this case if U is uniformly distributed between 0 and T then $Y(t)$ is a wide snese stationary random process. We desire then to compute the auto correlation of $Y(t)$ and also the spectrum of $Y(t)$. Assume that $\{b_\ell\}_{\ell=-\infty}^{\infty}$ is a sequences of i.i.d. random variables with zero mean and variance σ^2 (e.g.

$P\{b_\ell = +1\} = 1/2\; P\{b_\ell = -1\} = 1/2)$. Also assume U and b_ℓ are independent.

Claim:

$$
\begin{aligned}
R_Y(\tau) &= \frac{\sigma^2}{T}\int_{-\infty}^{\infty} x(t)x(t+\tau)dt \\
S_Y(\omega) &= \frac{1}{T}|F(\omega)|^2 \text{ where } F(\omega) = \mathcal{F}\{x(t)\}
\end{aligned}
$$

Derivation: For any t and τ

$$
\begin{aligned}
E[Y(t)Y(t+\tau)] &= E\left[\sum_{\ell=-\infty}^{\infty} b_\ell x(t-\ell T-U)\sum_{m=-\infty}^{\infty} b_m x(t+\tau-mT-U)\right] \\
&= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \underbrace{E\{b_\ell b_m\}}_{\delta_{\ell m}=\left\{\begin{smallmatrix}\sigma^2 \ell=m\\ 0 \ell\neq m\end{smallmatrix}\right.} E\left[x(t-\ell T-U)x(t+\tau-mT-U)\right] \\
&= \sum_{\ell=-\infty}^{\infty} \sigma^2 E\left[x(t-\ell T-U)x(t+\tau-\ell T-U)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\ell=-\infty}^{\infty} \sigma^2 \frac{1}{T} \int_{u=0}^{T} x(t-\ell T-u)x(t+\tau-\ell T-u)du \\
&= \frac{1}{T} \sum_{\ell=-\infty}^{\infty} \sigma^2 \int_{\ell T}^{(\ell+1)T} x(t-v)x(t+\tau-v)dv \ (v=\ell T+u \ \ dv=du) \\
&= \frac{\sigma^2}{T} \int_{-\infty}^{\infty} x(t-v)x(t+\tau-v)dv \ \ (w=t-v) \\
&= \frac{\sigma^2}{T} \int_{-\infty}^{\infty} x(w)x(w+\tau)dw = \frac{\sigma^2}{T} \int_{-\infty}^{\infty} x(t)x(t+\tau)dt
\end{aligned}
$$

Thus $Y(t)$ is wide sense stationary with

$$
R_Y(\tau) = \frac{\sigma^2}{T} \int_{-\infty}^{\infty} x(t)x(t+\tau)dt
$$

Now let $f_1(t) = x(t)$ $f_2(t) = x(-t)$ then

$$
f_1 * f_2(\tau) \ = \ \int_{-\infty}^{\infty} f_1(t) f_2(\tau - t)dt = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt
$$

$$= \int_{-\infty}^{\infty} x(t+\tau)x(t)dt$$

So

$$\begin{aligned}
R_Y(\tau) &= \frac{\sigma^2}{T}(f_1 * f_2)(\tau) \\
S_Y(\omega) &= \mathcal{F}\left\{\frac{\sigma^2}{T}(f_1 * f_2)(\tau)\right\} = \frac{\sigma^2}{T}F_1(\omega)F_2(\omega) \\
F_1(\omega) &= \mathcal{F}\{x(t)\} = F(\omega) \\
F_2(\omega) &= \mathcal{F}\{x(-t)\} = F^*(\omega) \\
S_Y(\omega) &= \frac{\sigma^2}{T}|F(\omega)|^2
\end{aligned}$$

Example

$$x(t) = p_T(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere} \end{cases}$$

$b_\ell \in \{\pm 1\} \Rightarrow \sigma^2 = 1.$

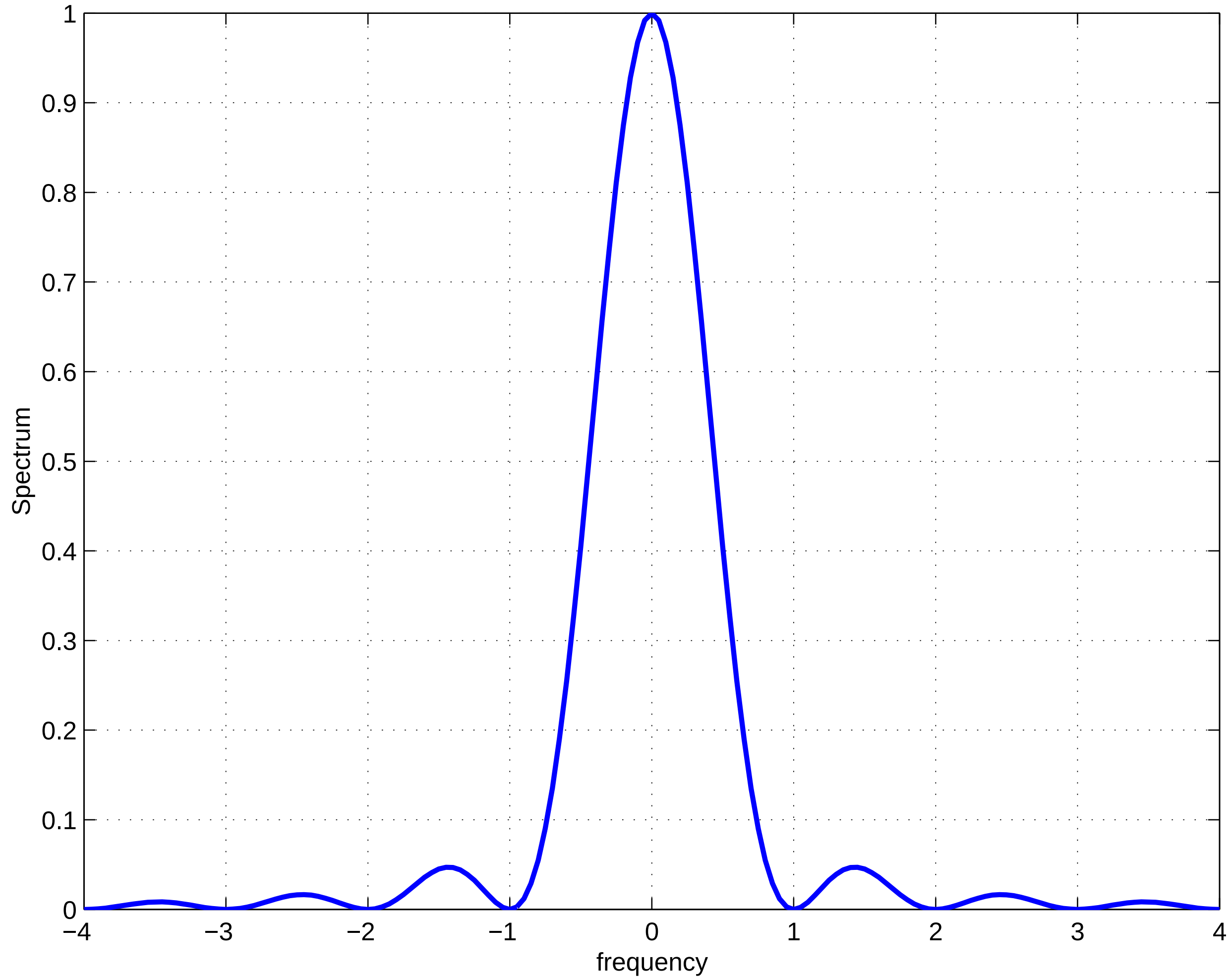
$$\int_{-\infty}^{\infty} x(t)x(t+\tau)dt = \int_0^T p_T(t+\tau)dt = \begin{cases} T-\tau & 0 \le \tau \le T \\ T+\tau & -T \le \tau \le 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$p_T(t+\tau) = \begin{cases} 1 & 0 \le t+\tau \le T \\ 0 & \text{elsewhere} \end{cases}$$

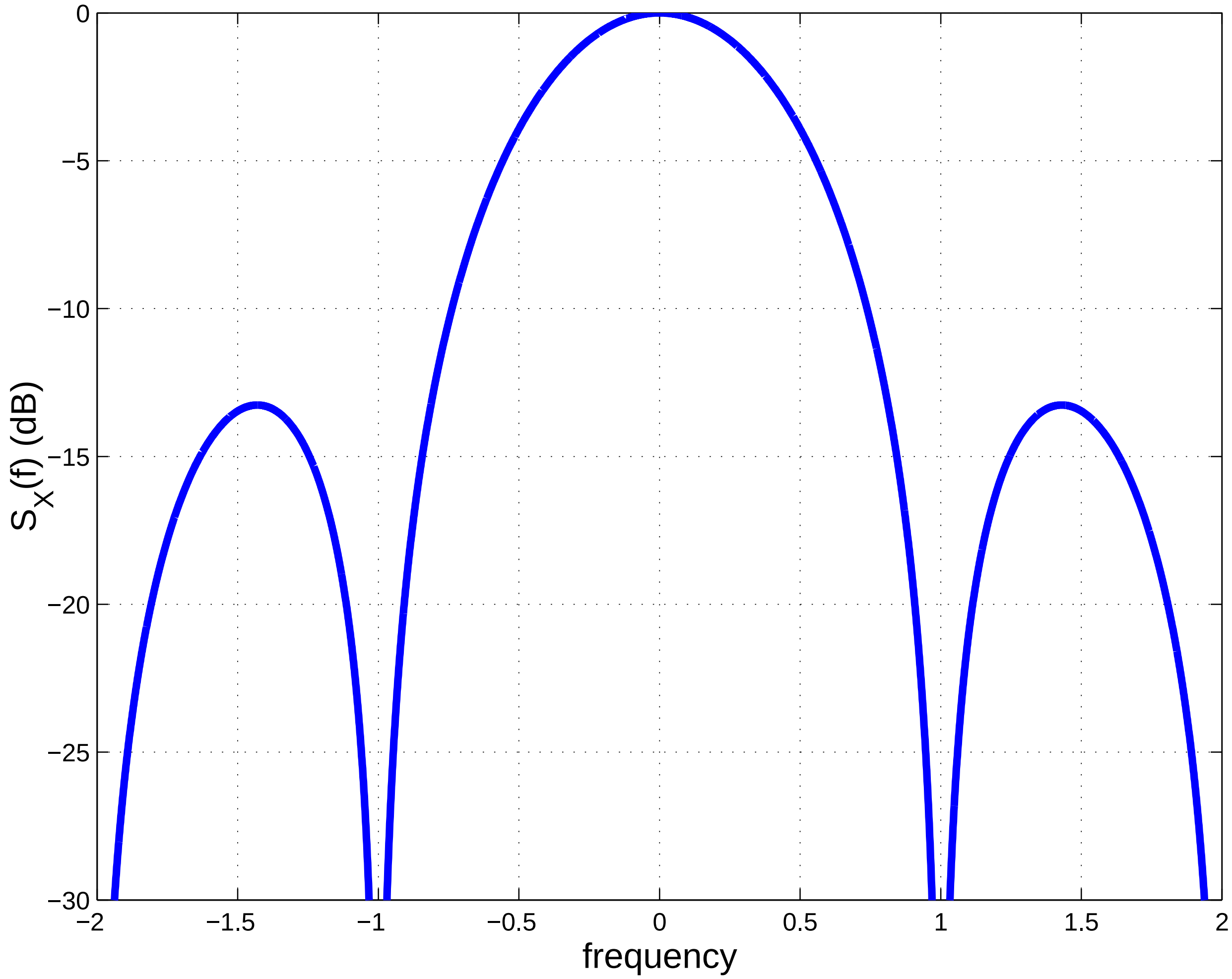
$$= \begin{cases} 1 & -\tau \le t \le T-\tau \\ 0 & \text{elsewhere} \end{cases}$$

$$
\begin{aligned}
R_Y(\tau) &= \begin{cases} \frac{1}{T}(T-|\tau|) & |\tau| \le T \\ 0 & \text{elsewhere} \end{cases} \\
S_Y(\omega) &= \frac{1}{T}|\mathcal{F}\{p_T(t)\}|^2 \\
F(\omega) &= \mathcal{F}\{p_T(t)\} = T\left[\frac{\sin \omega T/2}{\omega T/2}\right] e^{-j\omega T/2} \\
|F(\omega)|^2 &= T^2 \frac{\sin^2 \omega T/2}{(\omega T/2)^2} \\
S_Y(\omega) &= \frac{1}{T}|F(\omega)|^2 = T\frac{\sin^2 \omega T/2}{(\omega T/2)^2} = T\,\mathrm{sinc}^2 \\
S_Y(f) &= T\frac{\sin^2 \pi f T}{(\pi f T)^2} = T\,\mathrm{sinc}^2(fT)
\end{aligned}
$$

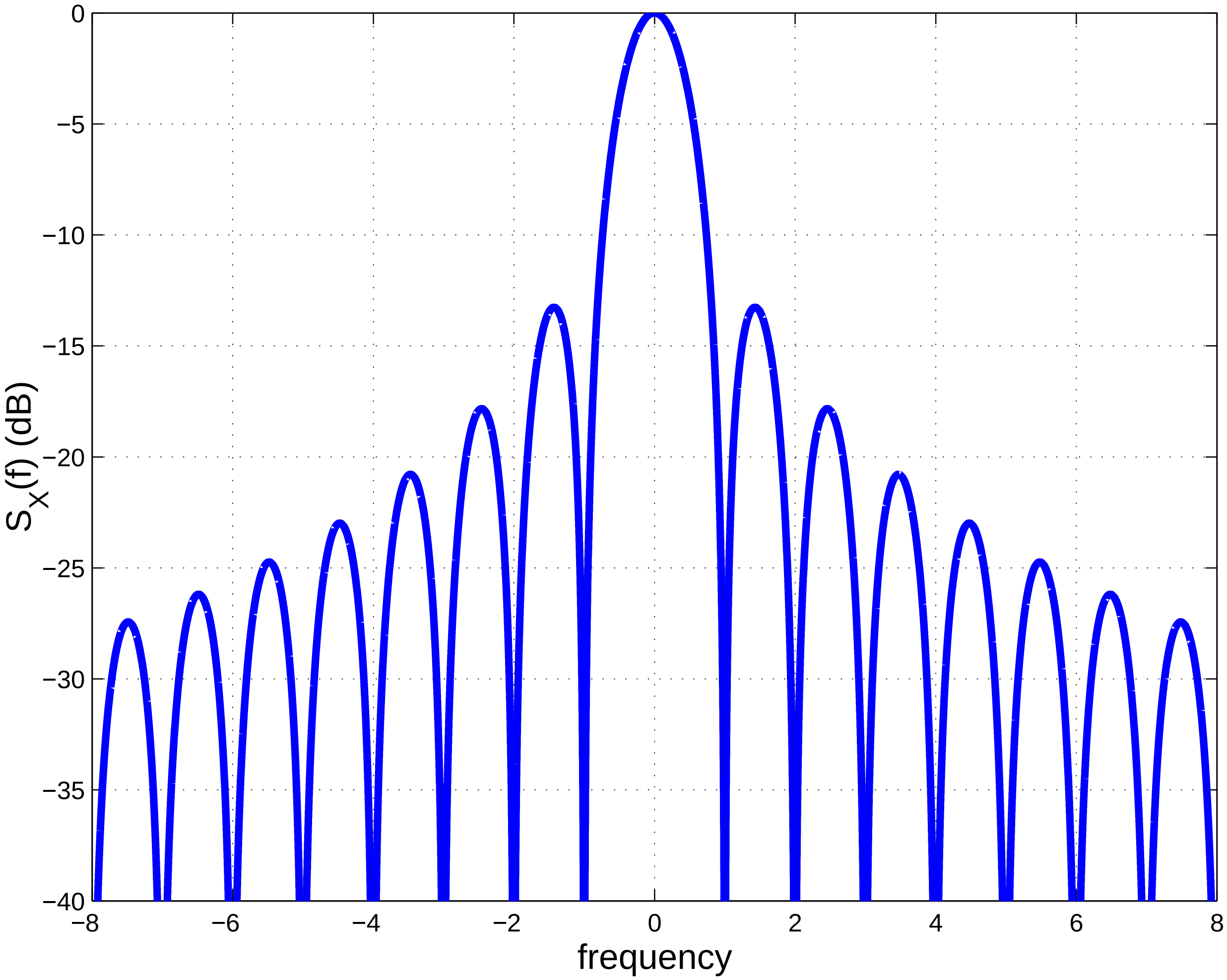
Spectrum for Rectangular Pulses



Spectrum for Rectangular Pulses

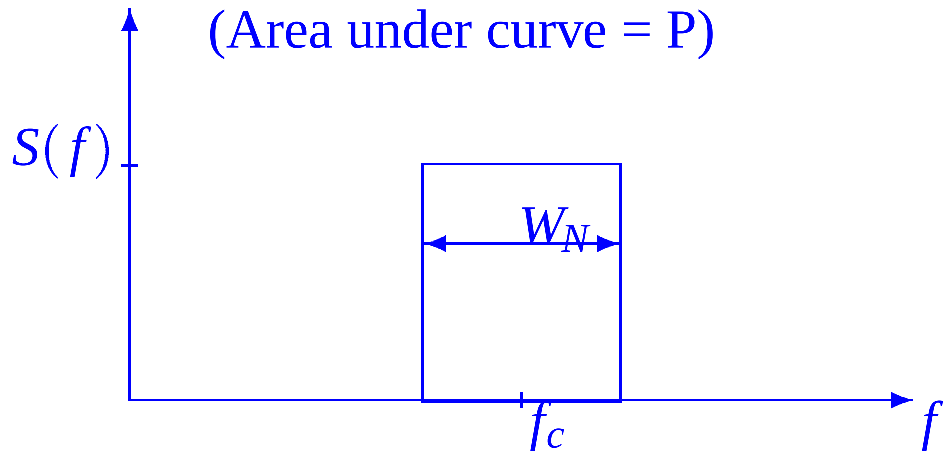


Spectrum for Rectangular Pulses



Definition of Bandwidth for Digital Signals

1. Null-to-Null bandwidth $\triangleq$ bandwidth of main lobe of power spectral density
2. 99% power bandwidth containtment $\triangleq$ bandwidth such that 1/2% of power lies above upper band limit and 1/2% lies below lower band limit
3. x dB bandwidth $\triangleq$ bandwidth such that spectrum is xdB below spectrum at center of band (e.g. 3dB bandwidth)
4. Noise bandwidth $\triangleq W_N = P/S(f_c)$ where P is total power and $S(f_c)$ is value of spectrum at $f = f_c$.



5. Gabor bandwidth $\triangleq \sigma$ where $\sigma^2 = \frac{\int_{-\infty}^{\infty}(f-f_c)^2 S(f)df}{\int_{-\infty}^{\infty}S(f)df}$

6. Absolute bandwidth $\triangleq W_A = \min\{W : S(f) = 0 \forall |f| > U\}$

Rectangular Pulse Example

$$S_Y(f) \quad = \quad T\frac{\sin^2 \pi f T}{(\pi f T)^2} = T\,\text{sinc}^2(fT)$$

$\text{sinc}(\pi(f-f_c)T = 0$ at $\pi(f-f_c)T = h\pi \; h = \pm 1, \pm 2, \pm 3, \ldots$
$f = f_c + \frac{n}{T}, \;\; n = \pm 1, \pm 2, \ldots$

Null to null bandwidth $\stackrel{\Delta}{=}$ width of main lobe of spectral density.

For PSK null to null bandwidth = $\frac{2}{T}$

Fractional Power containment Bandwidth $\stackrel{\Delta}{=}$ width of frequency band which leaves 1/2% of singal power above upper band limit and 1/2% of signal power below band limit

For PSK 99% energy bandwidth = $\frac{20.56}{T}$

We would like to find modulation schemes which decrease the bandwidth while retaining acceptable performance

Modulation	1	2	3 35dB	4	5	3 3dB
BPSK	2.0	20.56	35.12	1.00	∞	0.88
QPSK	1.0	10.28	17.56	0.50	∞	0.44
MSK	1.5	1.18	3.24	0.62	$(0.5)\frac{1}{2}$	0.59

These three modulation schemes all have same error probability.

MSK has minimum possible Gabor bandwidth over all modulation schemes whose basic pulse is limited to 2T seconds.

All of these have infinite absolute bandwidth.

Shannon's theorem revisited

Theorem: (Shannon) For a white Gaussian noise channel there exist signals with absolute bandwidth W conveying $R = \frac{1}{T}$ bits of information per second with arbitrarily small error probability provided

$$R = \frac{1}{T} < W \log_2(1 + \frac{E_b}{WTN_0})$$

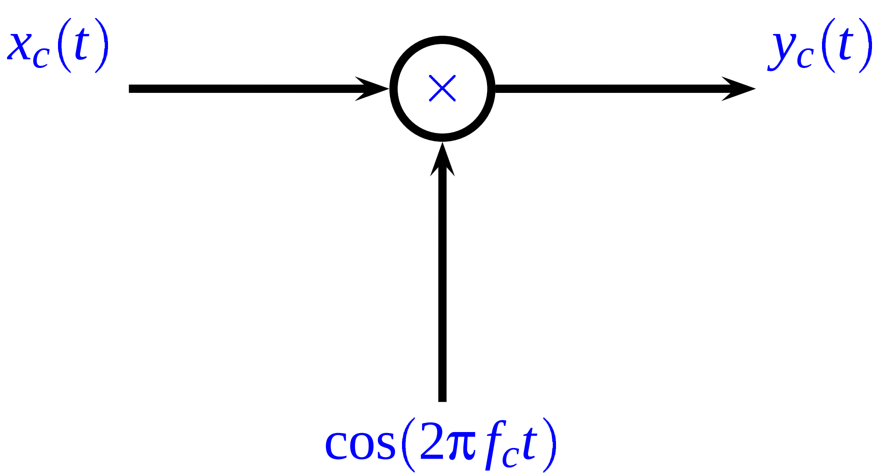
where $E_b = PT$ is the energy per data bit and P is the power of the signal

$$\begin{aligned} P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} s^2(t)dt \\ R &< W \log\left(1 + \frac{R}{W}\frac{E_b}{N_0}\right) \to \frac{P}{N_0 \ln 2} \text{ as } W \to \infty \end{aligned}$$

$$
\begin{aligned}
R/W &< \log\left(1+\frac{R}{W}\frac{E_b}{N_0}\right) \\
&\Rightarrow E_b/N_0 > \frac{2^{R/W}-1}{R/W} \to \ln 2 \text{ as } \frac{R}{W} \to 0 \\
E_b/N_0 &= \ln(2) \Rightarrow (E_b/N_0)_{\mathrm{dB}} = 10\log_{10}(\ln(2)) = -1.6dB
\end{aligned}
$$

Up and Down Conversion

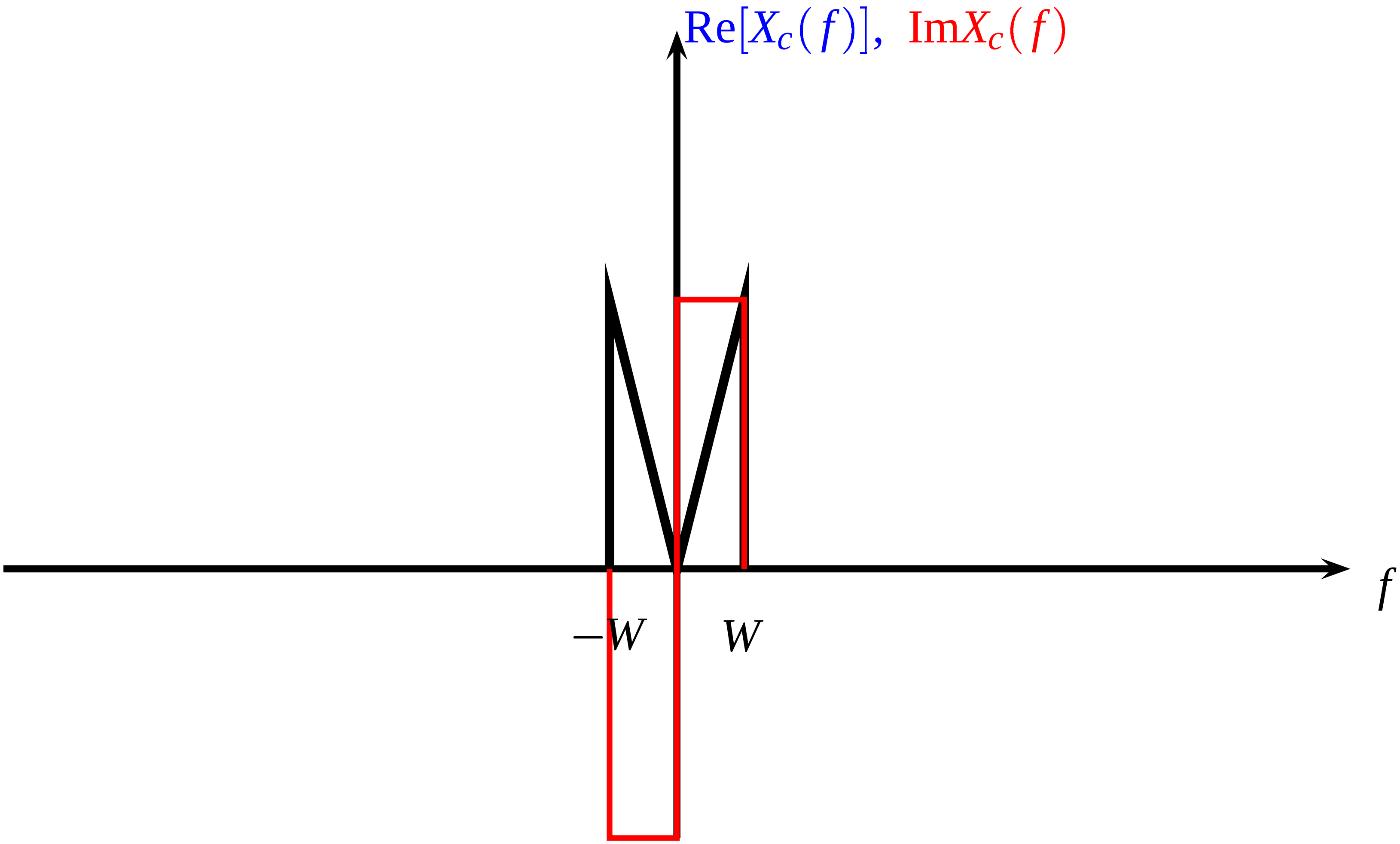
In communication systems typically the signal are generated at baseband and then up converted to the desired carrier frequency. At the receiver this process is reversed.

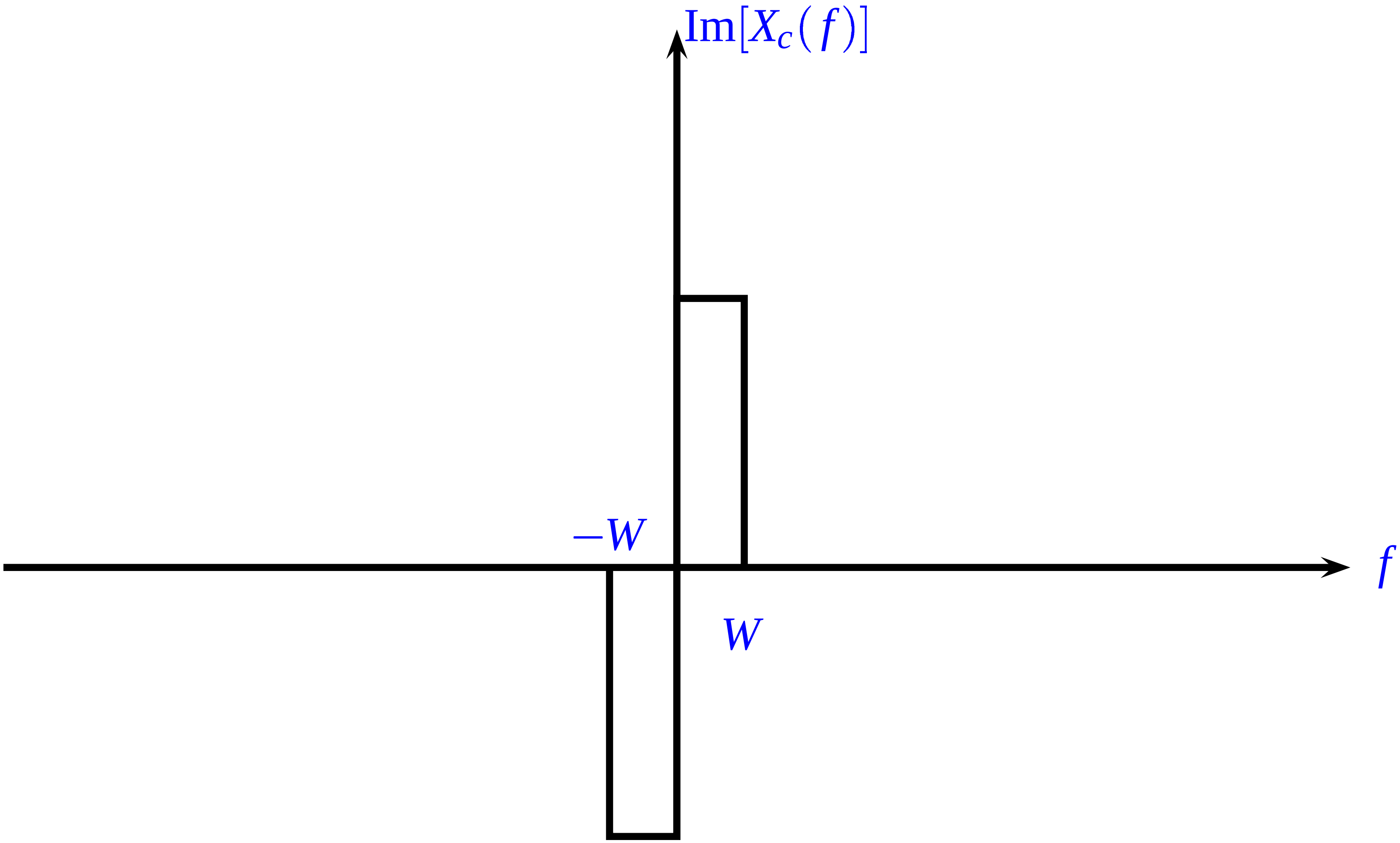


$$y_c(t) = x_c(t)\cos(2\pi f_c t)$$

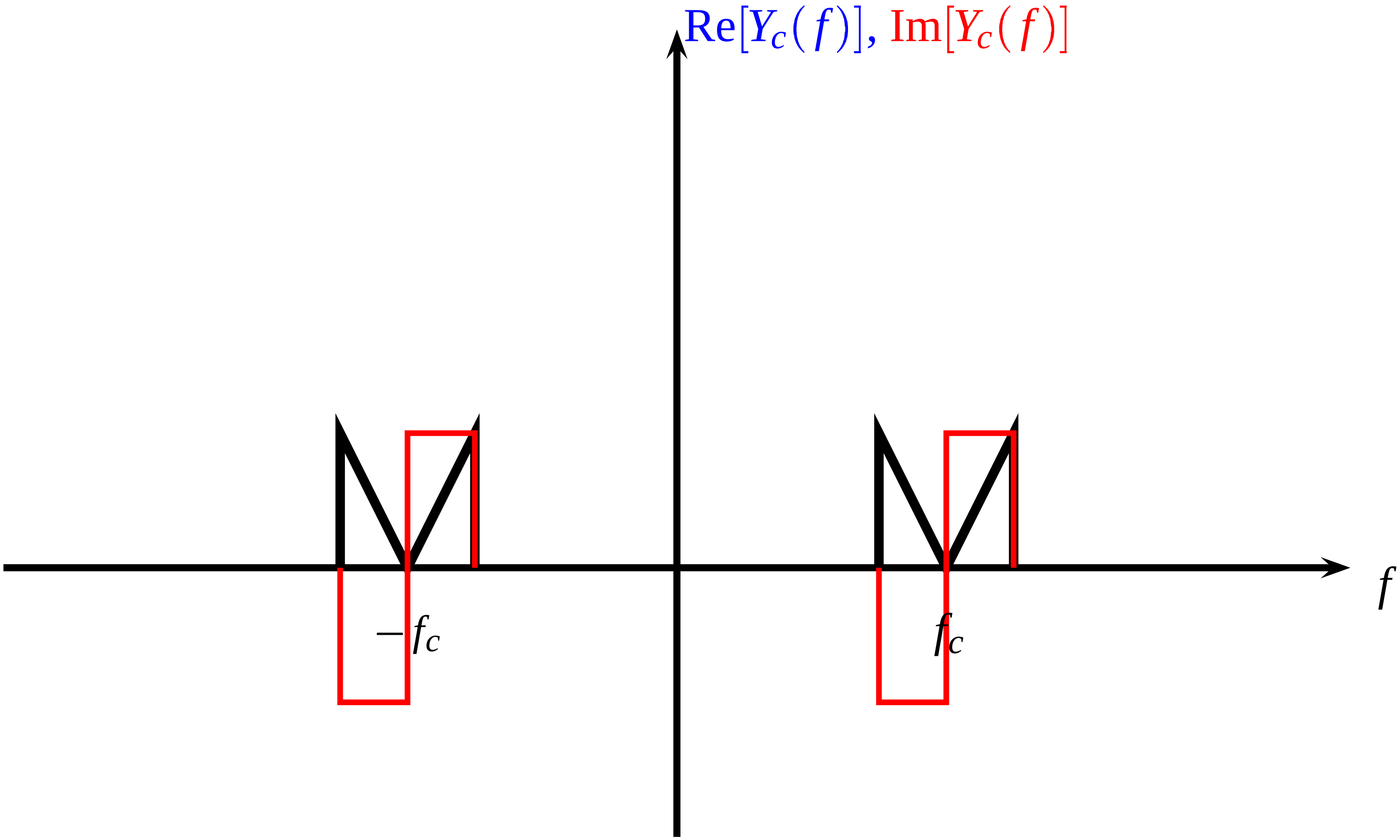
$$Y_c(f) = \frac{1}{2}[X_c(f - f_c) + X_c(f + f_c)]$$

Thus multiplication in the time domain by $\cos(2\pi f_c t)$ shifts the spectrum up and down by f_c and reduces each part by 1/2.

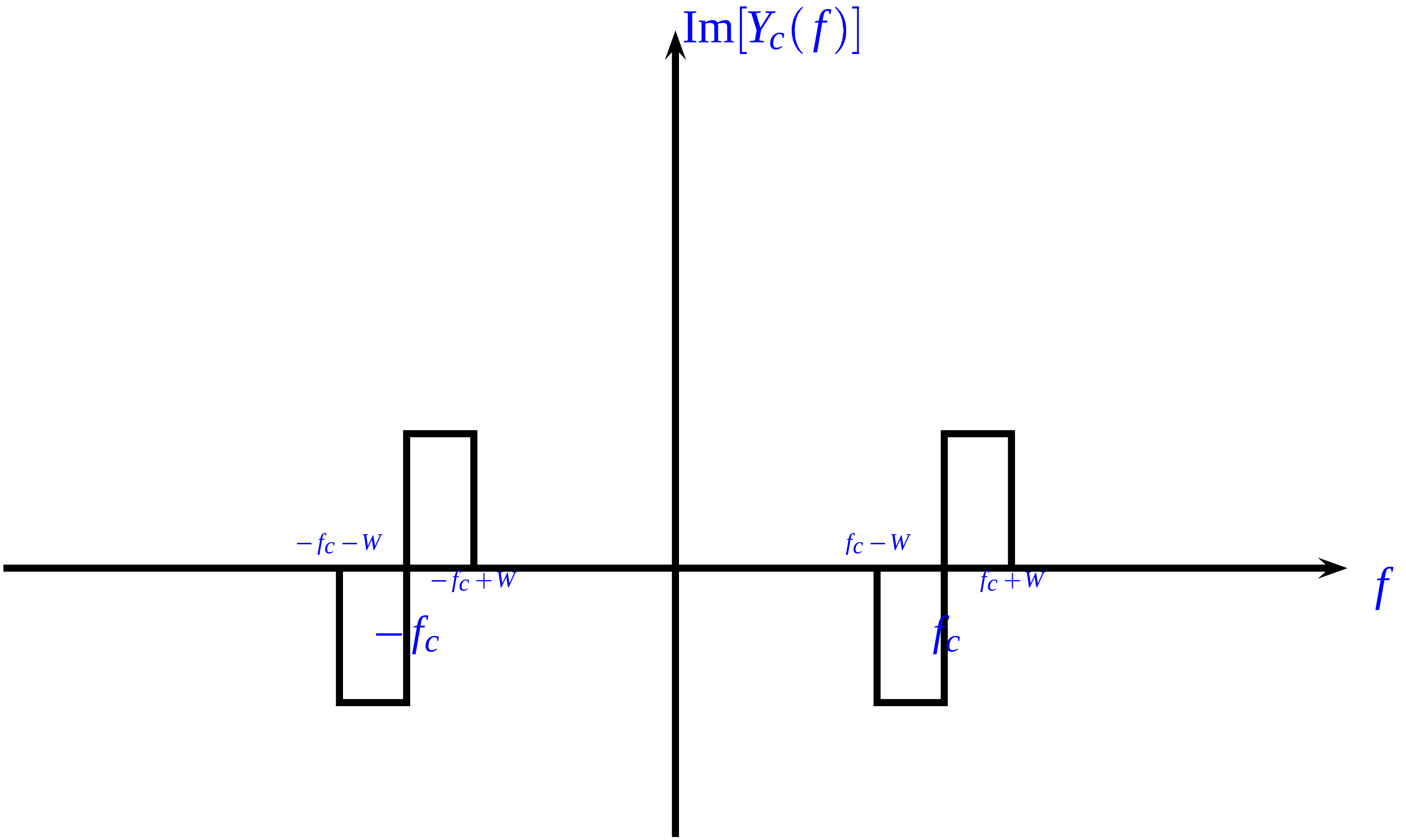

Re[Xc(f)], ImXc(f)
f
−W
W



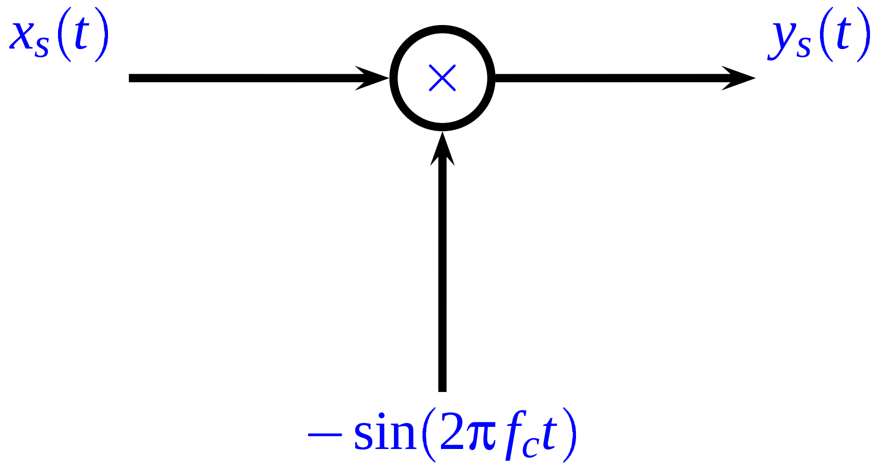
$\mathrm{Im}[X_c(f)]$
$-W$
W
f



$\mathrm{Re}[Y_c(f)]$, $\mathrm{Im}[Y_c(f)]$
f
$-f_c$
f_c


$\mathrm{Im}[Y_c(f)]$
$-f_c - W$
$-f_c + W$
$-f_c$
$f_c - W$
$f_c + W$
f_c
f

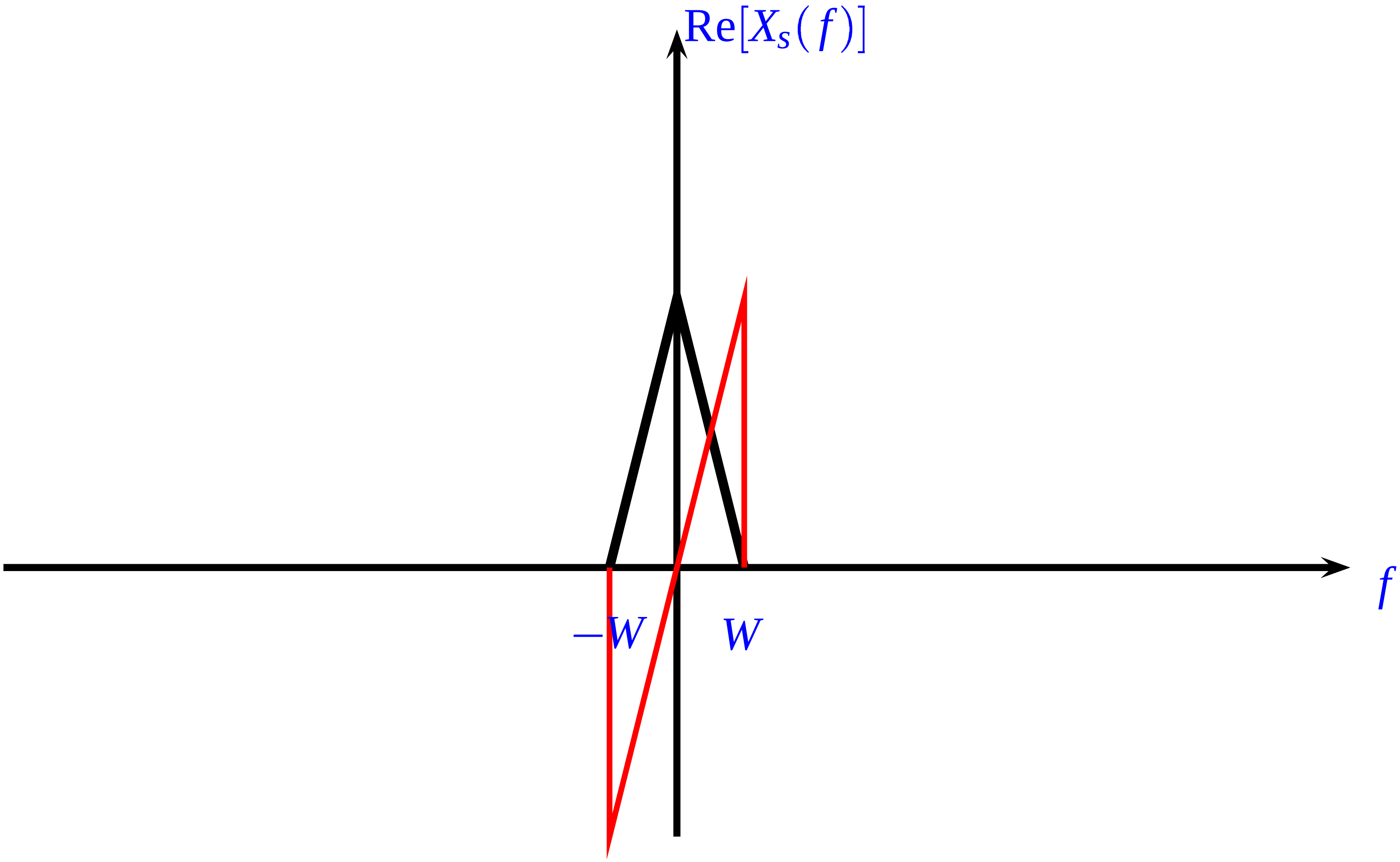
Now consider multiplication by $-\sin(2\pi f_c t)$.

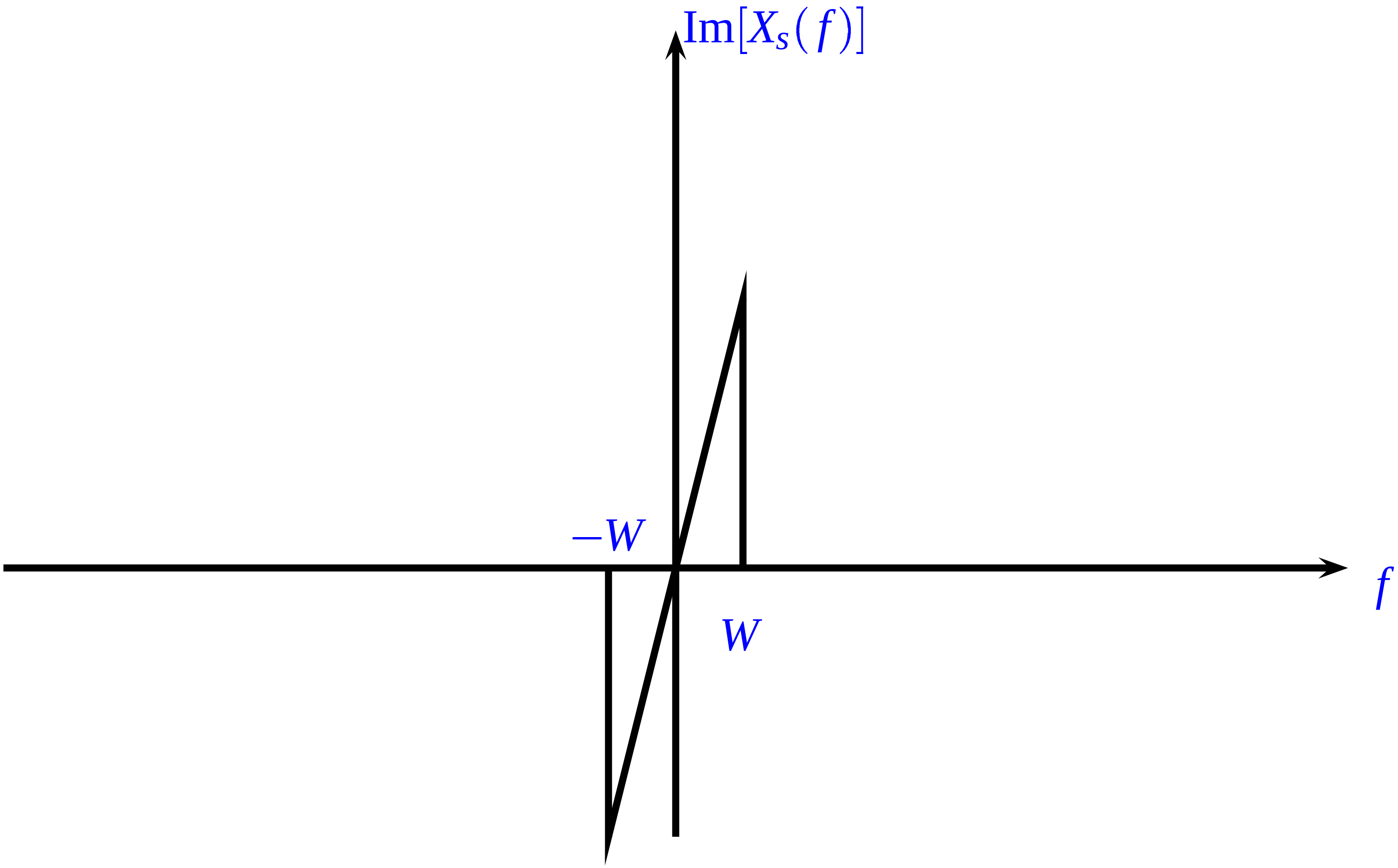


$$y_s(t) = -x_s(t)\sin(2\pi f_c t)$$

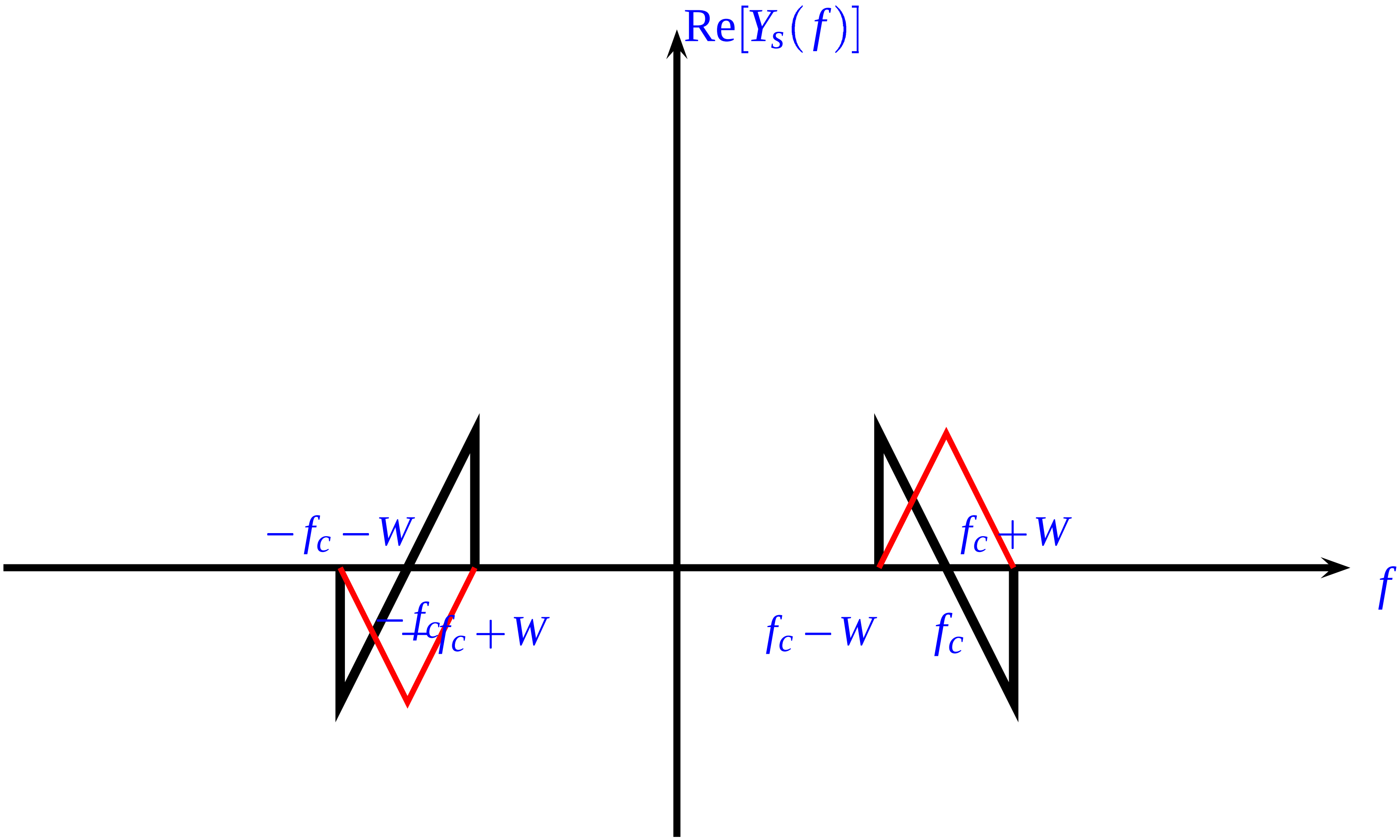
$$Y_c(f) = \frac{j}{2}[X_c(f - f_c) - X_c(f + f_c)]$$

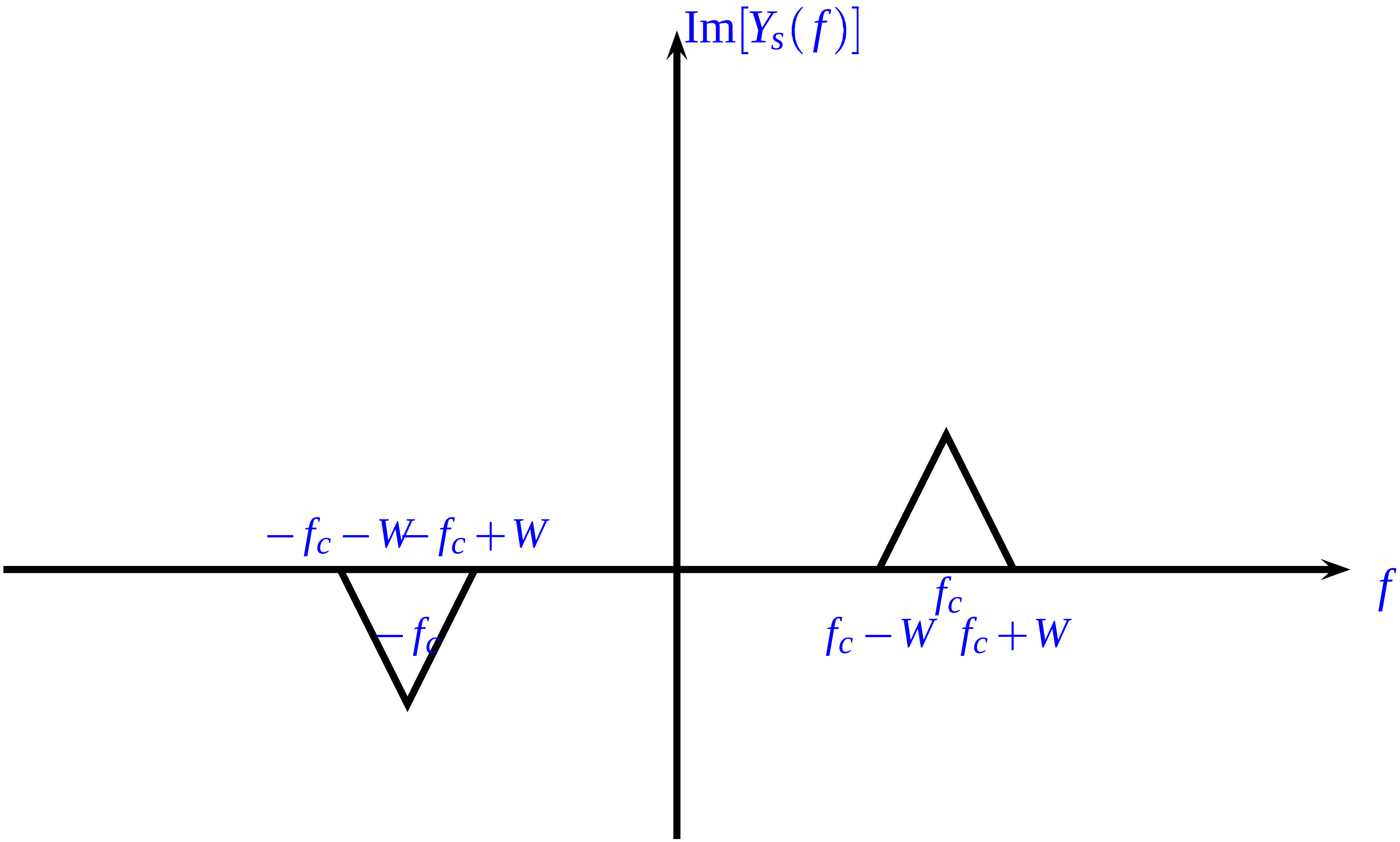
Thus multiplication by $-\sin(2\pi f_c t)$ shifts the spectrum up and down also except that the real part becomes the imaginary part and the imaginary part is inverted and becomes the real part in addition to a reduction by 1/2.


Re[$X_s(f)$]
f
$-W$
W



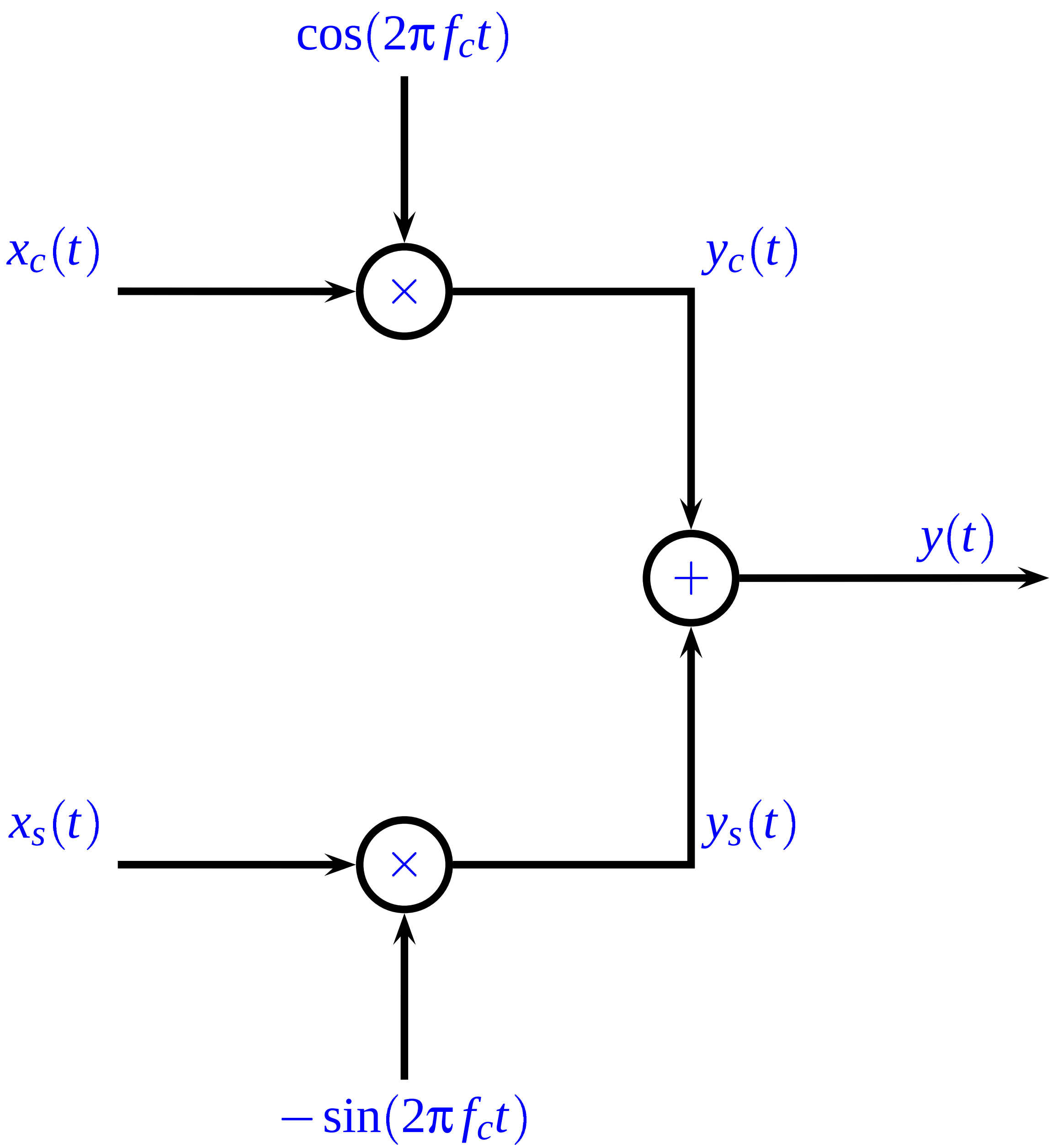
$\mathrm{Im}[X_s(f)]$
$-W$
W
f


$\mathrm{Re}[Y_s(f)]$
$-f_c - W$
$-f_c$
$-f_c + W$
$f_c - W$
f_c
$f_c + W$
f



$\mathrm{Im}[Y_s(f)]$
$-f_c - W$
$-f_c + W$
$-f_c$
f_c
$f_c - W$
$f_c + W$
f

Now consider adding these two functions together.


$\cos(2\pi f_c t)$
$x_c(t)$
$y_c(t)$
$y(t)$
$x_s(t)$
$y_s(t)$
$-\sin(2\pi f_c t)$

$$
\begin{aligned}
y(t) &= y_c(t) + y_s(t) \\
&= x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t) \\
&= x_e(t)\cos(2\pi f_c t + \theta(t))
\end{aligned}
$$

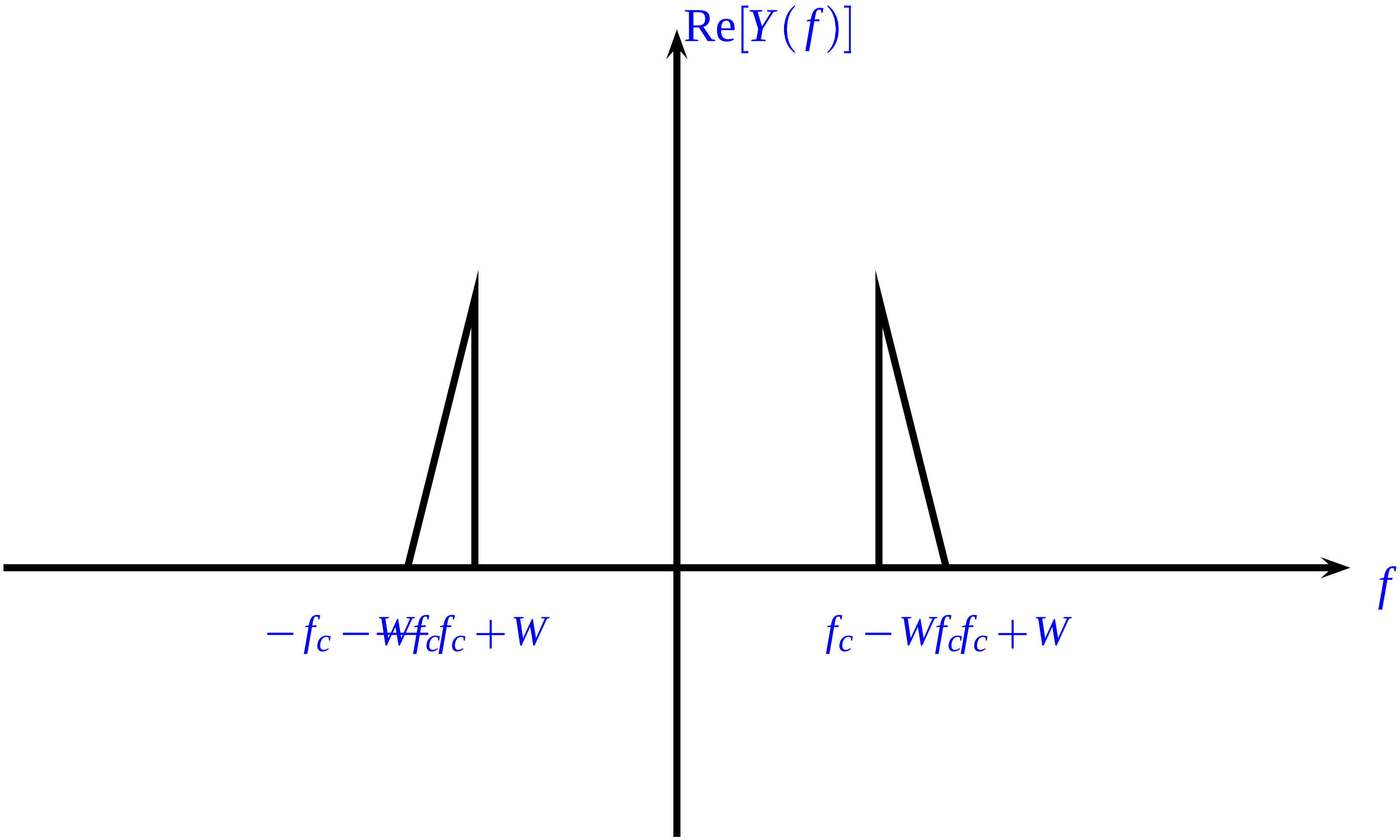
The signal $x_e(t)$ is called the envelope and $\theta(t)$ is called the phase.

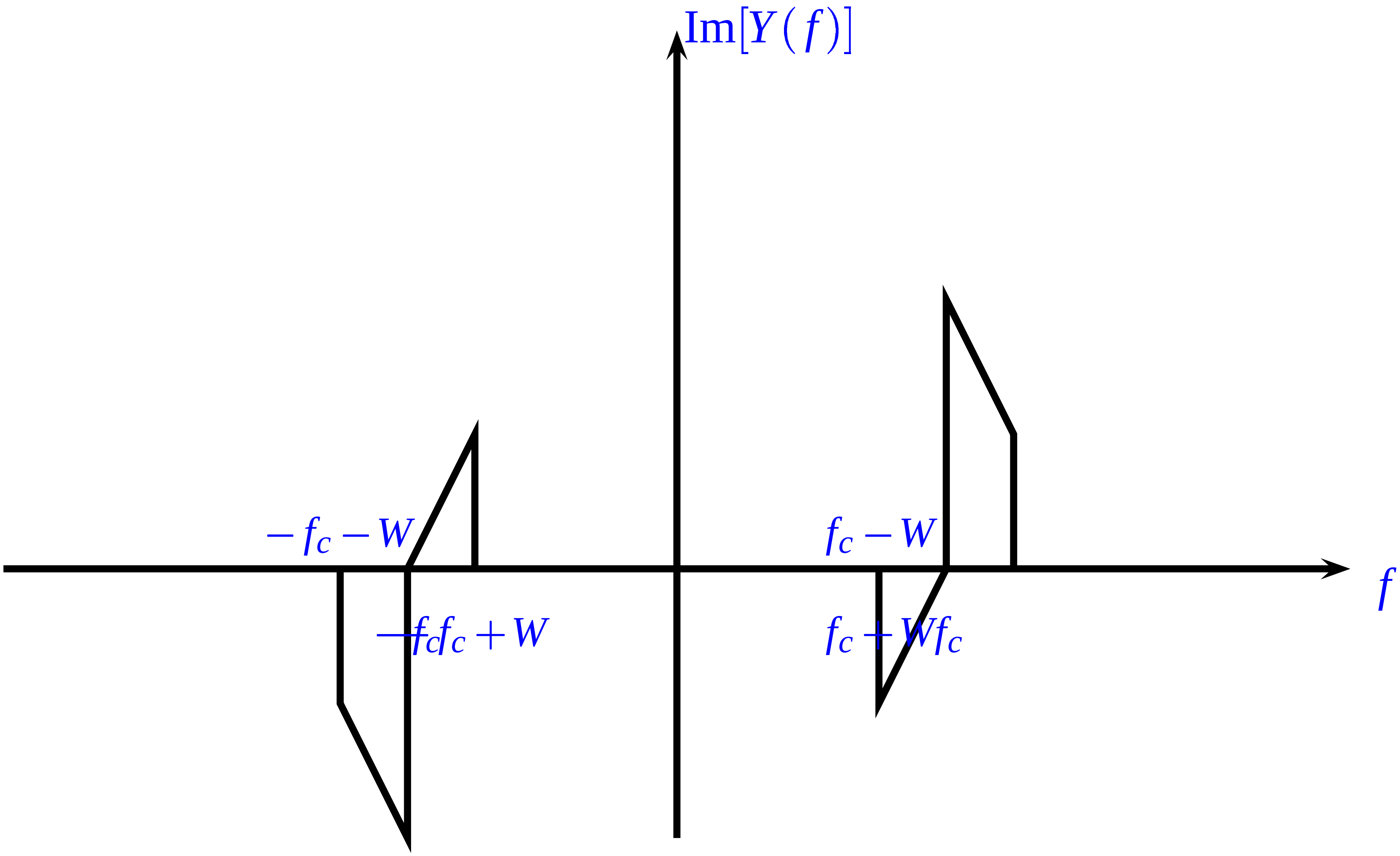
$$Y(f) = Y_c(f) + Y_s(f)$$

$$\theta(t) = \tan^{-1}\left[\frac{x_s(t)}{x_c(t)}\right]$$

$$x_e(t) = (x_c^2(t) + x_s^2(t))^{1/2}$$

$$y(t) = \mathrm{Re}[(x_c(t) + jx_s(t))e^{j2\pi f_c t}]$$

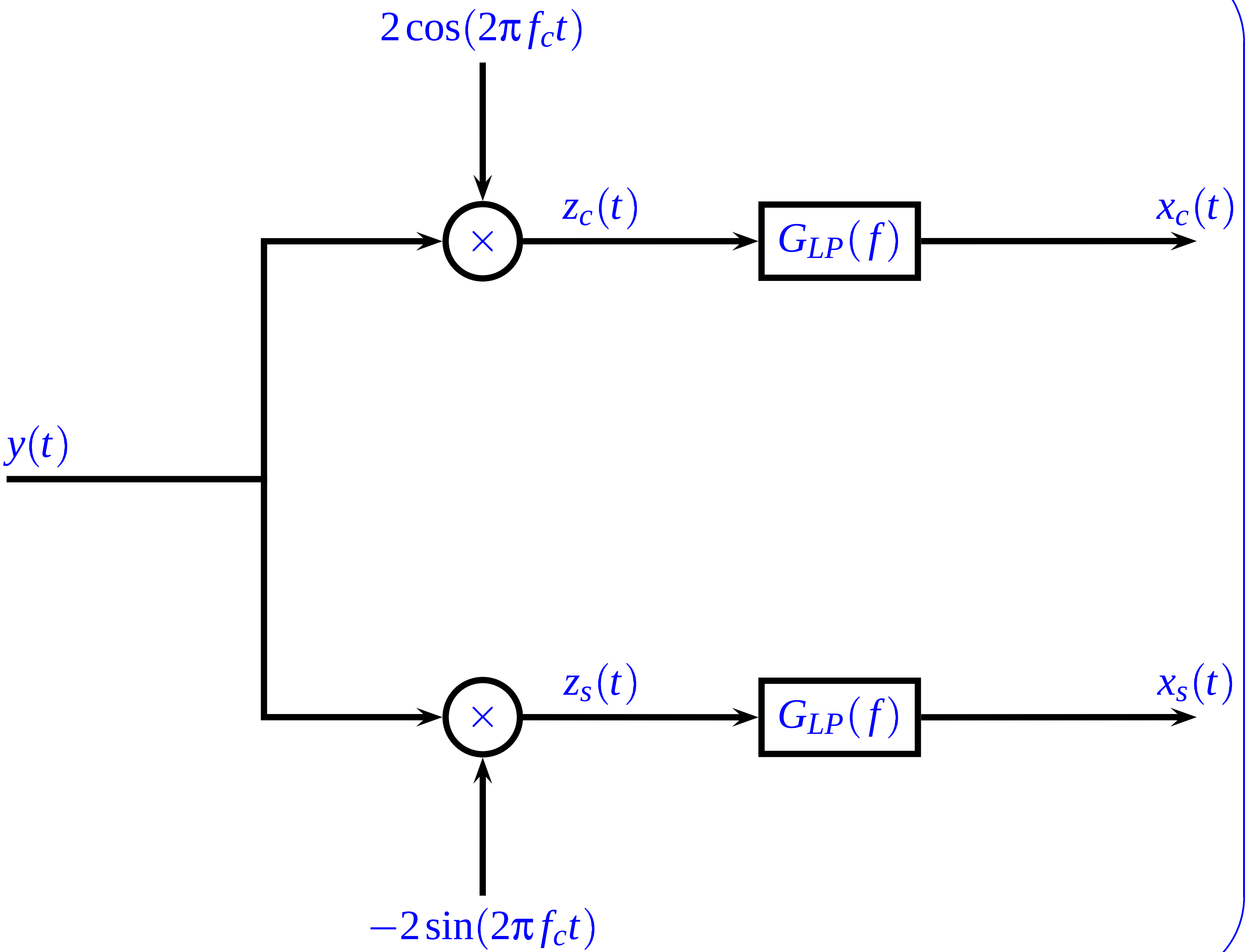

$\mathrm{Re}[Y(f)]$
$-f_c - W$
$-f_c$
$f_c + W$
$f_c - W$
f_c
$f_c + W$
f



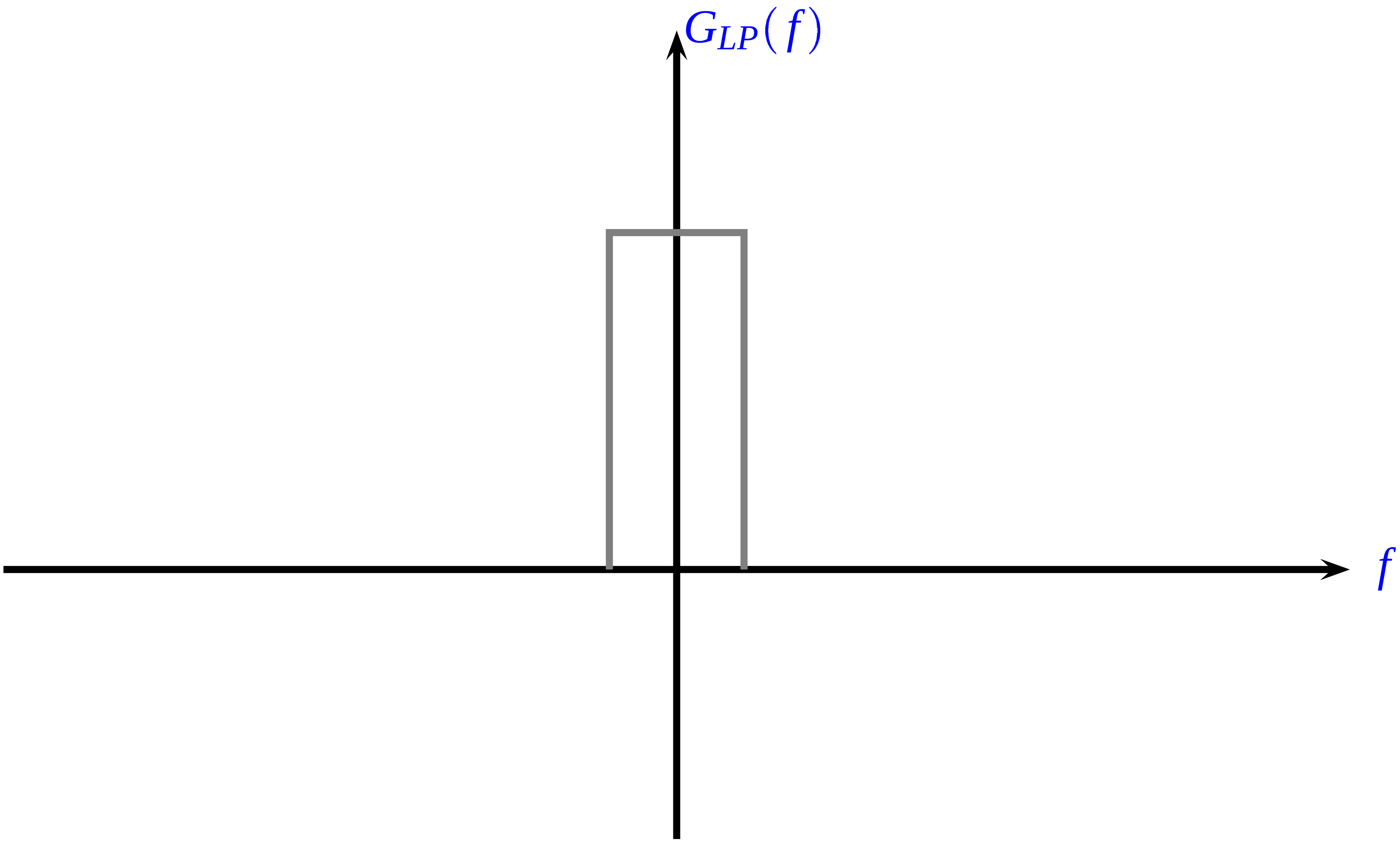
$\mathrm{Im}[Y(f)]$
$-f_c - W$
$f_c - W$
f
$-f_c$
$-f_c + W$
$f_c + W$
f_c

Signal Decomposition

The signals $x_c(t)$ and $x_s(t)$ can be recovered from $y(t)$ by mixing down to baseband and filtering out the double frequency terms. Note that we need the exact phase of the local oscillators to do this perfectly.


$2\cos(2\pi f_c t)$
$z_c(t)$
$G_{LP}(f)$
$x_c(t)$
$y(t)$
$z_s(t)$
$G_{LP}(f)$
$x_s(t)$
$-2\sin(2\pi f_c t)$

$G_{LP}(f)$ is an ideal low pass filter with transfer function $G_{LP}(f) = 1\ |f| \leq W$ and $G_{LP}(f) = 0$ otherwise.

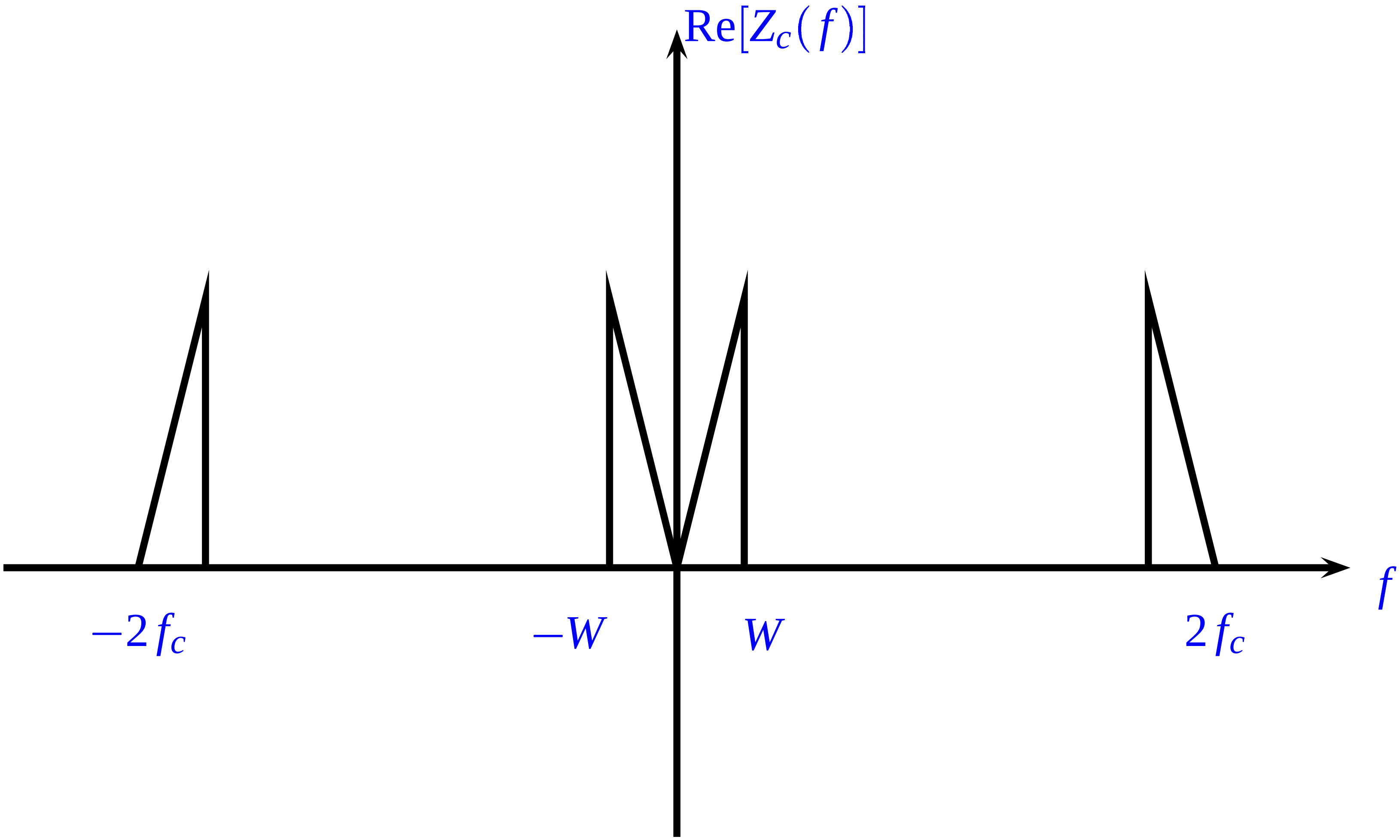

$G_{LP}(f)$
f

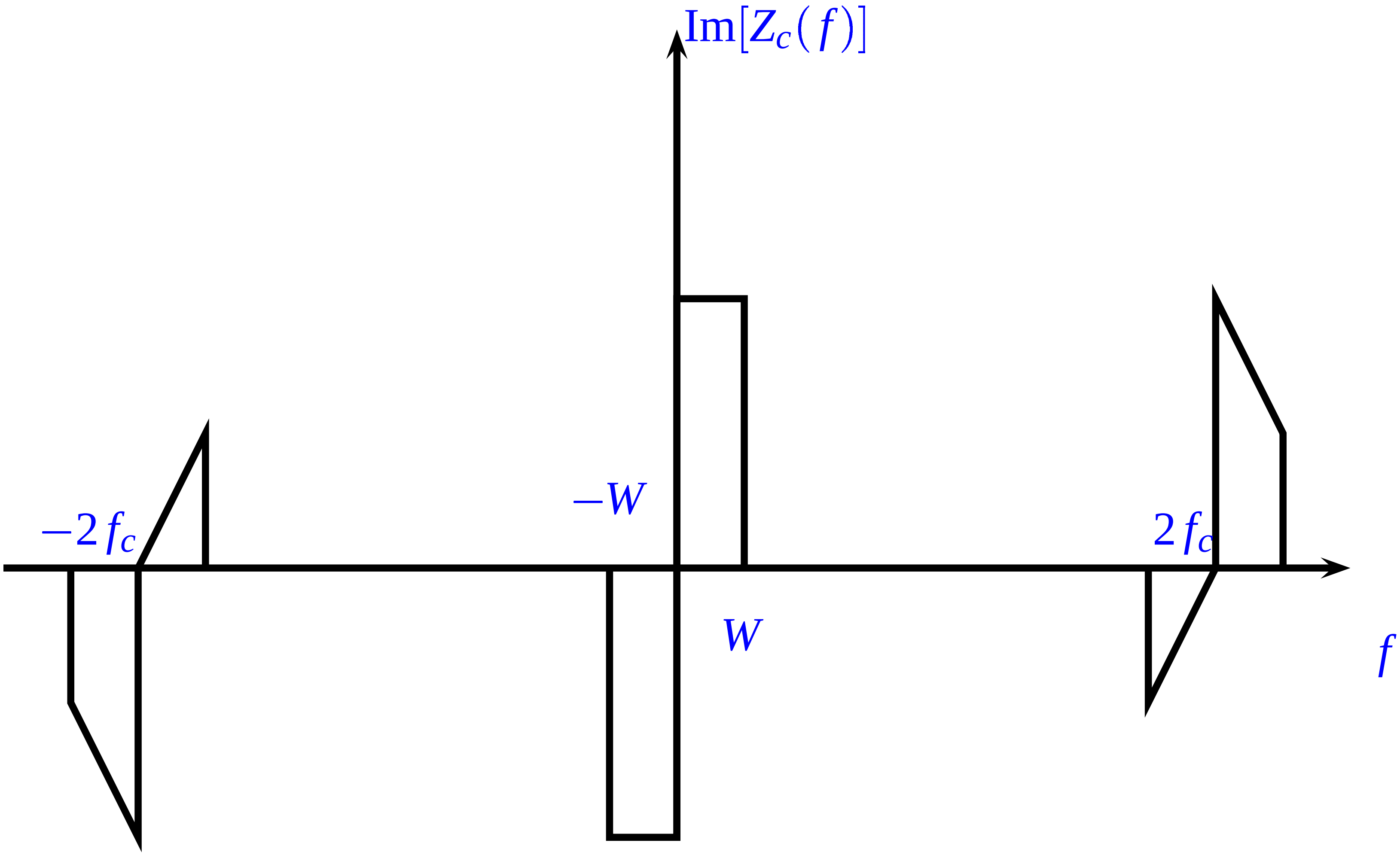
Consider the spectrum of $z_c(t)$. This is given by

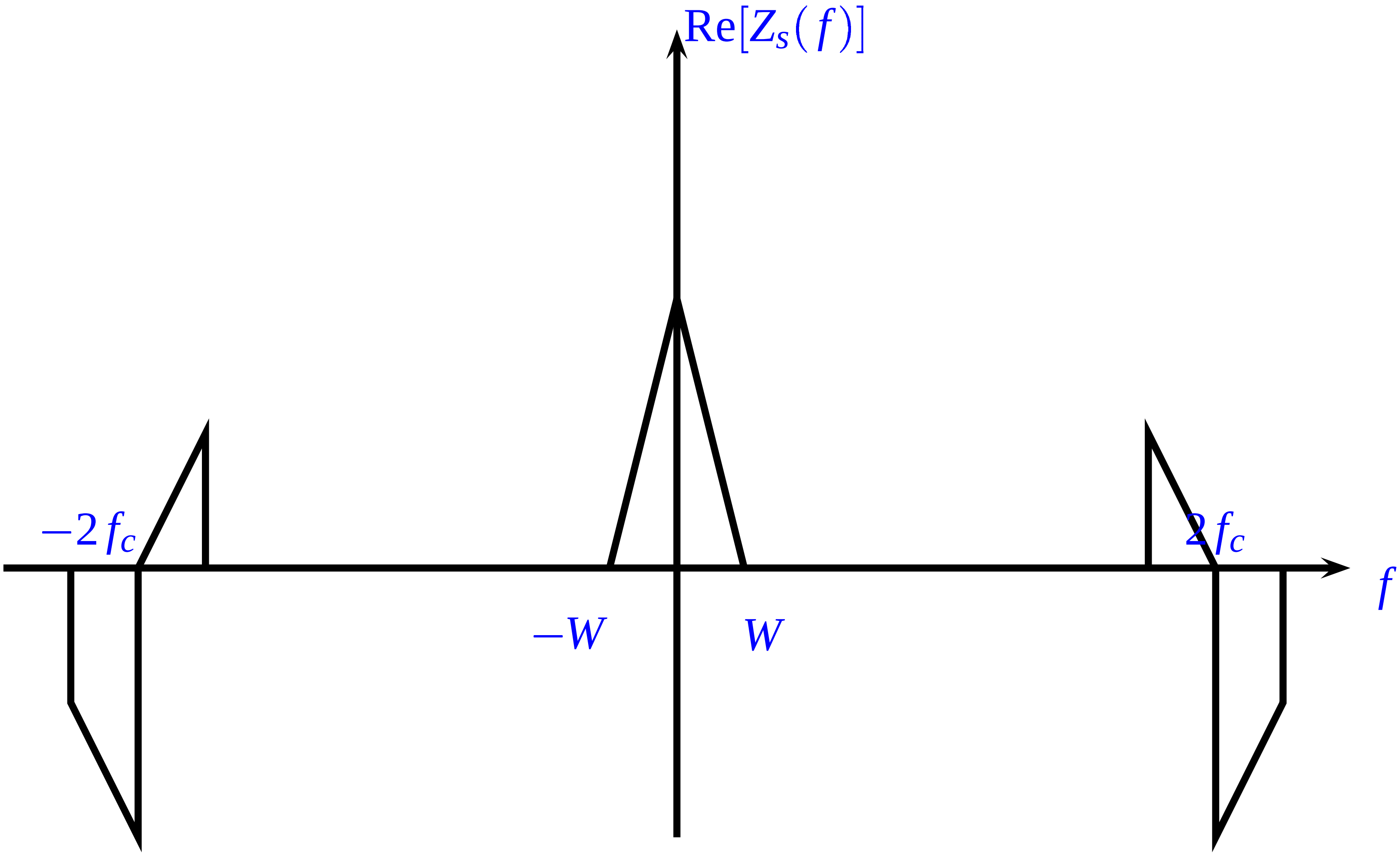
$$Z_c(f) = Y(f - f_c) + Y(f + f_c).$$

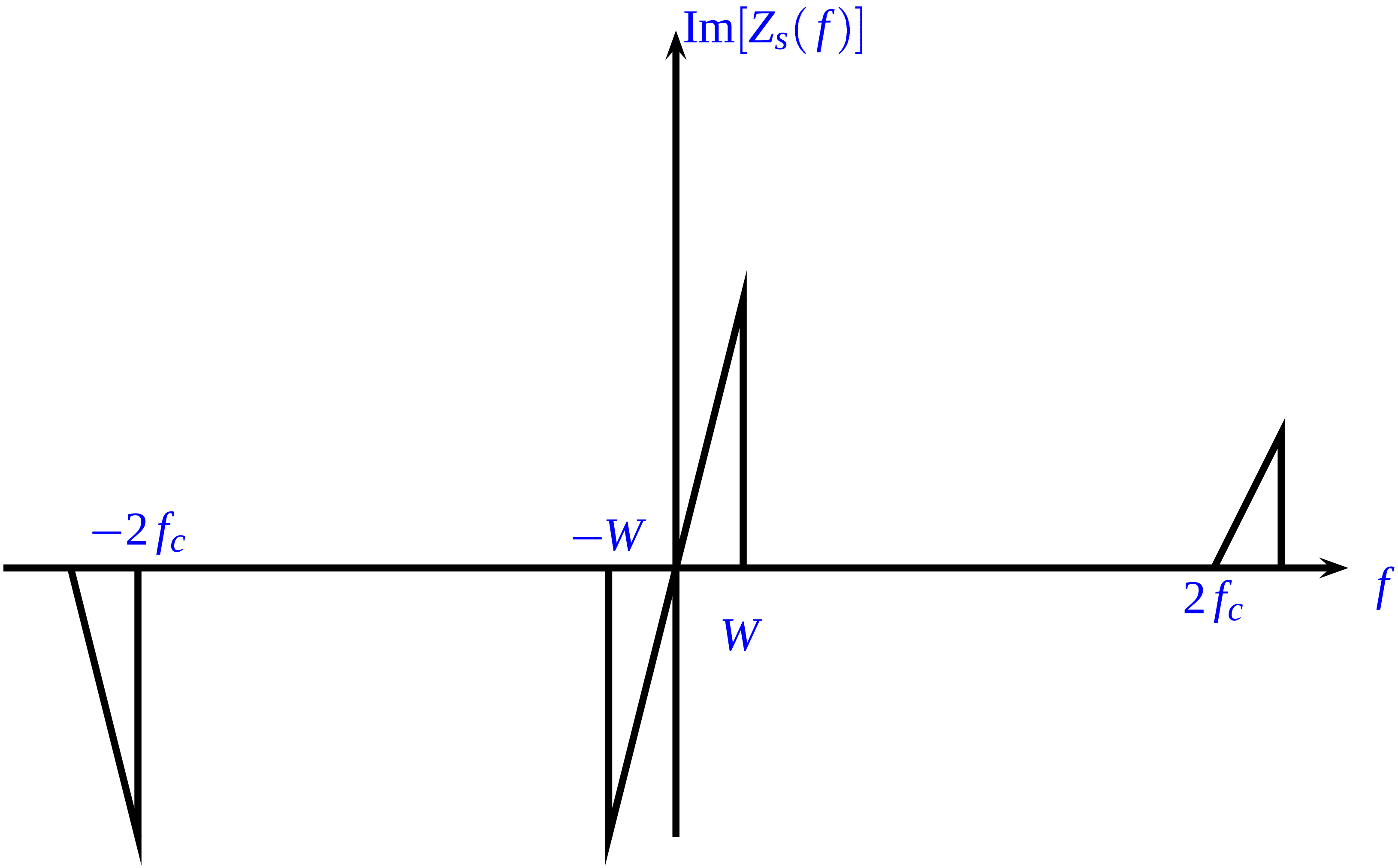
Similarly the spectrum of $z_s(t)$ is

$$Z_s(f) = j[Y(f - f_c) - Y(f + f_c)].$$


$\mathrm{Re}[Z_c(f)]$
f
$-2f_c$
$-W$
W
$2f_c$


$\mathrm{Im}[Z_c(f)]$
$-2f_c$
$-W$
$2f_c$
W
f


$\mathrm{Re}[Z_s(f)]$
$-2f_c$
$2f_c$
f
$-W$
W



Im[$Z_s(f)$]
$-2f_c$
$-W$
W
$2f_c$
f

Example

$$
\begin{aligned}
x_c(t) &= a_{c,1}\cos(2\pi f_1 t) + a_{c,2}\sin(2\pi f_1 t) + a_{c,3}\cos(2\pi f_2 t) \\
x_s(t) &= a_{s,1}\cos(2\pi f_1 t) + a_{s,2}\sin(2\pi f_1 t) + a_{s,3}\sin(2\pi f_2 t)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{c,1} &= 0.25, a_{c,2} = 0.5, a_{c,3} = 1, \\
a_{s,1} &= -1.0, a_{c,2} = 0.25, a_{c,3} = 1, \\
f_1 &= 1, f_2 = 2.
\end{aligned}
$$

The signals are upconverted with a quadrature modulator to produce

$$
\begin{aligned}
&= x_c(t)\cos(2\pi f_c t) - x_s(t)\sin(2\pi f_c t) \\
&= x_e(t)\cos(2\pi f_c t + \theta(t))
\end{aligned}
$$

where $f_c = 16$.

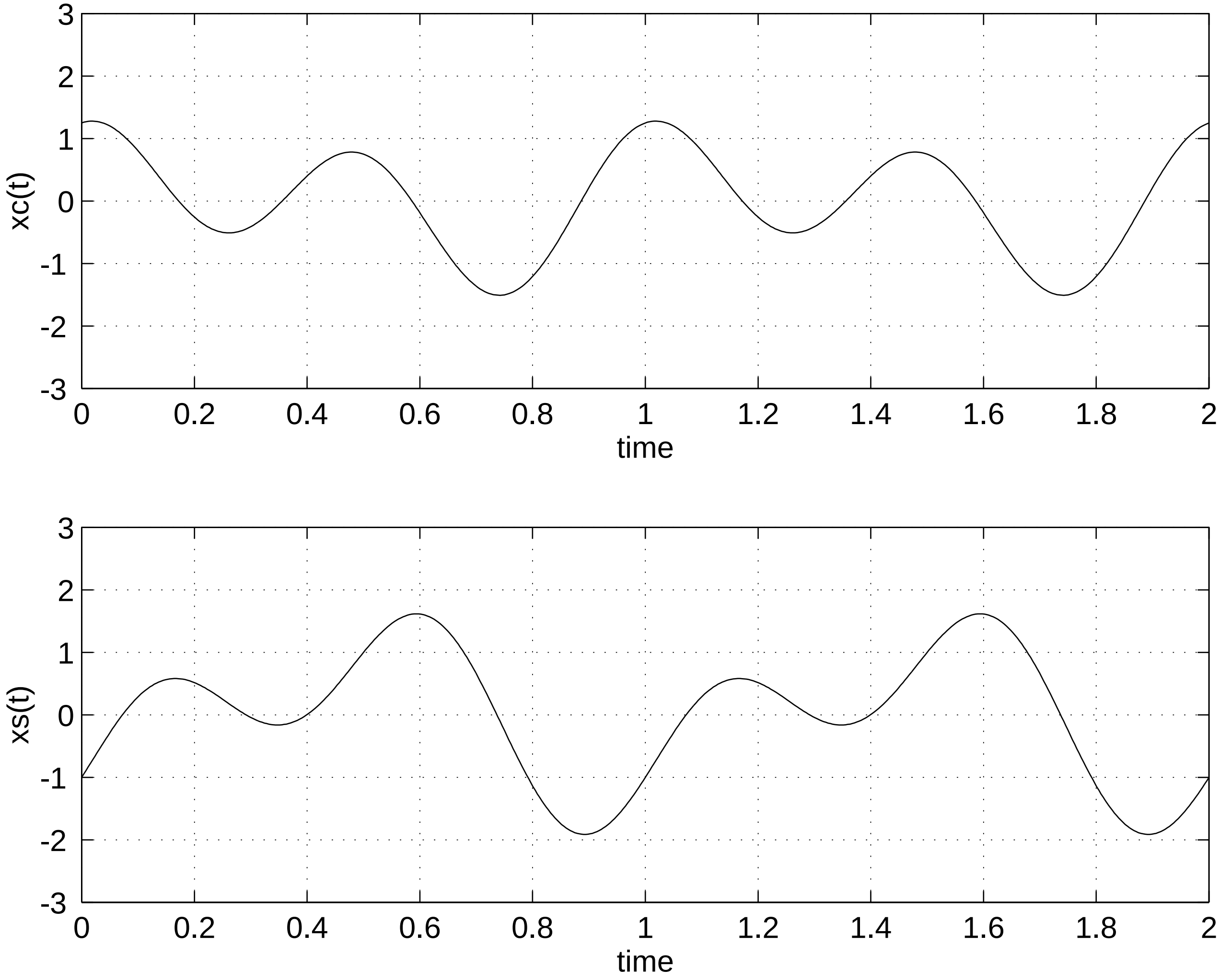


Figure 28: Example of Up and Down Conversion of Signals

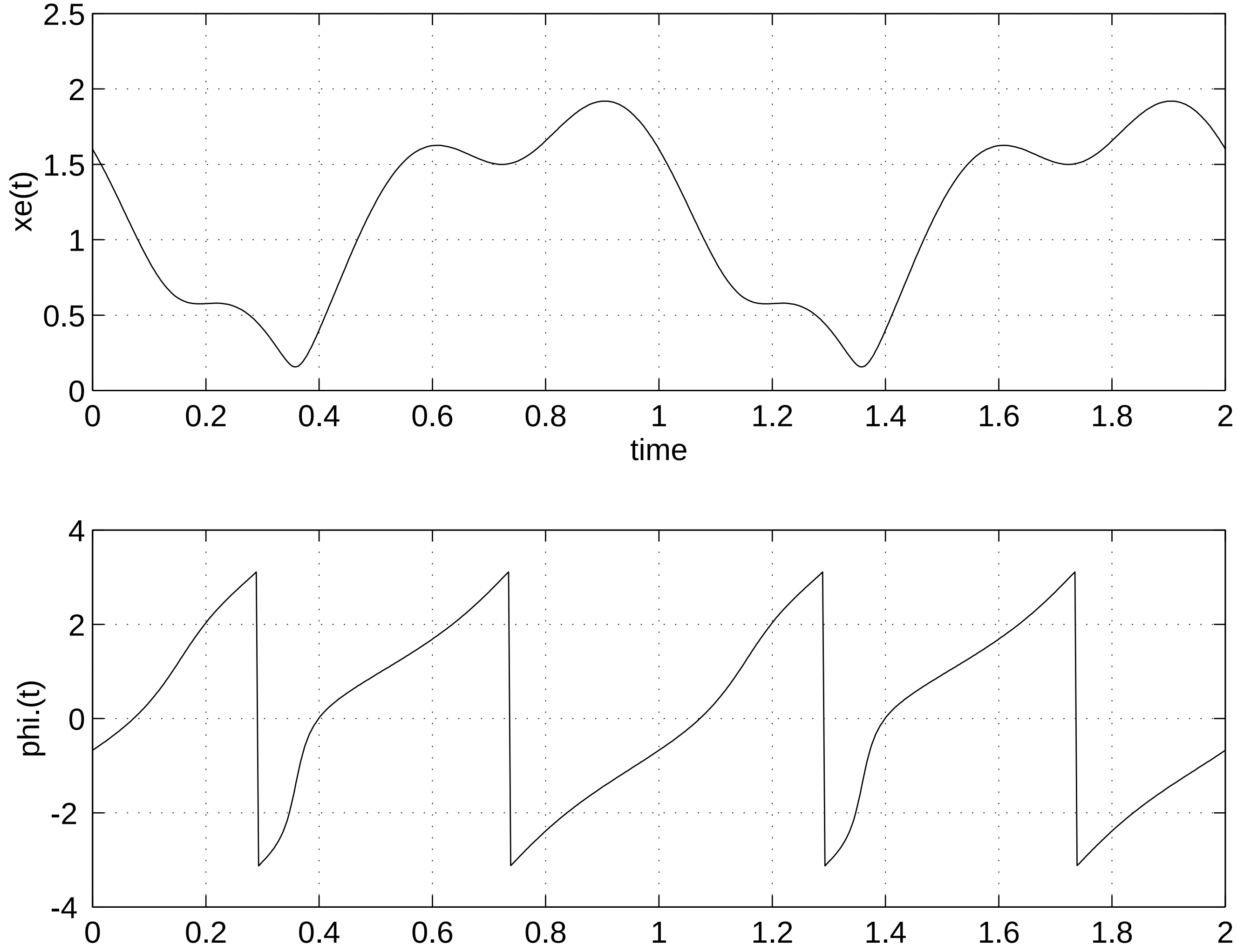


Figure 29: Example of Up and Down Conversion of Signals

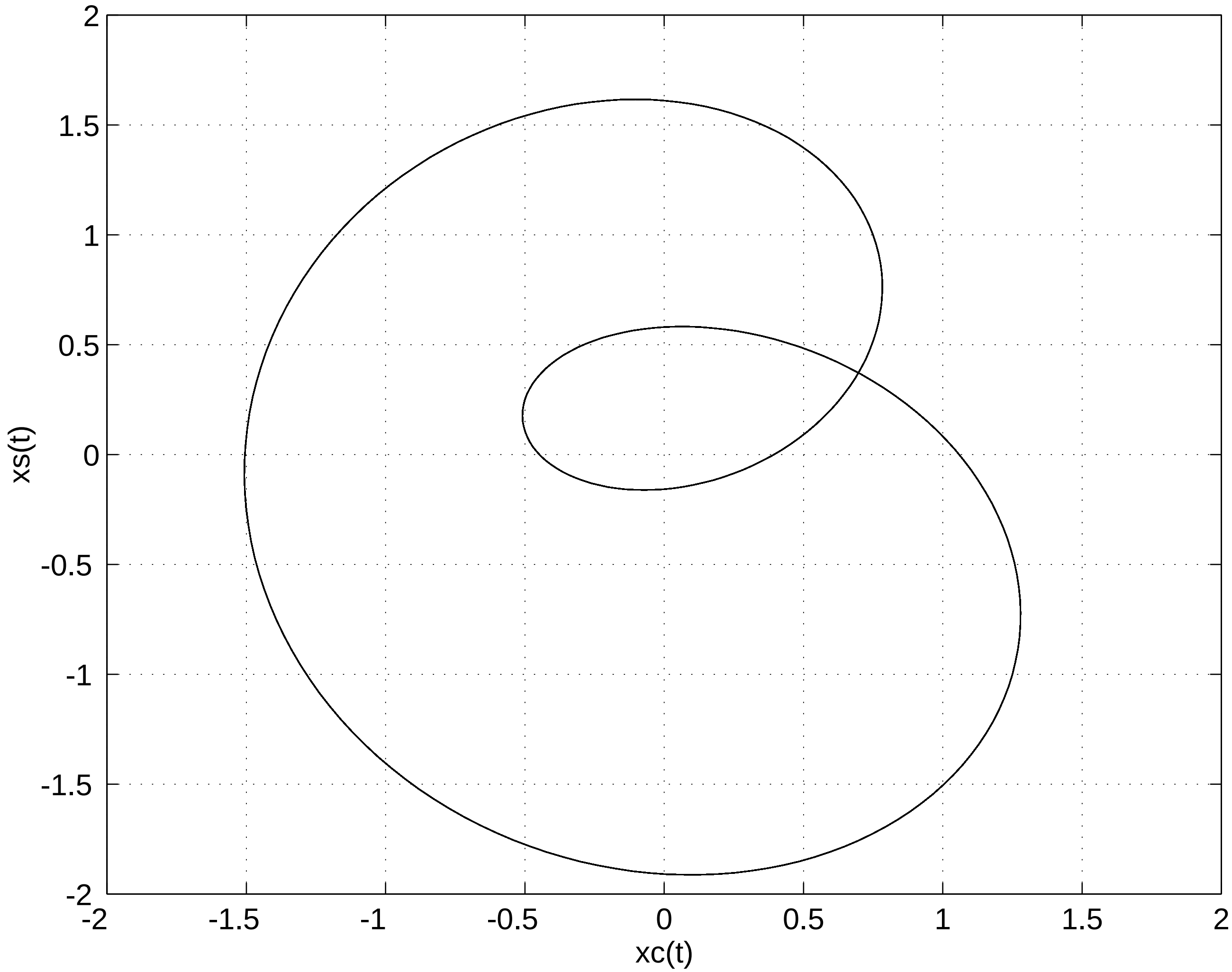


Figure 30: Real and Imaginary Part of Complex Signal

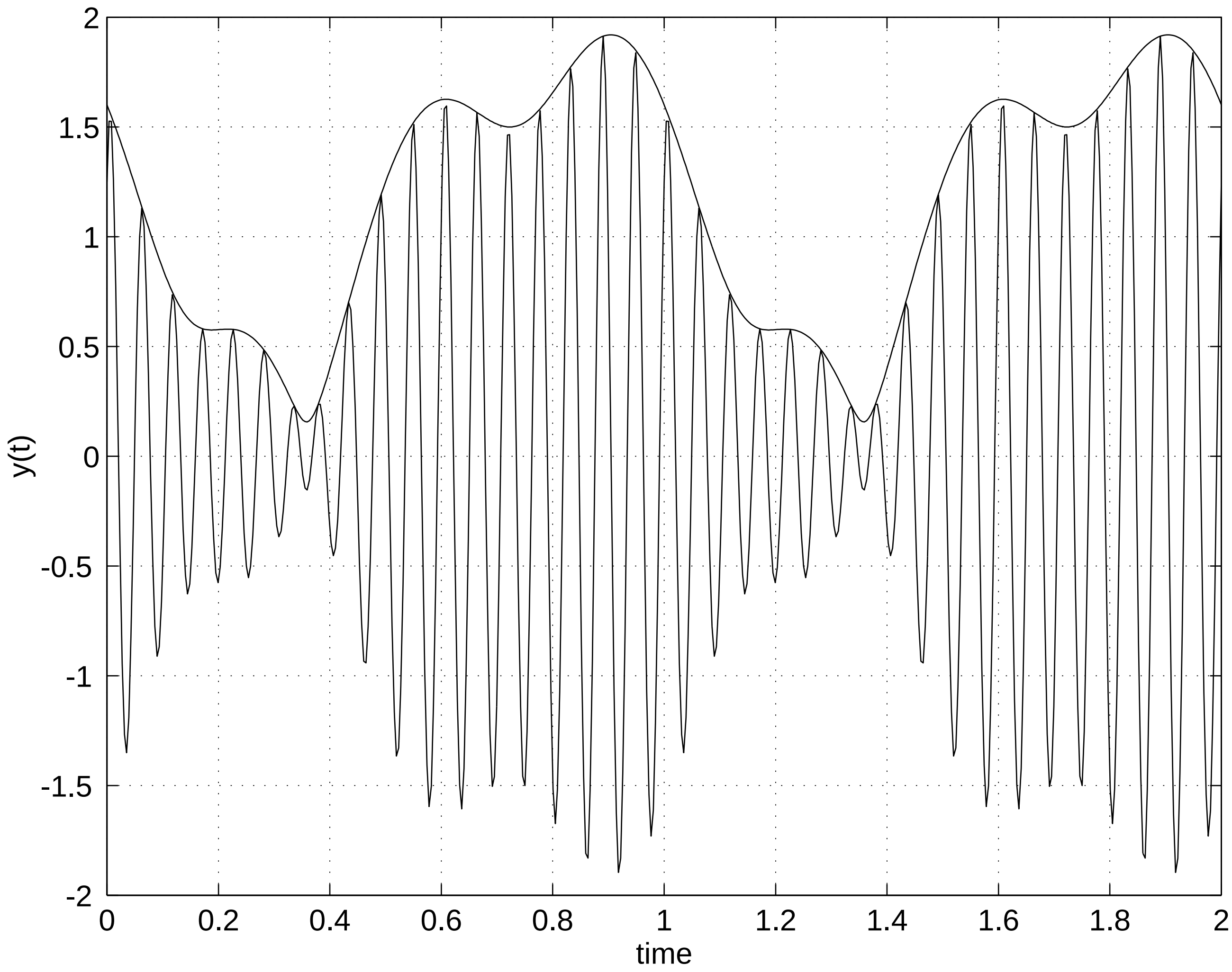


Figure 31: Example of Up and Down Conversion of Signals

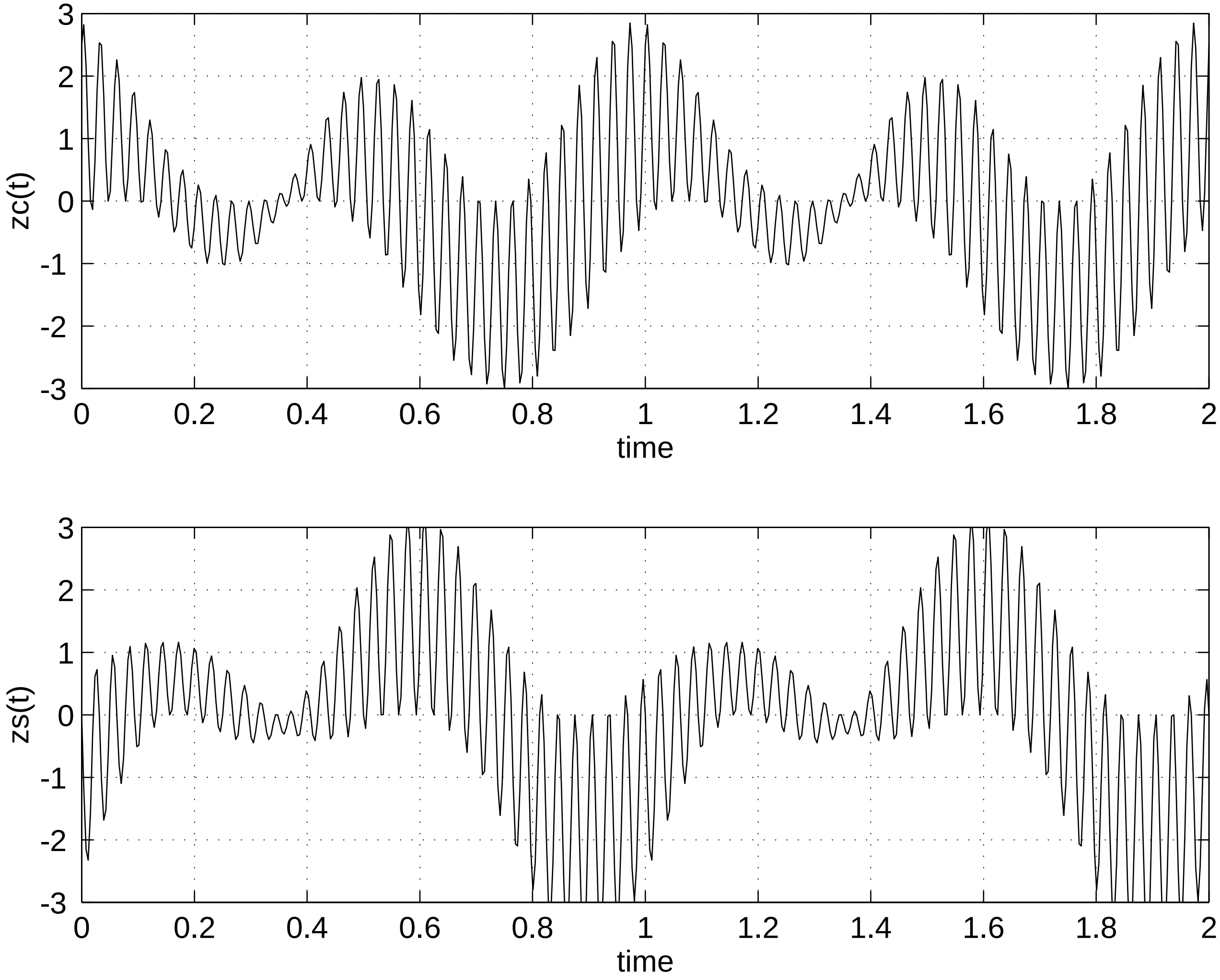
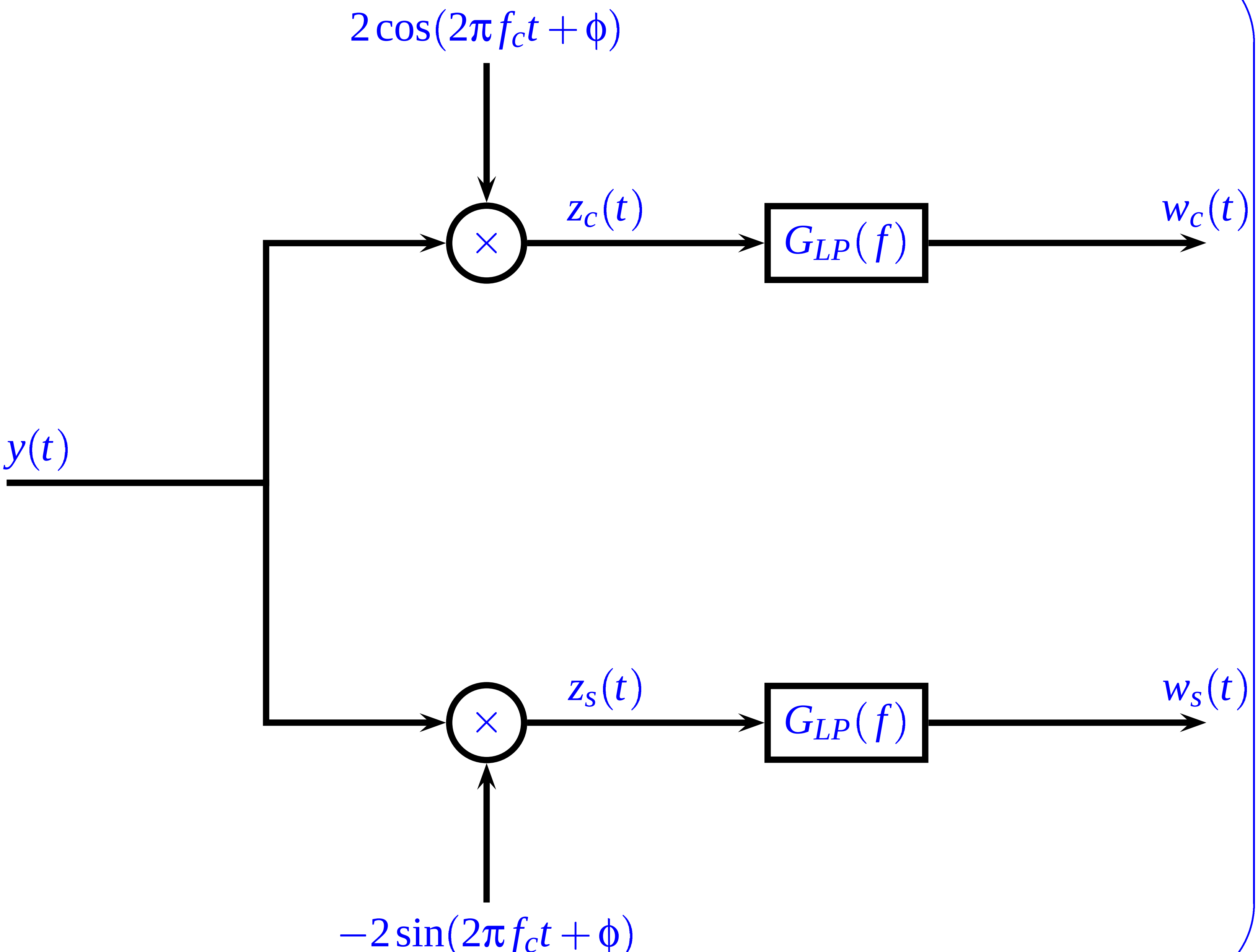


Figure 32: Example of Up and Down Conversion of Signals

Now consider what happens if there is an imperfect local oscillator



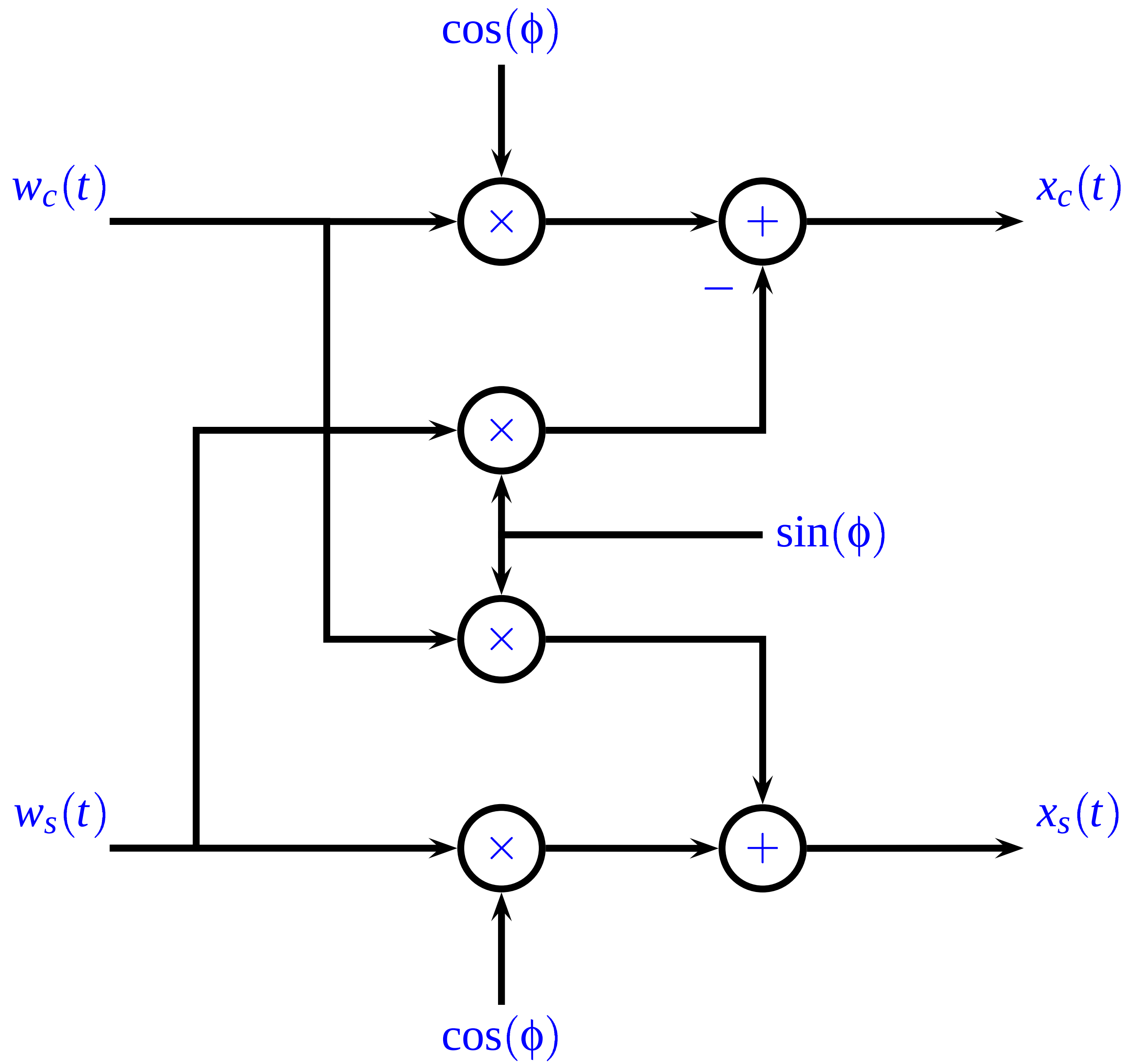
$2\cos(2\pi f_c t + \phi)$
$z_c(t)$
$G_{LP}(f)$
$w_c(t)$
$y(t)$
$z_s(t)$
$G_{LP}(f)$
$w_s(t)$
$-2\sin(2\pi f_c t + \phi)$

$$
\begin{aligned}
w_c(t) &= x_c(t)\cos(\phi) + x_s(t)\sin(\phi) \\
w_s(t) &= -x_c(t)\sin(\phi) + x_s(t)\cos(\phi) \\
w_c(t) + j w_s(t) &= (x_c(t) + j x_s(t))e^{-j\phi}
\end{aligned}
$$

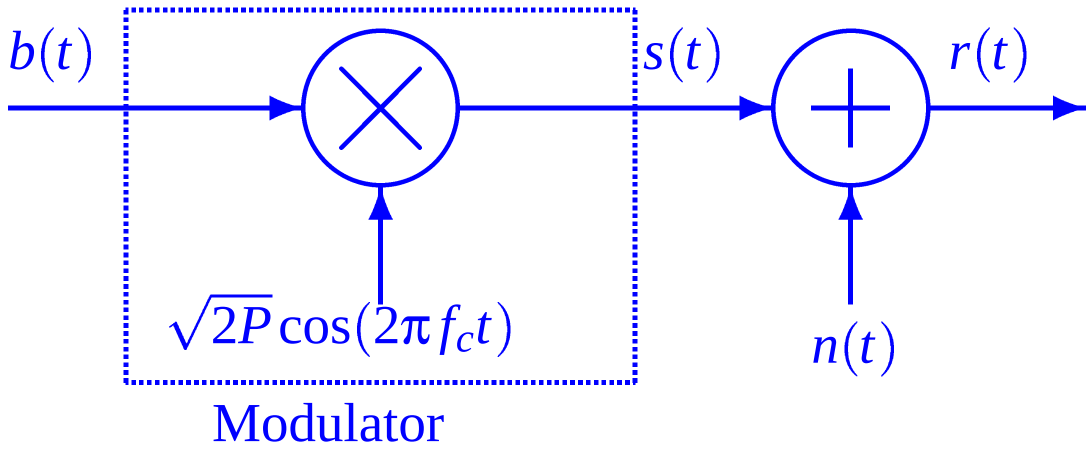
Note that this is equivalent to a phase rotation by angle ϕ.

We can recover the orignial signal by rotating the signal.

$$x_c(t) + jx_s(t) = (w_c(t) + jw_s(t))e^{+j\phi}$$

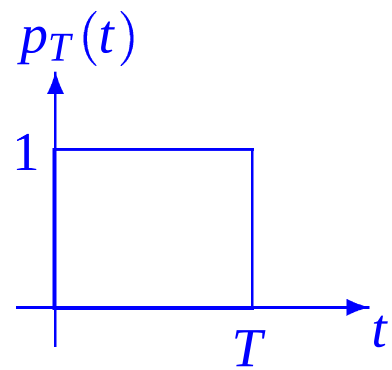

cos(ϕ)
$w_c(t)$
×
+
$x_c(t)$
−
×
sin(ϕ)
×
$w_s(t)$
×
+
$x_s(t)$
cos(ϕ)

Binary Phase Shift Keying (BPSK)



$$b(t) = \sum_{l=-\infty}^{\infty} b_l \, p_T(t - lT), \qquad b_l \in \{+1, -1\}.$$

$$p_T(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise.} \end{cases}$$



$$
\begin{aligned}
s(t) &= \sqrt{2P} \sum_{l=-\infty}^{\infty} b_l \cos(2\pi f_c t) p_T(t - lT) \\
&= \sqrt{2P}\, b(t) \cos(2\pi f_c t) = \sqrt{2P} \cos(2\pi f_c t + \phi(t))
\end{aligned}
$$

where $\phi(t)$ is the phase waveform. The signal has power P. The energy of the transmitted bit is $E = PT$.

The phase of a BPSK signal can take on one of two values as shown below.

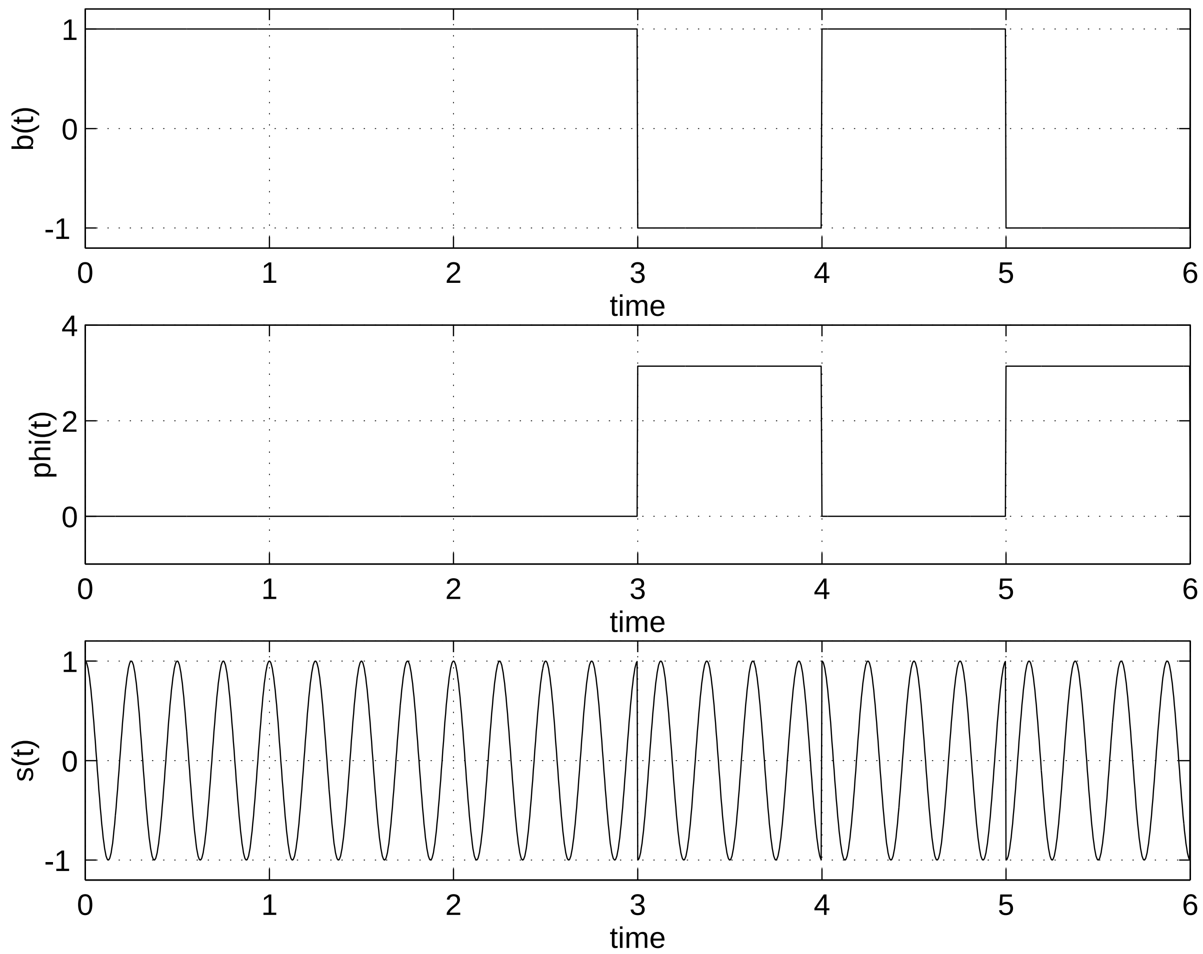
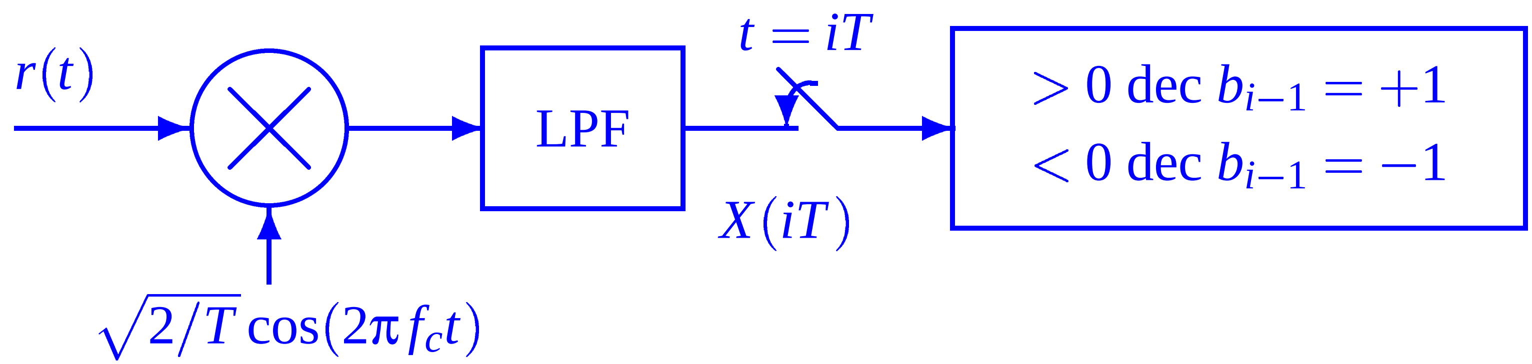


Figure 33: Signals for BPSK Modulation

Demodulator



The low pass filter (LPF) is a filter "matched" to the baseband signal being transmitted. For BPSK this is just a rectangular pulse of duration T. The impulse response is $h(t) = p_T(t)$.

$$
\begin{aligned}
X(t) &= \int_{-\infty}^{\infty} \sqrt{2/T}\cos(2\pi f_c \tau) h(t-\tau) r(\tau) d\tau \\
X(iT) &= \int_{-\infty}^{\infty} \sqrt{2/T}\cos(2\pi f_c \tau) p_T(iT-\tau) r(\tau) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{(i-1)T}^{iT} \sqrt{2/T}\cos(2\pi f_c\tau)\left[\sqrt{2P}\,b(\tau)\cos(2\pi f_c\tau)\right. \\
&\qquad\qquad \left. + n(\tau)\right] d\tau \\
&= \int_{(i-1)T}^{iT} 2\sqrt{P/T} b_{i-1}\cos(2\pi f_c\tau)\cos(2\pi f_c\tau) d\tau + \eta_i
\end{aligned}
$$

η_i is Gaussian random variable, mean 0 variance $N_0/2$. Assuming $2\pi f_c T = 2\pi n$

$$
\boxed{X(iT) = \sqrt{PT}\, b_{i-1} + \eta_i = \sqrt{E}\, b_{i-1} + \eta_i.}
$$

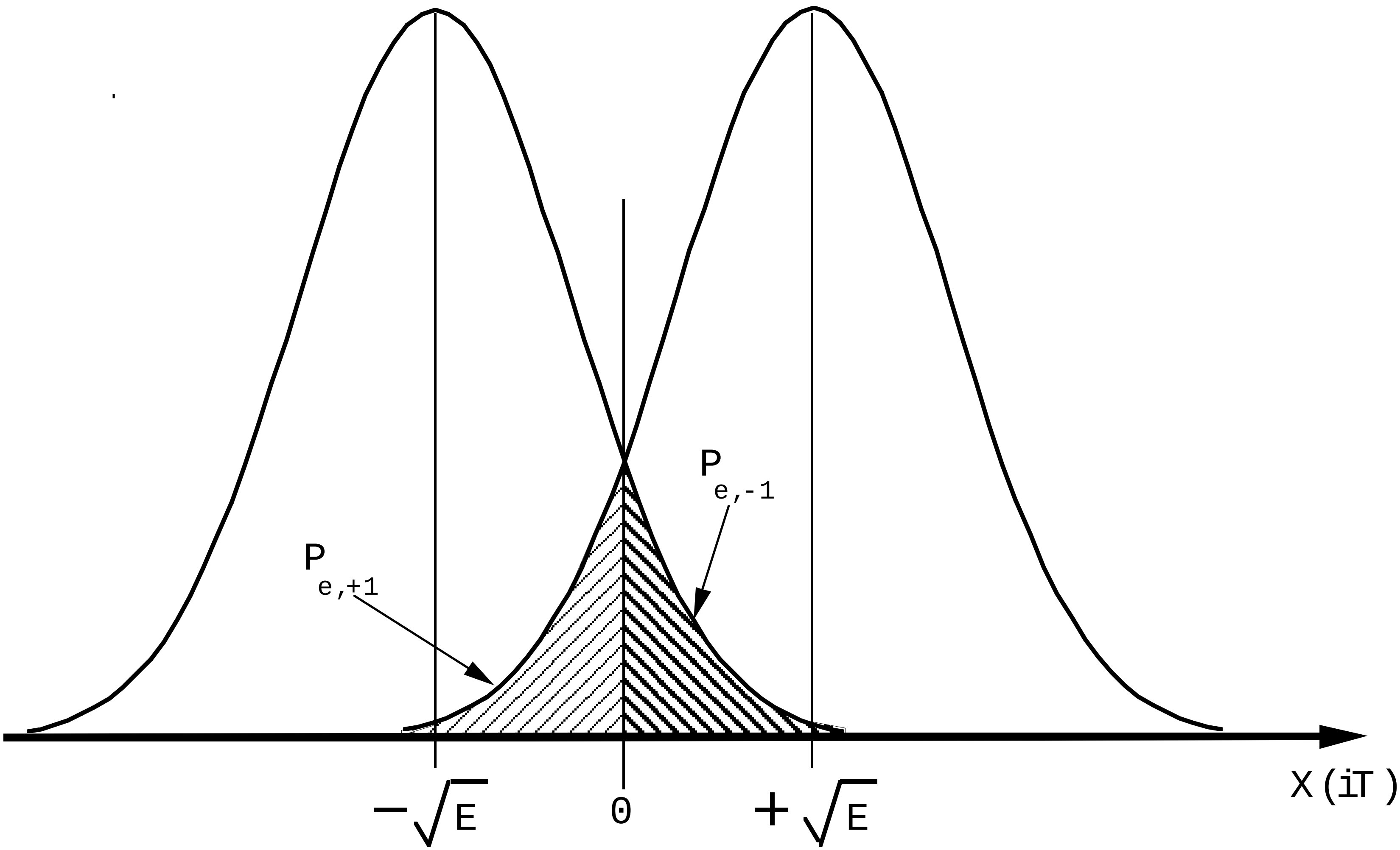


Figure 34: Probability Density of Decision Statistic for Binary Phase Shift Keying

Bit Error Probability of BPSK

$$P_{e,b} = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where

$$Q(x) = \int_x^\infty \frac{1}{2\pi} e^{-u^2/2} du$$

For binary signals this is the smallest bit error probability, i.e. BPSK are optimal signals and the receiver shown above is optimum (in additive white Gaussian noise). For binary signals the energy transmitted per information bit E_b is equal to the energy per signal E. For $P_{e,b} = 10^{-5}$ we need a bit-energy, E_b to noise density N_0 ratio of $E_b/N_0 = 9.6$dB. **Note:** $Q(x)$ is a decreasing function which is 1/2 at $x = 0$. There are efficient algorithms (based on Taylor series expansions) to calculate $Q(x)$. Since $Q(x) \leq e^{\{-x^2/2\}}/2$ the error probability can be upper bounded by

$$P_{e,b} \leq \frac{1}{2} e^{\{-E_b/N_0\}}$$

which decreases exponentially with signal-to-noise ratio.

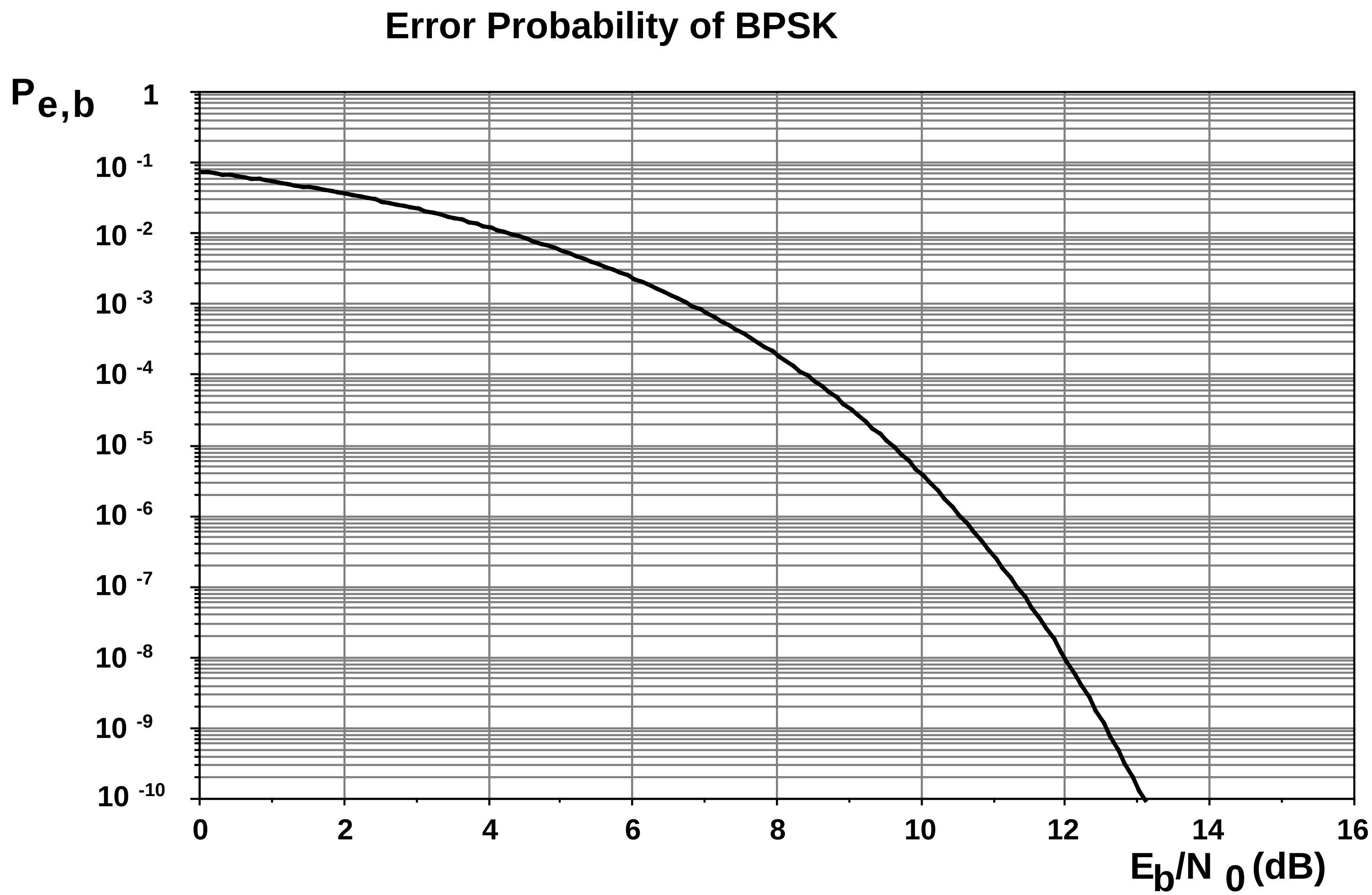


Figure 35: Error Probability of BPSK.

Bandwidth of BPSK

The power spectral density is a measure of the distribution of power with respect to frequency. The power spectral density for BPSK has the form

$$S(f) = \frac{PT}{2}\left[\mathrm{sinc}^2((f-f_c)T) + \mathrm{sinc}^2((f+f_c)T)\right]$$

where

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Notice that

$$\int_{-\infty}^{\infty} S(f)\,df = P.$$

The power spectrum has zeros or nulls at $f - f_c = i/T$ except for $i = 0$; that is there is a null at $f - f_c = \pm 1/T$ called the first null; a null at $f - f_c = \pm 2/T$ called the second null; etc. The bandwidth between the first nulls is called the null-to-null bandwidth. For BPSK the null-to-null bandwidth is $2/T$. Notice that the spectrum falls off as $(f - f_c)^2$ as f moves away from f_c. (The spectrum of MSK falls off as the fourth power, versus the second power for BPSK).

It is possible to reduce the bandwidth of a BPSK signal by filtering. If the filtering is done properly the (absolute) bandwidth of the signal can be reduced to $1/T$ without causing any intersymbol interference; that is all the power is concentrated in the frequency range $-1/(2T) \leq |f - f_c| \leq 1/(2T)$. The drawbacks are that the signal loses its constant envelope property (useful for nonlinear amplifiers) and the sensitivity to timing errors is greatly increased. The timing sensitivity problem can be greatly alleviated by filtering to a slightly larger bandwidth $-(1+\alpha)/(2T) \leq |f - f_c| \leq (1+\alpha)/(2T)$.

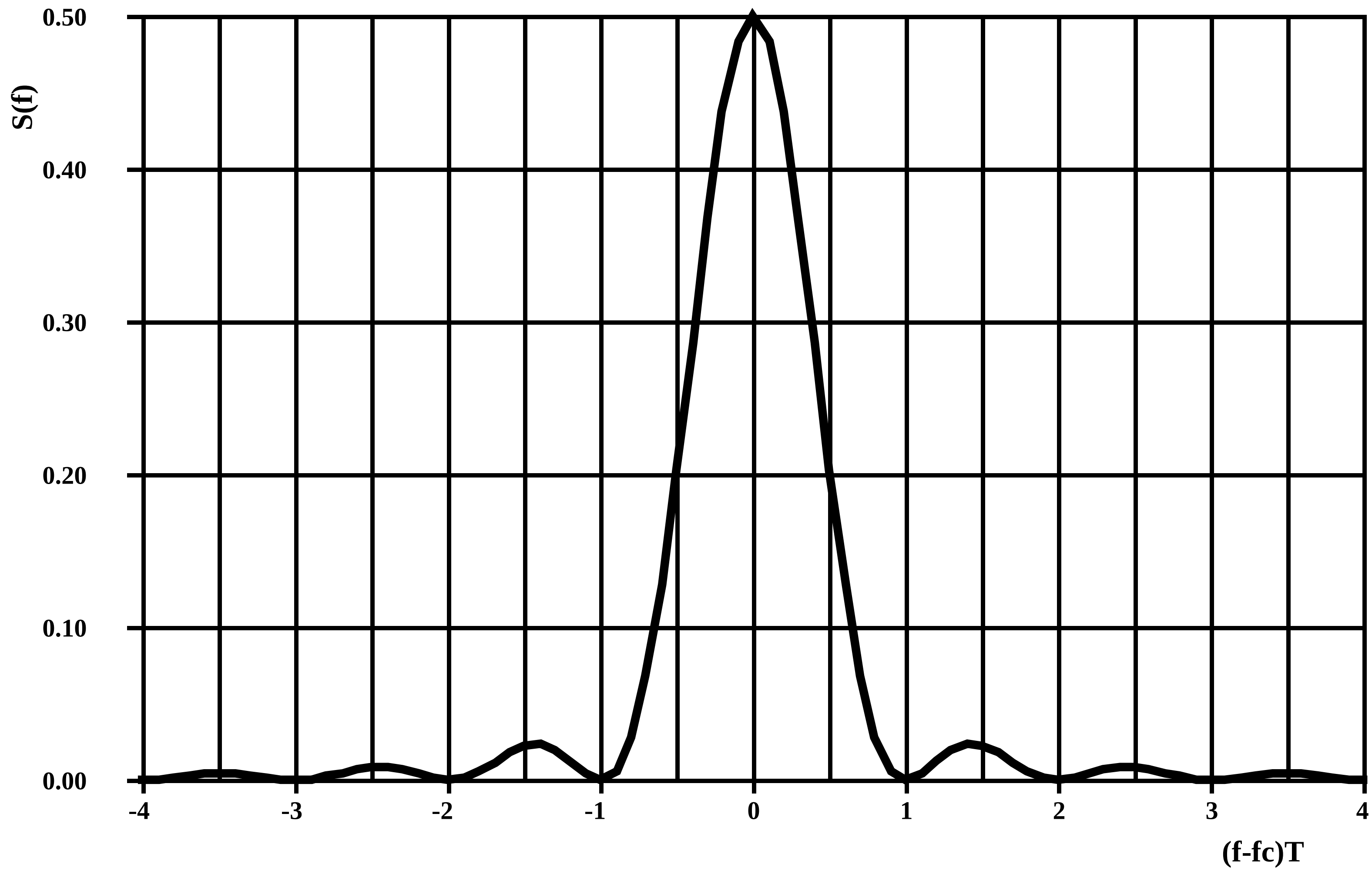


Figure 36: Spectrum of BPSK

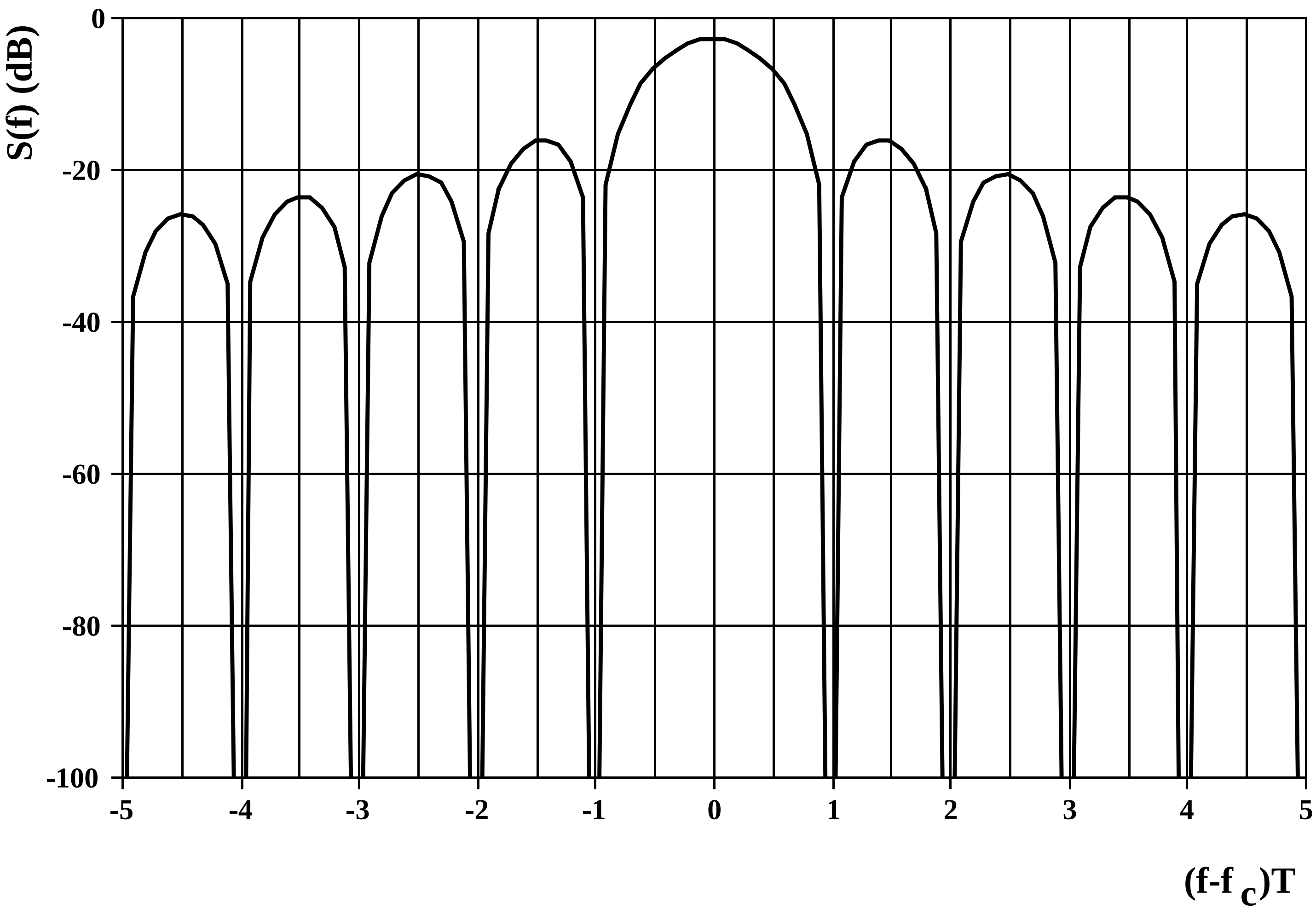


Figure 37: Spectrum of BPSK

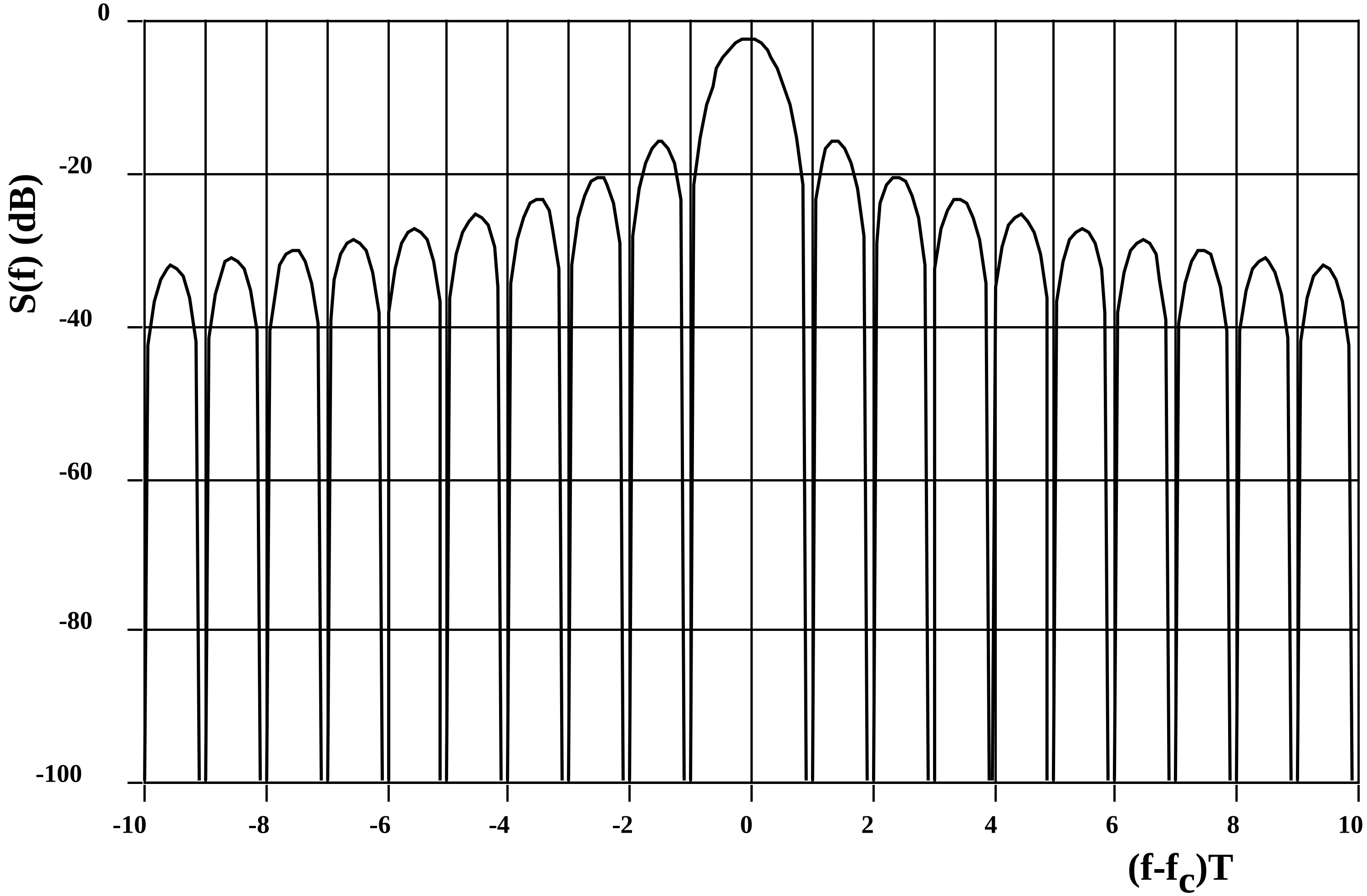


Figure 38: Spectrum of BPSK

Example

Given:

- Noise power spectral density of $N_0/2 = -150$ dBm/Hz $=10^{-18}$ Watts/Hz.
- $P_r = 3 \times 10^{-10}$ Watts
- Desired $P_e = 10^{-7}$.

Find: The data rate that can be used and the bandwidth that is needed.

Solution: Need $Q(\sqrt{2E_b/N_0}) = 10^{-7}$ or $E_b/N_0 = 11.3$dB or $E_b/N_0 = 13.52$. But $E_b/N_0 = P_r T/N_0 = 13.52$. Thus the data bit must be at least $T = 9.0 \times 10^{-8}$ seconds long, i.e. the data rate $1/T$ must be less than 11 Mbits/second. Clearly we also need a (null-to-null) bandwidth of 22 MHz.

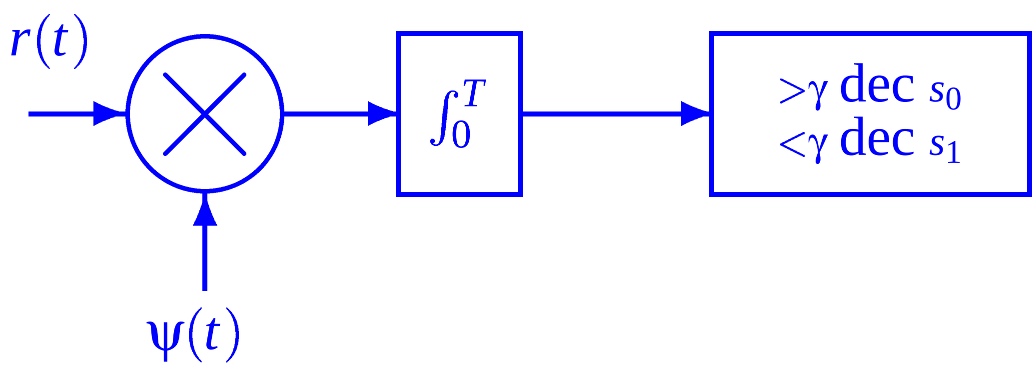
An alternative view of BPSK is that of two antipodal signals; that is

$$s_0(t) = \sqrt{E}\psi(t), \quad 0 \le t \le T$$

and

$$s_1(t) = -\sqrt{E}\psi(t), \quad 0 \le t \le T$$

where $\psi(t) = \sqrt{2/T}\cos(2\pi f_c t)$, $0 \le t \le T$ is a unit energy waveform. The above describes the signals transmitted only during the interval $[0, T]$. Obviously this is repeated for other intervals. The receiver correlates with $\psi(t)$ over the interval $[0, T]$ and compares with a threshold (usually 0) to make a decision. The correlation receiver is shown below.



This is called the “Correlation Receiver.” Note that synchronization to the symbol timing and oscillator phase are required.

Spectrum for Passband Pulses

Now Let $Z(t) = Y(t)\cos(\omega_c t + \theta)$ with θ uniform on $[0, 2\pi]$ and independent of $Y(t)$. Then

$$\begin{aligned} R_Z(\tau) &= \frac{1}{2} R_Y(\tau)\cos\omega_c\tau \\ S_Z(\omega) &= \frac{1}{4}\left[S_Y(\omega-\omega_c) + S_Y(\omega+\omega_c)\right] \end{aligned}$$

If $Z(t) = Y_1(t)\cos(\omega_c t + \theta_1) + Y_2(t)\cos(\omega_c t + \theta_2)$ with $Y_1(t)$ and $Y_2(t)$ independent then

$$R_Z(\tau) = \frac{1}{2}\left[R_{Y_1}(\tau) + R_{Y_2}(\tau)\right]\cos\omega_c\tau$$

Application: for PSK $x(t) = Ap_T(t)$. The spectrum is

$$S_Z(f) = \frac{A^2 T}{4}\left\{\operatorname{sinc}^2(f - f_c)T + \operatorname{sinc}^2(f + f_c)T\right\}.$$