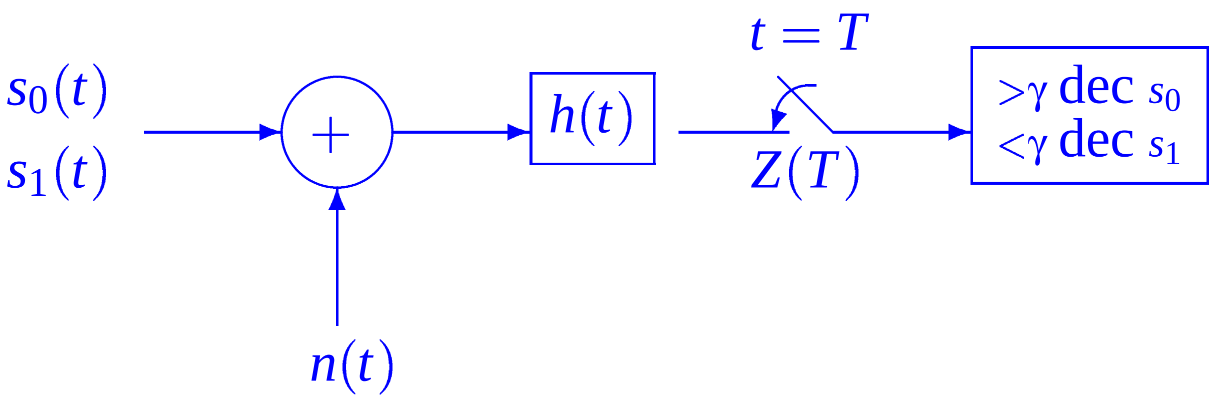
Lecture 6

Goals:

- Determine the optimal threshold, filter, signals for a binary communications problem

Minimum Average Error Probability

Problem: Find the optimum filter, threshold and signals to minimize the average error probability.



$$
\begin{aligned}
P_{e,0} &= P\{\text{error}|s_0 \text{ transmitted}\}. \\
P_{e,1} &= P\{\text{error}|s_1 \text{ transmitted}\}. \\
\pi_0 &= \text{Probability } s_0 \text{ transmitted.}
\end{aligned}
$$

$$\pi_1 = \text{Probability } s_1 \text{ transmitted.}$$

$$(\pi_0 + \pi_1 = 1).$$

The average probability of error is

$$\bar{P_e} = P_{e,0}\pi_0 + P_{e,1}\pi_1. \tag{1}$$

Let

$$\hat{s}_0(T) = \int_{-\infty}^{\infty} h(T-\tau)s_0(\tau)d\tau \leftarrow \text{output due to } s_0 \text{ alone,}$$

$$\hat{s}_1(T) = \int_{-\infty}^{\infty} h(T-\tau)s_1(\tau)d\tau \leftarrow \text{output due to } s_1 \text{ alone,}$$

Since we assume that the receiver will decide s_0 if the output of the filter is larger than a threshold and s_1 if it is smaller, we need to assume that $\hat{s}_0(T) > \hat{s}_1(T)$.

$$P_{e,0} = P\left\{Z(T) < \gamma | s_0 \text{ transmitted}\right\}.$$

If s_0 is transmitted then $Z(T)$ takes the form

$$Z(T) = \hat{s}_0(T) + \eta$$

where η is a Gaussian random variable with mean 0 and variance σ_N^2;

$$\sigma_N^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df.$$

Thus

$$\begin{aligned} P_{e,0} &= P\{\hat{s}_0(T) + \eta < \gamma\} \\ &= P\{\eta < \gamma - \hat{s}_0(T)\} \\ &= Q\left(\frac{\hat{s}_0(T) - \gamma}{\sigma_N}\right). \end{aligned} \tag{2}$$

$$\begin{aligned} P_{e,1} &= P\{Z(T) > \gamma | s_1 \text{ transmitted}\} \\ &= P\{\hat{s}_1(T) + \eta > \gamma | s_1 \text{ transmitted}\} \\ &= P\{\eta > \gamma - \hat{s}_1(T)\} \end{aligned}$$

$$= \quad Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma_N}\right). \tag{3}$$

Substituting (2) and (3) into (1) yields

$$\bar{P}_e(\gamma, h(t), s_0, s_1) = \pi_0 Q\left(\frac{\hat{s}_0(T) - \gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma_N}\right). \tag{4}$$

The problem is to minimize the error probability over all choices of $\gamma, h(t)$ and $s_0(t), s_1(t)$.

Step 1: Minimize $\bar{P}_e$ over γ

Facts used:

$$
\begin{aligned}
Q(x) &= \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du, \\
Q'(x) &= \frac{-e^{-x^2/2}}{\sqrt{2\pi}}.
\end{aligned}
$$

Method: Set the derivative of $\bar{P}_e$ with respect to γ equal to 0.

$$
\begin{aligned}
\frac{d\bar{P}_e}{d\gamma} &= \pi_0 \left(\frac{-\exp\{-\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)^2/2\}}{\sqrt{2\pi}} \left(-\frac{1}{\sigma_N}\right) \right) \\
&\quad + \pi_1 \left(\frac{-\exp\{-\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)^2/2\}}{\sqrt{2\pi}} \left(\frac{1}{\sigma_N}\right) \right) = 0
\end{aligned}
$$

$$\pi_0 \exp\{-\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)^2/2\} = \pi_1 \exp\{-\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)^2/2\}$$

$$\exp\left\{\left[(\gamma-\hat{s}_1(T))^2-(\hat{s}_0(T)-\gamma)^2\right]/2\sigma_N^2\right\} = \frac{\pi_1}{\pi_0}$$

$$\gamma^2 - 2\gamma\hat{s}_1(T) + \hat{s}_1^2(T) - \hat{s}_0^2(T) + 2\gamma\hat{s}_0(T) - \gamma^2 = 2\sigma_N^2 \ln\frac{\pi_1}{\pi_0}$$

$$2\gamma[\hat{s}_0(T)-\hat{s}_1(T)] = 2\sigma_N^2 \ln\frac{\pi_1}{\pi_0} + \hat{s}_0^2(T) - \hat{s}_1^2(T)$$

$$\gamma = \frac{\sigma_N^2 \ln\frac{\pi_1}{\pi_0} + \frac{\hat{s}_0^2(T)-\hat{s}_1^2(T)}{2}}{\hat{s}_0(T)-\hat{s}_1(T)}$$

$$\boxed{\gamma_{opt} = \frac{\sigma_N^2 \ln\frac{\pi_1}{\pi_0}}{\hat{s}_0(T)-\hat{s}_1(T)} + \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2}.} \qquad (5)$$

Special Case: If $\pi_1 = \pi_0 = 1/2$ then

$$\gamma_{opt} = \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2}.$$

What is $\overline{P_e}$ for the optimal threshold?

$$
\begin{aligned}
\hat{s}_0(T) - \gamma_{opt} &= \hat{s}_0(T) - \left[\frac{\sigma_N^2 \ln \frac{\pi_1}{\pi_0}}{\hat{s}_0(T) - \hat{s}_1(T)} + \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2} \right] \\
&= \frac{\hat{s}_0(T) - \hat{s}_1(T)}{2} - \frac{\sigma_N^2}{\hat{s}_0(T) - \hat{s}_1(T)} \ln \frac{\pi_1}{\pi_0} \\
\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} &= \frac{\hat{s}_0(T) - \hat{s}_1(T)}{2\sigma_N} - \frac{\sigma_N}{\hat{s}_0(T) - \hat{s}_1(T)} \ln \frac{\pi_1}{\pi_0}. \qquad (6)
\end{aligned}
$$

Definition:

$$
\begin{aligned}
(f(t), g(t)) &\triangleq \int_{-\infty}^{\infty} f(t) g(t) dt \\
s_T(t) &\triangleq s_0(T-t) - s_1(T-t) \\
\hat{s}_0(T) - \hat{s}_1(T) &= \int_{-\infty}^{\infty} h(\tau) \left[s_0(T-\tau) - s_1(T-\tau) \right] d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) s_T(\tau) d\tau = (h, s_T).
\end{aligned}
$$

Thus from (6)

$$\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} = \frac{(h, s_T)}{2\sigma_N} - \frac{\sigma_N}{(h, s_T)} \ln \frac{\pi_1}{\pi_0}. \qquad (7)$$

Similarly

$$\frac{\gamma_{opt} - \hat{s}_1(T)}{\sigma_N} = \frac{(h, s_T)}{2\sigma_N} + \frac{\sigma_N}{(h, s_T)} \ln \frac{\pi_1}{\pi_0}. \qquad (8)$$

Remember that

$$\sigma_N^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t)dt \qquad (9)$$

$$= \frac{N_0}{2} \|h\|^2 \qquad (\|h\|^2 = \int_{-\infty}^{\infty} h^2(t)dt).$$

Let $\lambda \stackrel{\Delta}{=} \frac{(h, s_T)}{\|h\| \; \|s_T\|}$ (10)

$$\|s_T\|^2 = \int_{-\infty}^{\infty} [s_0(T-t) - s_1(T-t)]^2 dt$$

$$= \int_{-\infty}^{\infty} s_0^2(T-t) - 2s_0(T-t)s_1(T-t) + s_1^2(T-t)dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} s_0^2(t) - 2(s_0, s_1) + \int_{-\infty}^{\infty} s_1^2(t)\,dt \\
&= E_0 + E_1 - 2r\bar{E}. \\
r &= (s_0, s_1)/\bar{E}, \quad \bar{E} = \frac{E_0 + E_1}{2}.
\end{aligned}
$$

$$
\|s_T\|^2 = 2\bar{E}(1-r) \Rightarrow \|s_T\| = \sqrt{2\bar{E}(1-r)}. \tag{11}
$$

Combining (7), (8), (9), (10), and (11)

$$
\begin{aligned}
\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} &= \frac{(h, s_T)}{2\sqrt{\frac{N_0}{2}}\|h\|} - \frac{\sqrt{\frac{N_0}{2}}\|h\|}{(h, s_T)} \ln \frac{\pi_1}{\pi_0} \\
&= \frac{(h, s_T)}{\|h\|\ \|s_T\|} \frac{\sqrt{2\bar{E}(1-r)}}{\sqrt{2N_0}} - \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \frac{\|h\|\ \|s_T\|}{(h, s_T)} \ln \frac{\pi_1}{\pi_0}.
\end{aligned}
$$

Let $\lambda = \frac{(h, s_T)}{\|h\| \, \|s_T\|}$. Then

$$\frac{\hat{s}_0(T) - \gamma_{opt}}{\sigma_N} = \lambda\alpha - \beta\frac{1}{\lambda},$$

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln\frac{\pi_1}{\pi_0}.$$

Similarly

$$\frac{\gamma_{opt} - \hat{s}_1(T)}{\sigma_N} = \lambda\alpha + \beta\frac{1}{\lambda}.$$

Summary of Step 1:

$$\gamma_{opt} = \frac{\sigma_N^2 \ln \frac{\pi_1}{\pi_0}}{\hat{s}_0(T) - \hat{s}_1(T)} + \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}.$$

$$\bar{P}_e(\gamma_{opt}, h(t), s_0(t), s_1(t)) = \pi_0 Q\left(\lambda\alpha - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\lambda\alpha + \frac{\beta}{\lambda}\right).$$

$$\lambda = \frac{(h, s_T)}{||h|| \; ||s_T||}.$$

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln \frac{\pi_1}{\pi_0}.$$

$$\bar{E} = \frac{E_0 + E_1}{2}, \quad r = (s_0, s_1)/\bar{E}.$$

Step 2:

Find the optimal filter $h(t)$ to minimize the average probability of error

Method: First show that $\bar{P}_e$ is an decreasing function of λ by showing the derivative is negative. Then find the h that maximizes λ (thus minimizing $\bar{P}_e$).

$$\begin{aligned} \bar{P}_e(h, s_0, s_1) &= \bar{P}_e(\gamma_{opt}, h, s_0, s_1) \\ &= \pi_0 Q(\left(\alpha\lambda - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\alpha\lambda + \frac{\beta}{\lambda}\right). \end{aligned}$$

$$\begin{aligned} \frac{\partial \bar{P}_e}{\partial \lambda} &= \pi_0 \left[-e^{-(\alpha\lambda - \frac{\beta}{\lambda})^2/2} \frac{1}{\sqrt{2\pi}} \left(\alpha + \frac{\beta}{\lambda^2}\right)\right] + \pi_1 \left[-e^{-(\alpha\lambda + \frac{\beta}{\lambda})^2/2} \frac{1}{\sqrt{2\pi}} \left(\alpha - \frac{\beta}{\lambda^2}\right)\right] \\ &= -\pi_0 \left[\left(\alpha + \frac{\beta}{\lambda^2}\right) \exp\left\{-\frac{1}{2}\left(\alpha^2\lambda^2 - 2\alpha\beta + \beta^2/\lambda^2\right)\right\}\right] \frac{1}{\sqrt{2\pi}} \\ &\quad -\pi_1 \left[\left(\alpha - \frac{\beta}{\lambda^2}\right) \exp\left\{-\frac{1}{2}\left(\alpha^2\lambda^2 + 2\alpha\beta + \beta^2/\lambda^2\right)\right\}\right] \frac{1}{\sqrt{2\pi}} \end{aligned}$$

$$
\begin{aligned}
&= -\exp\left\{-\frac{1}{2}\left(\alpha^2\lambda^2+\beta^2/\lambda^2\right)\right\}\left[\pi_0\left(\alpha+\frac{\beta}{\lambda^2}\right)e^{\alpha\beta}+\pi_1\left(\alpha-\frac{\beta}{\lambda^2}\right)e^{-\alpha\beta}\right] \\
\alpha\beta &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}\sqrt{\frac{N_0}{4\bar{E}(1-r)}}\ln\frac{\pi_1}{\pi_0} \\
&= \frac{1}{2}\ln\frac{\pi_1}{\pi_0} = \ln\sqrt{\frac{\pi_1}{\pi_0}}. \\
e^{\alpha\beta} &= \sqrt{\frac{\pi_1}{\pi_0}}, \quad e^{-\alpha\beta} = \sqrt{\frac{\pi_0}{\pi_1}}. \\
\pi_0 e^{\alpha\beta} &= \sqrt{\pi_0\pi_1}, \quad \pi_1 e^{-\alpha\beta} = \sqrt{\pi_0\pi_1}. \\
\frac{d\bar{P}_e}{d\lambda} &= -e^{-1/2(\alpha^2\lambda^2+\beta^2/\lambda^2)}\left[\sqrt{\pi_0\pi_1}\left(\alpha+\frac{\beta}{\lambda^2}+\alpha-\frac{\beta}{\lambda^2}\right)\right] \\
&= -\frac{1}{\sqrt{2\pi}}e^{-1/2(\alpha^2\lambda^2+\beta^2/\lambda^2)}\sqrt{\pi_0\pi_1}(2\alpha).
\end{aligned}
$$

Since $\alpha > 0$, $\frac{d\bar{P}_e}{d\lambda} < 0$ so that $\bar{P}_e$ is minimized by maximizing λ.

$$\lambda = \frac{(h, s_T)}{\|h\| \; \|s_T\|}.$$

From Schwartz's inequality

$$-\|h\| \; \|s_T\| \leq (h, s_T) \leq \|h\| \; \|s_T\|.$$

Thus $-1 \leq \lambda \leq 1$ with equality if $h = s_T$. Choose $\lambda = 1$ $(h = s_T = s_0(T-t) - s_1(T-t))$. For optimal threshold and optimal filter

$$\bar{P}_e(\gamma_{opt}, h_{opt}, s_0, s_1) = \pi_0 Q(\alpha - \beta) + \pi_1 Q(\alpha + \beta).$$

$h(t) = s_0(T-t) - s_1(T-t)$ is called the matched filter because it is matched to the signals.

$$\gamma_{opt}(h_{opt}, s_0, s_1) = \frac{1}{2}(E_0 - E_1) + \frac{1}{2} N_0 \ln \frac{\pi_1}{\pi_0}.$$

For the optimal filter the output due to signal alone are

$$\hat{s}_0(T) = E_0 - r\bar{E}$$

$$\hat{s}_1(T) = r\bar{E} - E_1$$

If $\pi_0 = \pi_1$ then $\bar{P_e} = Q(\alpha) = Q\left(\sqrt{\frac{\bar{E}(1-r)}{N_0}}\right)$.

Step 3:

Find the optimal signals $s_0(t)$ and $s_1(t)$ to minimize the average probability of error.

Method: $\bar{P}_e$ depends on the signal only through $\bar{E}$ and r.

$$\left(\bar{E} = \frac{E_0 + E_1}{2}, r = (s_0, s_1)/\bar{E}\right).$$

It is obvious that we could just increase the energy to infinity and get error probability 0. Instead we will fix $\bar{E}$ and vary the signals to vary r. Again we show that $\bar{P}_e$ is an increasing function of r and then choose the signals to minimize r.

$$\bar{P}_e = \pi_0 Q(\alpha - \beta) + \pi_1 Q(\alpha + \beta).$$

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \; \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln \frac{\pi_1}{\pi_0}.$$

$$
\begin{aligned}
\frac{d\bar{P}_e}{dr} &= \pi_0 \left[\frac{-e^{-(\alpha-\beta)^2/2}}{\sqrt{2\pi}} \frac{\partial \alpha}{\partial r} - \frac{\partial \beta}{\partial r} \right] + \pi_1 \left[\frac{-e^{-(\alpha+\beta)^2/2}}{\sqrt{2\pi}} \frac{\partial \alpha}{\partial r} + \frac{\partial \beta}{\partial r} \right] \\
&= -e^{-(\alpha^2+\beta^2)/2} \left[\pi_0 e^{\alpha\beta} \left(\frac{\partial \alpha}{\partial r} - \frac{\partial \beta}{\partial r} \right) + \pi_1 e^{\alpha\beta} \left(\frac{\partial \alpha}{\partial r} + \frac{\partial \beta}{\partial r} \right) \right] \\
&= -e^{-(\alpha^2+\beta^2)/2} \left[\sqrt{\pi_0 \pi_1} 2 \frac{\partial \alpha}{\partial r} \right]. \\
\frac{\partial \alpha}{\partial r} &= \sqrt{\frac{\bar{E}}{N_0} \frac{1}{2}} \left(\frac{-1}{\sqrt{1-r}} \right) < 0 \\
&\Rightarrow \frac{d\bar{P}_e}{dr} > 0
\end{aligned}
$$

From Schwartz's inequality

$$
r = \frac{(s_0, s_1)}{\bar{E}} \geq \frac{-\|s_0\| \, \|s_1\|}{\bar{E}}
$$

with equality if $s_0 = -Ks_1, \;\; K > 0$. For $s_0 = -Ks_1$

$$\begin{aligned} r &= -\frac{\sqrt{E_0 E_1}}{\left(\frac{E_0+E_1}{2}\right)} \\ &\geq -1 \end{aligned}$$

with equality if $E_0 = E_1$. (Arithmetic mean $\geq$ Geometric mean).

Two signals $s_0(t)$ and $s_1(t)$ are said to be antipodal if

$$s_0(t) = -s_1(t).$$

Optimal signals are antipodal.

$$\begin{aligned} r = -1 \quad \Rightarrow \quad \alpha &= \sqrt{\frac{2E}{N_0}}, \\ \beta &= \sqrt{\frac{N_0}{8E}} \ln \frac{\pi_1}{\pi_0}. \end{aligned}$$

If $\pi_1 = \pi_0 = 1/2$ then

$$\bar{P_e} = Q(\alpha) = Q\left(\sqrt{\frac{2E}{N_0}}\right).$$

Aside: Schwartz's inequality:

For any α

$$||f-\alpha g||^2 \geq 0$$
$$||f||^2 - 2\alpha(f,g) + \alpha^2||g||^2 \geq 0$$

Since the polynomial in α is never negative there must be either no zeros or a double zero. Thus the discrimanant must be not be positive.

$$\begin{aligned} 4(f,g)^2 - 4||f||^2||g||^2 &\leq 0 \\ -||f||\,||g|| &\leq (f,g) \leq ||f||\,||g|| \end{aligned}$$

Equality occurs when $f(x) = Kg(x)$. If K is positive the inequality on the right side becomes equality and if K is negative the inequality on the right side becomes equality. This is Schwartz's inequality.

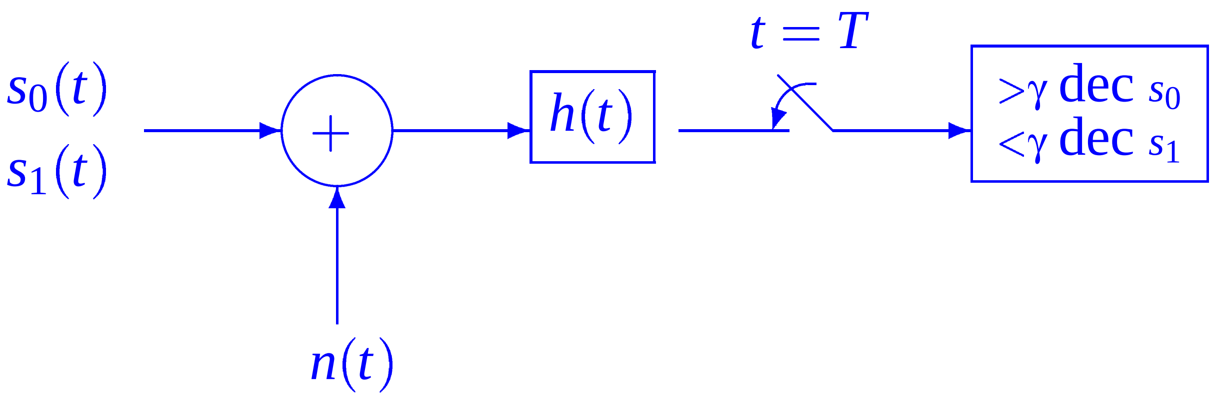
Aside: Arithmetic mean $\geq$ Geometric mean:

Let a_0 and a_1 be real nonnegative numbers.

$$(a_0 - a_1)^2 \geq 0 \text{ with equaltiy if } a_0 = a_1.$$

$$\begin{aligned}
a_0^2 - 2a_0a_1 + a_1^2 &\geq 0 \\
a_0^2 + 2a_0a_1 + a_1^2 &\geq 4a_0a_1 \\
(a_0 + a_1)^2 &\geq 4a_0a_1 \\
a_0 + a_1 &\geq 2\sqrt{a_0a_1} \\
\frac{a_0 + a_1}{2} &\geq \sqrt{a_0a_1} \text{ with equaltiy if } a_0 = a_1.
\end{aligned}$$

Summary



$$
\begin{aligned}
\hat{s}_0(T) &= \int_{-\infty}^{\infty} h(\tau) s_0(T-\tau)\, d\tau. \\
\hat{s}_1(T) &= \int_{-\infty}^{\infty} h(\tau) s_1(T-\tau)\, d\tau. \\
\sigma_N^2 &= \frac{N_0}{2} \int_{-\infty}^{\infty} h(\tau)\, d\tau = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df.
\end{aligned}
$$

$$\bar{P}_e(\gamma, h(t), s_0(t), s_1(t)) \quad = \quad \pi_0 Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right) + \pi_1 Q\left(\frac{\gamma - \hat{s}_1(T)}{\sigma_N}\right).$$

Step 1: Optimize with respect to γ.

$$\bar{P}_e(\gamma_{opt}, h, s_0, s_1) \quad = \quad \pi_0 Q\left(\alpha\lambda - \frac{\beta}{\lambda}\right) + \pi_1 Q\left(\alpha\lambda + \frac{\beta}{\lambda}\right).$$

$$\alpha \quad = \quad \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4E(1-r)}\ln\frac{\pi_1}{\pi_0}}, \quad \lambda = \frac{(h, s_T)}{\|h\|\ \|s_T\|}.$$

$$s_T(t) \quad = \quad s_0(T-t) - s_1(T-t).$$

$$\gamma_{opt} \quad = \quad \frac{\hat{s}_0(T)+\hat{s}_1(T)}{2} + \frac{\sigma_N^2}{\hat{s}_0(T)-\hat{s}_1(T)} \ln\frac{\pi_1}{\pi_0}.$$

Step 2: Optimize with respect to $h(t)$.

$$\bar{P}_e(\gamma_{opt}, h_{opt}, s_0, s_1) \quad = \quad \pi_0 Q(\alpha+\beta) + \pi_1 Q(\alpha+\beta).$$

$$
\begin{aligned}
\alpha &= \sqrt{\frac{\bar{E}(1-r)}{N_0}}, \quad \beta = \sqrt{\frac{N_0}{4\bar{E}(1-r)}} \ln \frac{\pi_1}{\pi_0}. \\
h_{opt}(t) &= s_0(T-t) - s_1(T-t), \quad \textbf{the matched filter.} \\
\gamma_{opt}|_{h=h_{opt}} &= \frac{1}{2}(E_0 - E_1) + \frac{1}{2} N_0 \ln \frac{\pi_1}{\pi_0}.
\end{aligned}
$$

Step 3: Optimize with respect to s_0 and s_1.

$$
\begin{aligned}
\bar{P}_e(\gamma_{opt}, h_{opt}, s_{0,opt}, s_{1,opt}) &= \pi_0 Q\left(\hat{\alpha} + \hat{\beta}\right) + \pi_1 Q\left(\hat{\alpha} - \hat{\beta}\right). \\
\hat{\alpha} &= \sqrt{\frac{2\bar{E}}{N_0}}, \quad \hat{\beta} = \sqrt{\frac{N_0}{8\bar{E}}} \ln \frac{\pi_1}{\pi_0}. \\
s_0(t) &= -s_1(t). \\
h_{opt}(t)\Big|_{\substack{s_0 = s_{0,opt} \\ s_1 = s_{1,opt}}} &= 2 s_0(T-t).
\end{aligned}
$$

$$\gamma_{opt}\big|_{h=h_{opt},s_{0,opt},s_{1,opt}} = \frac{1}{2}(N_0)\ln\frac{\pi_1}{\pi_0}.$$

SPECIAL CASE

$$\pi_0 = \pi_1 = 1/2$$

$$\bar{P_e} = 1/2Q\left(\frac{\hat{s}_0(T)-\gamma}{\sigma_N}\right)+1/2Q\left(\frac{\gamma-\hat{s}_1(T)}{\sigma_N}\right)$$

Step 1: Optimize with respect to γ

$$\bar{P_e} = Q(\alpha\lambda),$$

$$\alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}} \quad \lambda = \frac{(h, s_T)}{\|h\| \, \|S_T\|}$$

$$\bar{E} = \frac{E_0+E_1}{2} \quad r = (s_0, s_1)/\bar{E}$$

$$E_0 = \int_{-\infty}^{\infty} s_0^2(t)dt \quad E_1 = \int_{-\infty}^{\infty} s_1^2(t)dt$$

$$\gamma_{opt} = \frac{\hat{s}_0(T) + \hat{s}_1(T)}{2}$$

Step 2: Optimize with respect to $h(t)$

$$\bar{P}_e = Q(\alpha), \quad \alpha = \sqrt{\frac{\bar{E}(1-r)}{N_0}}$$

$$h_{opt} = s_0(T-t) - s_1(T-t) \text{ matched filter}$$

$$\gamma_{opt} = 1/2(E_0 - E_1)$$

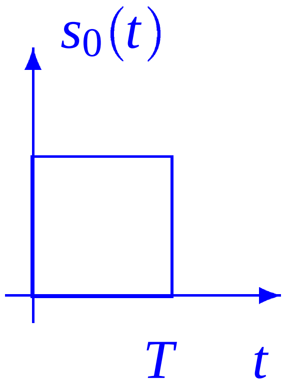
Step 3: Optimize with respect to $s_0(t)$ and $s_1(t)$.

$$\bar{P}_e = Q(\hat{\alpha}) \quad \hat{\alpha} = \sqrt{\frac{2\bar{E}}{N_0}}$$

$$s_0(t) = -s_1(t)$$

$$h_{opt} = 2s_0(T-t)\ , \ \gamma_{opt} = 0\ .$$

Example:



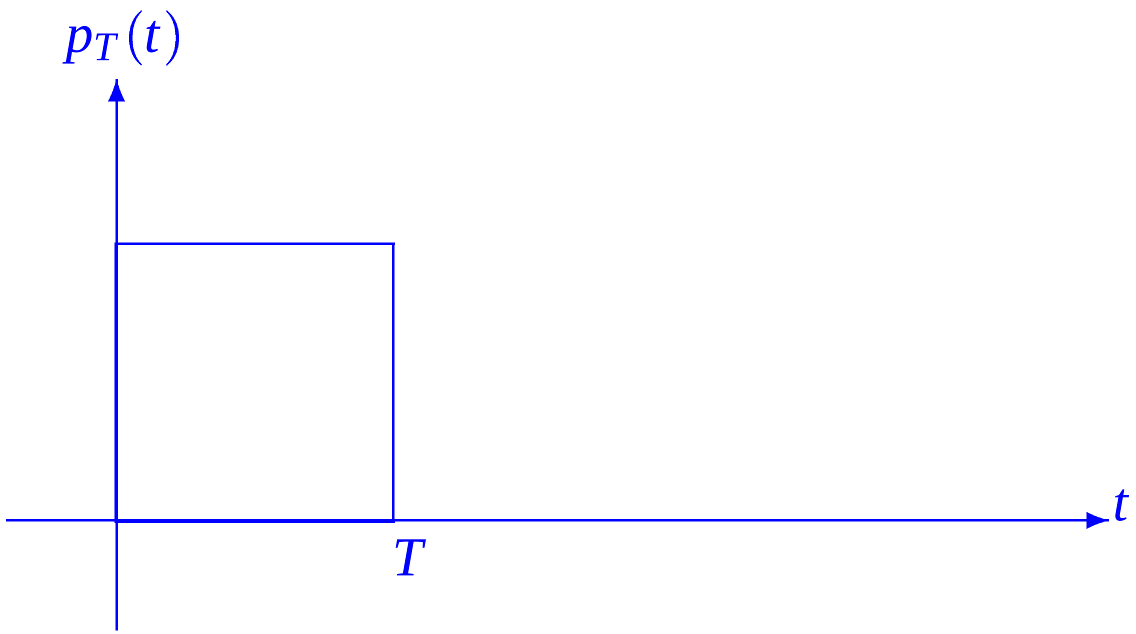
$$
\begin{aligned}
s_0(t) &= Ap_T(t) \\
s_1(t) &= -Ap_T(t)
\end{aligned}
$$

Baseband signals

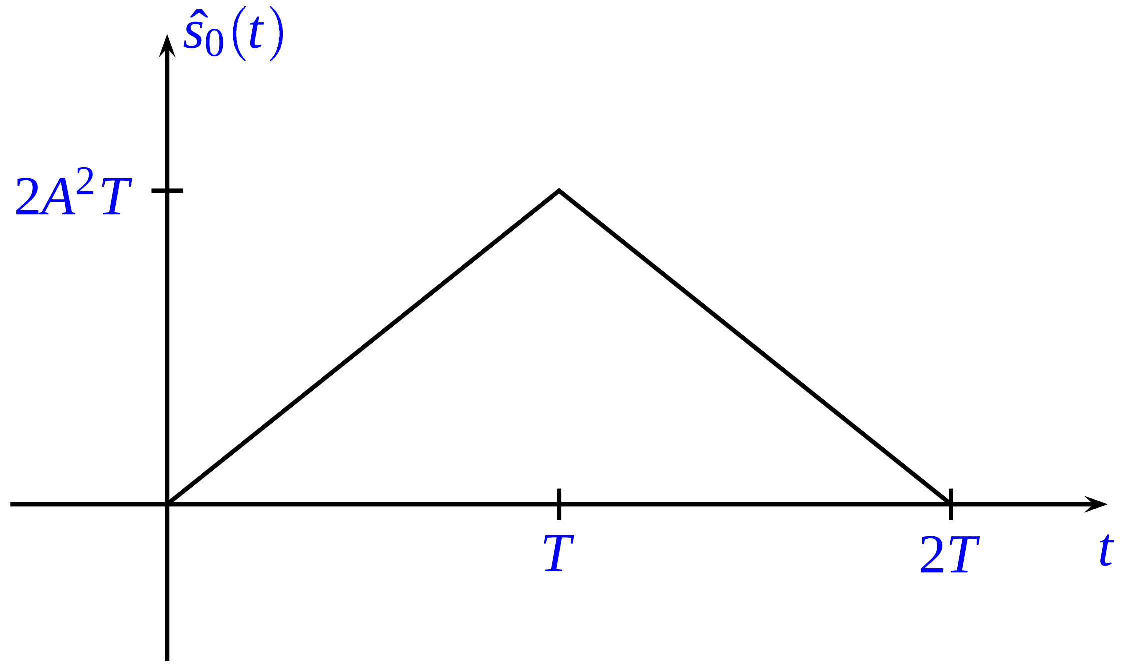
$$h_{opt}(t) = 2Ap_T(t)$$

Assume $s_0(t)$ transmitted

$$
\begin{aligned}
\int_{-\infty}^{\infty} h(t-\tau)s_0(\tau) &= \int_{-\infty}^{\infty} 2Ap_T(t-\tau)Ap_T(\tau)d\tau \\
&= 2A^2 \int_{-\infty}^{\infty} p_T(t-\tau)p_T(\tau)d\tau \\
&= 2A^2 \int_{t-T}^{t} p_T(\tau)d\tau
\end{aligned}
$$



The output due to signal alone:

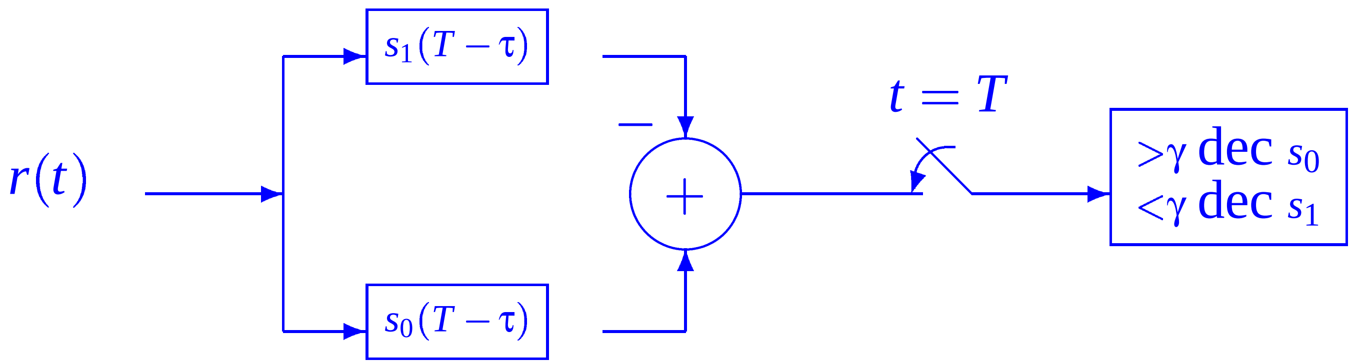


The output due to noise is a Gaussian random variable with mean zero and variance

$$\sigma_N^2 = \frac{1}{2} N_0 T (4A^2) = 2A^2 N_0 T$$

Let T_0 be the sampling time. Since the signal out is a maximum when $T_0 = T$ and the noise variance does not depend on the sample time the optimum sampling time is $T_0 = T$.

Equivalent form of optimal receiver



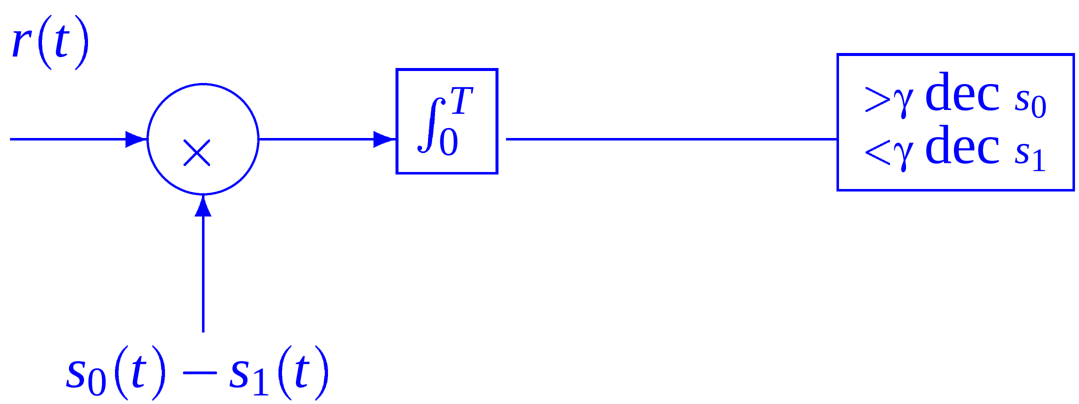
$$
\begin{aligned}
Z(t) &= \int_{-\infty}^{\infty} h(t-\tau) r(\tau) d\tau, \\
& \qquad h(t) = s_0(T-t) - s_1(T-t) \\
&= \int_{-\infty}^{\infty} r(\tau) \left[s_0(T-(t-\tau)) - s_1(T-(t-\tau)) \right] d\tau
\end{aligned}
$$

$$= \int_{-\infty}^{\infty} r(\tau)\,[s_0(\tau+T-t)-s_1(\tau+T-t)]\,d\tau$$

$$Z(T) = \int_{-\infty}^{\infty} r(\tau)\,[s_0(\tau)-s_1(\tau)]\,d\tau$$

If $s_0(t)$ and $s_1(t)$ are time limited to $[0,T]$ then

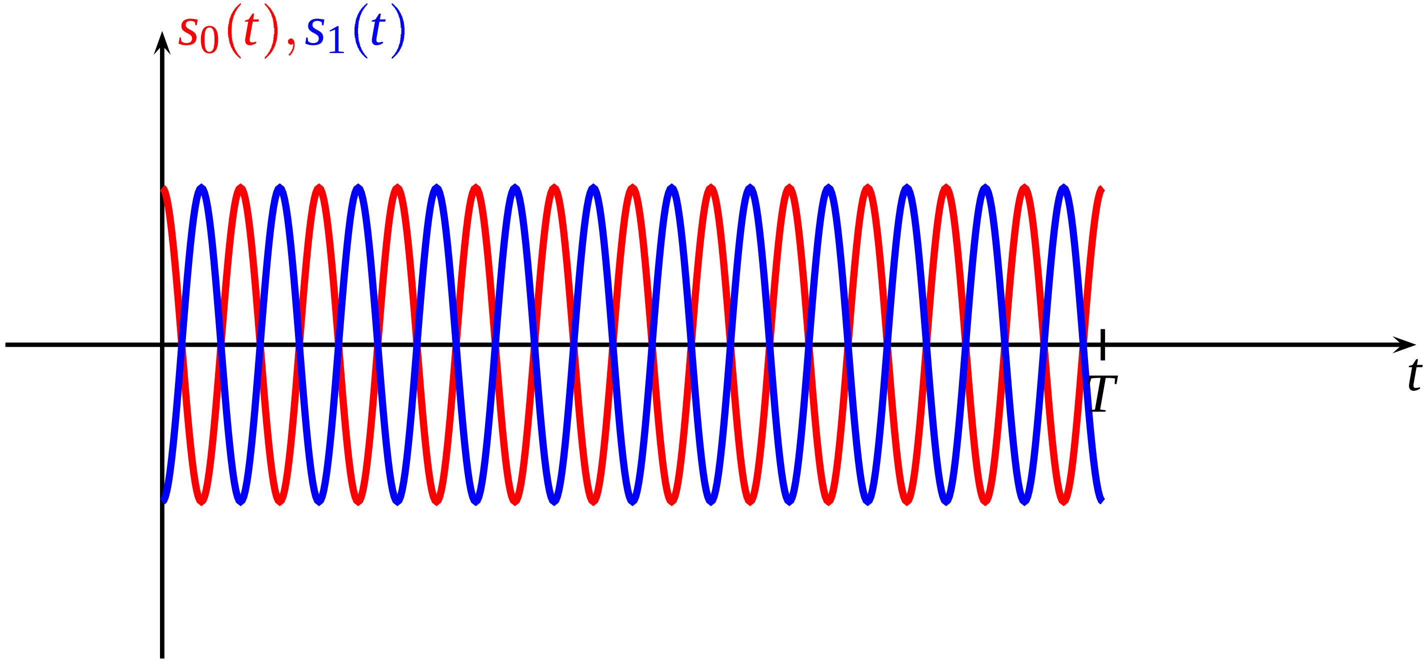
$$Z(T) = \int_{0}^{T} r(\tau)\,[s_0(\tau)-s_1(\tau)]\,d\tau$$

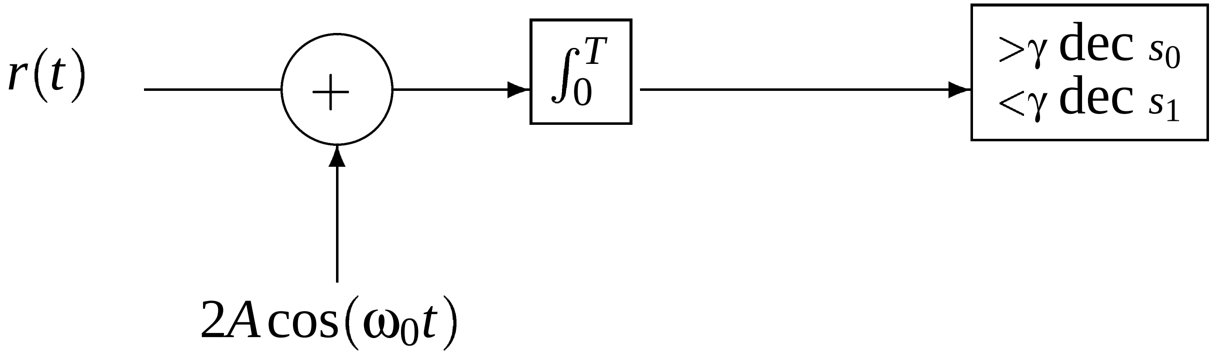


This is called the "Correlation Receiver."

Example: Binary Phase Shift Keying (RF signals)

$$
\begin{aligned}
s_0(t) &= A\cos(\omega_0 t)p_T(t) \\
s_1(t) &= -s_0(t) \\
s_i(t) &= (-1)^i A\cos(\omega_0 t)p_T(t) \\
&= A\cos(\omega_0 t + i\pi)p_T(t)
\end{aligned}
$$


$s_0(t), s_1(t)$
T
t


$r(t)$
$+$
$\int_0^T$
$>\gamma$ dec s_0
$<\gamma$ dec s_1
$2A\cos(\omega_0 t)$

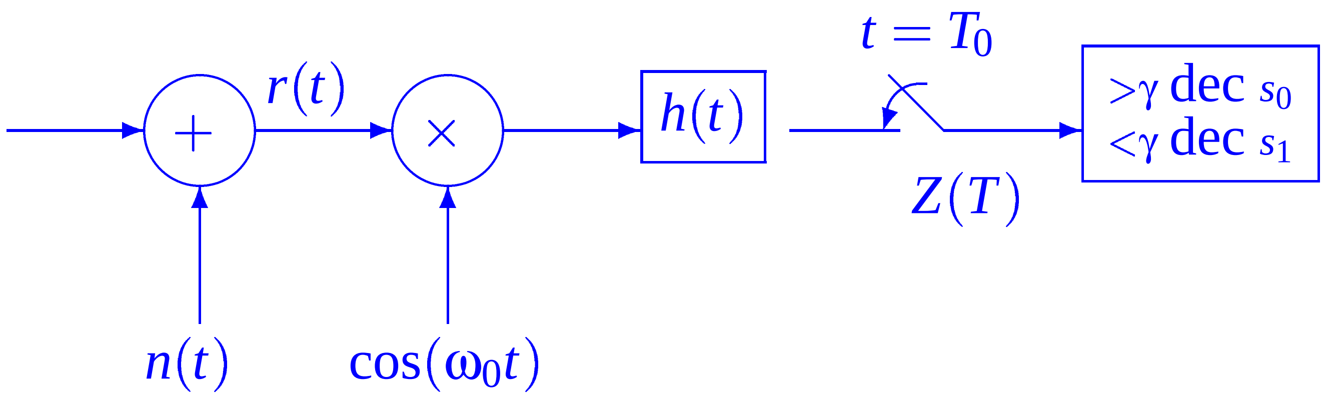
Assume $\omega_0 T >> 1$ or $\omega_0 T = 2n\pi$

$$\begin{aligned} E_i = \int_{-\infty}^{\infty} s_i^2(t)dt &= \int_0^T A^2 \cos^2(\omega_0 t)dt \\ &= A^2 \int_0^T 1/2 + 1/2\cos(2\omega_0 t)dt \\ &= \frac{A^2 T}{2}[1 + \frac{\sin(2\omega_0 T)}{2\omega_0 T}] \\ &= \frac{A^2 T}{2}. \end{aligned}$$

$$P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{A^2 T}{N_0}}\right) \quad (\pi_1 = \pi_0, \gamma = 0).$$

Suboptimal Receivers

$$s_i(t) = (-1)^i A\cos(\omega_0 t)p_T(t), \quad \omega_0 T >> 2n\pi$$



$$P_e = 1/2Q\left(\frac{\hat{s}_0(T_0)}{\sigma_N}\right) + 1/2Q\left(\frac{-\hat{s}_1(T_0)}{\sigma_N}\right)$$

Claim: $\sigma_n^2 = N_0\|h\|^2/4$. The factor of 1/2 is due to multiplying by $\cos(\omega_0 t)$ (power equals 1/2).

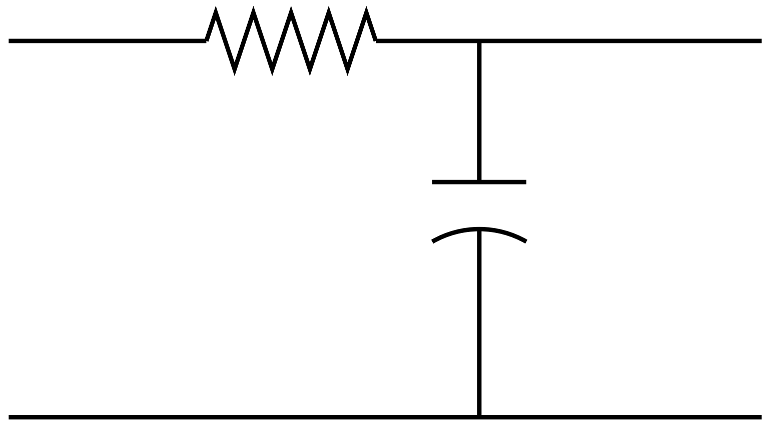
Proof:

$$
\begin{aligned}
\sigma_n^2 &= E[\int_0^T \int_0^T n(t)\cos(\omega_0 t) n(s)\cos(\omega_0 s)\,dt\,ds] \\
&= \int_0^T \int_0^T E[n(t)n(s)]\cos(\omega_0 t)\cos(\omega_0 s)\,dt\,ds \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(t-s)\cos(\omega_0 t)\cos(\omega_0 s)\,dt\,ds \\
&= \int_0^T \frac{N_0}{2}\cos^2(\omega_0 t)\,dt \\
&= \frac{N_0}{4}\int_0^T (1+\cos(2\omega_0 t))\,dt \\
&= \frac{N_0 T}{4}\left(1+\frac{\sin(2\omega_0 T)}{2\omega_0 T}\right) \\
&= \frac{N_0 T}{4}
\end{aligned}
$$

provide that $\omega_0 T >> 1$ or $2\omega_0 T = n\pi$.

$$
\begin{aligned}
\hat{s}_i(T_0) &= \int_{-\infty}^{\infty} h(T_0-\tau)(-1)^i A\cos^2\omega_0\tau\, p_T(\tau)\,d\tau \\
&= (-1)^i \frac{A}{2}\int_0^T \underbrace{h(T_0-\tau)}_{\text{low pass}}[1+\underbrace{\cos(2w_0\tau)}_{\text{high freq.}}]\,d\tau \\
&= (-1)^i \frac{A}{2}\int_0^T h(T_0-\tau)\,d\tau \\
P_e &= Q\left(\frac{|\hat{s}_i(T_0)|}{\sigma_N}\right)
\end{aligned}
$$

Example: Single pole RC filter



$$
\begin{aligned}
h(t) &= \frac{1}{RC} e^{-t/RC} u(t) \\
&= \alpha e^{-\alpha t} u(t), \ \alpha = 1/RC \\
\|h\| &= \int_{-\infty}^{\infty} h^2(t) dt = \int_0^{\infty} \alpha^2 e^{-2\alpha t} dt \\
&= \frac{\alpha^2}{-2\alpha} e^{-2\alpha t} |_0^{\infty} = \frac{\alpha}{2}
\end{aligned}
$$

$$\sigma_N^2 = \frac{N_0}{4}\left(\frac{\alpha}{2}\right)$$

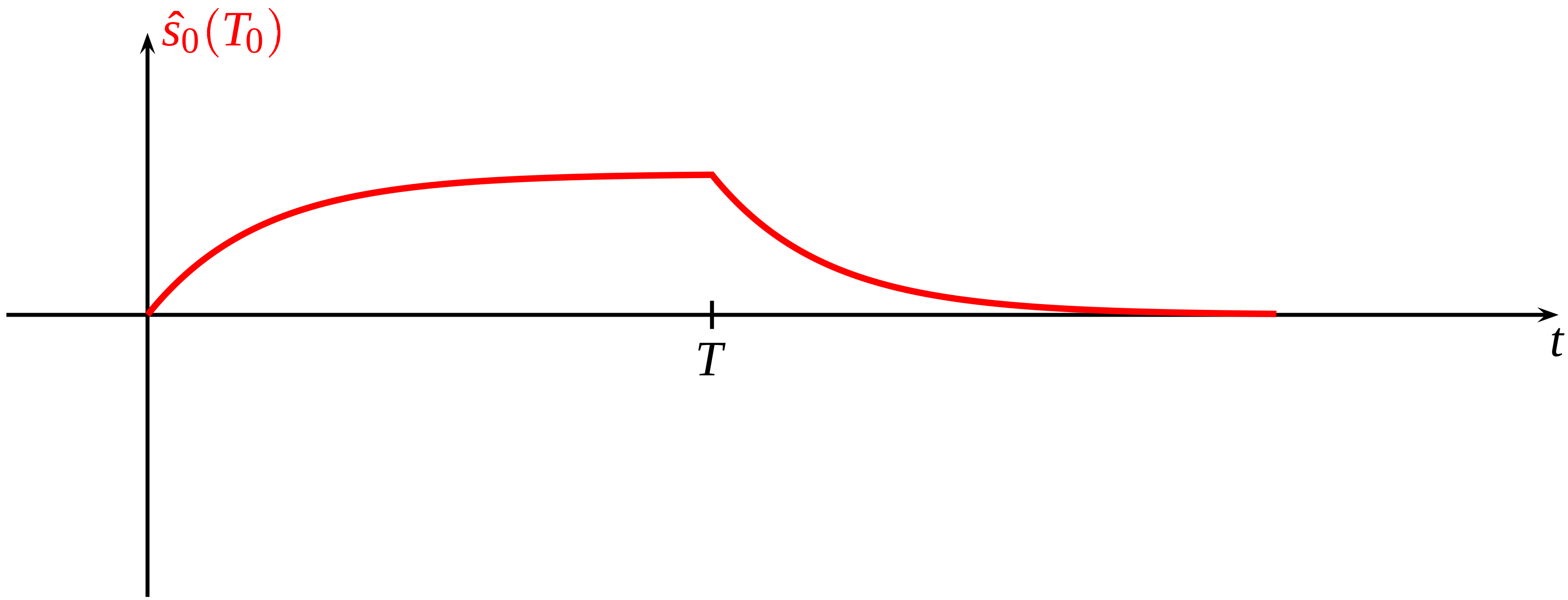
$$\int_0^T h(T_0-\tau)d\tau = \int_0^T \alpha e^{-\alpha(T_0-\tau)}u(T_0-\tau)d\tau$$

$$u(T_0-\tau) = \begin{cases} 0, & T_0-\tau < 0 \quad (\tau > T_0) \\ 1, & T_0-\tau > 0 \quad (\tau < T_0) \end{cases}$$

$$\int_0^T h(T_0-\tau)d\tau = \begin{cases} 0, & T_0 < 0 \\ \int_0^{T_0} \alpha e^{-\alpha(T_0-\tau)}d\tau, & 0 \le T_0 \le T \\ \int_0^T \alpha e^{-\alpha(T_0-\tau)}d\tau, & T_0 > T \end{cases}$$

$$= \begin{cases} 0 & T_0 < 0 \\ +e^{-\alpha(T_0-\tau)}\big|_0^{T_0}, & 0 \le T_0 \le T \\ +e^{-\alpha(T_0-\tau)}\big|_0^{T}, & T_0 \ge T \end{cases}$$

$$= \begin{cases} 0, & T_0 < 0 \\ 1 - e^{-\alpha T_0}, & 0 \le T_0 \le T \\ (1 - e^{-\alpha T}) e^{-\alpha T_0 - T}, & T_0 > T \end{cases}$$



We would like to maximize $|\hat{s}_0(T_0)|/\sigma_n^2$. Since $|\hat{s}_0(T_0)|$ is maximized at $T_0 = T$ and $\sigma_N^2 = \frac{N_0}{4}\frac{\alpha}{2}$ is constant the optimal sampling time is $T_0 = T$. This results in a signal-to-noise ratio of

$$SNR = \frac{(\hat{s}_0(T))}{\sigma_N} = \frac{(1 - e^{-\alpha T})\frac{A}{2}}{\sqrt{\alpha/2}\sqrt{\frac{N_0}{4}}}$$

Maximize w.r.t. α

$$\begin{aligned} f(\alpha) &= \sqrt{2}(1 - e^{-\alpha T})\alpha^{(-1/2)} \\ f'(\alpha) &= \sqrt{2}Te^{-\alpha T}\alpha^{(-1/2)} - 1/2(1 - e^{-\alpha T})\alpha^{(-3/2)} = 0 \\ & T\alpha e^{-\alpha T} - 1/2(1 - e^{-\alpha T}) = 0 \end{aligned}$$

Let $x = \alpha T$ then

$$\begin{aligned} xe^{-x} - 1/2(1 - e^{-x}) &= 0 \\ e^{-x}(x + 1/2) &= 1/2 \\ x + 1/2 &= 1/2e^{x} \end{aligned}$$

$$2x = e^x - 1$$

We can numerically solve this to get $x = 1.256$ and so $\alpha = \frac{1.256}{T} \quad RC = .8T$.
This yeilds a signal-to-noise ratio of

$$SNR = \sqrt{\frac{A^2 T}{N_0}}.9025$$

Loss due to suboptimal receiver = <u>0.89 dB</u>